RAMIFIED CAUCHY PROBLEM FOR A CLASS OF FUCHSIAN OPERATORS WITH TANGENT CHARACTERISTICS

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Abstract. – We consider a ramified Cauchy problem for Fuchsian operators of the form

\[ a(x; D) = x_0(D_0 + q \frac{x_0^{q-1}}{1} D_1)D_0 + \sum_{j=1}^n a_j(x)D_j + b(x). \]

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0. Introduction

A partial differential operator:

\[ a(x; D) = x_0(D_0 + q \frac{x_0^{q-1}}{1} D_1)D_0 + \sum_{j=1}^n a_j(x)D_j + b(x) \]

has three characteristic hypersurfaces \( x_0 = 0, x_1 = 0 \) and \( x_1 - x_0^q = 0 \) and is Fuchsian along \( x_0 = 0 \). We consider a Cauchy problem for \( a(x; D) \) with ramified right-hand side.

For operators of the type \( (D_0 + q \frac{x_0^{q-1}}{1} D_1)D_0 + \text{lower order terms} \), the ramified Cauchy problem has been studied by Wagschal [9]. He gave an integral representation of the solution by using results of Kobayashi [4]. Their techniques are employed after some modification in the present paper.

Ramified Cauchy problems for Fuchsian operators have been studied by Urabe [8], Ouchi [6] and Fujiié [3]. They treated cases where characteristic hypersurfaces are mutually transversal or tangent with an contact of order 1.

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1. Main theorem

In an open neighborhood of \(0 \in \mathbb{C}^{n+1} \setminus \mathbb{C}^1\), \(x = (x_0, x') = (x_0, x_1, \ldots, x_n)\), we consider the following second order partial differential operator with holomorphic coefficients:

\[
a(x, D) = x_0(D_0 + qx_0^{q-1}D_1)D_0 + \sum_{j=1}^{n} a_j(x)D_j + b(x).
\]

Here \(q\) is an integer \(\geq 2\) and \(D_j\) denotes the differentiation with respect to \(x_j\). It is a Fuchsian operator along \(S\): \(x_0 = 0\) of weight 1 and its characteristic exponents are 0 and 1. Hence it induces the following isomorphism:

\[
a(x, D) : x_0^2 \mathbb{C}[x] \sim x_0 \mathbb{C}[x],
\]

where \(\mathbb{C}[x]\) denotes the stalk at the origin of the sheaf of holomorphic functions. Indeed, \(x_0^{-1}a(x, D)x_0^q\) is a Fuchsian operator of weight 0 and its characteristic exponents are \(-1\) and \(-2\). So by the results of [1], it induces an automorphisms of \(\mathbb{C}[x]\).

Following [9], we put

\[
T : x_0 = x_1 = 0, \quad K_0 : x_1 = 0, \quad K_1 : x_1 - x_0^q = 0.
\]

It is easy to see that \(K_0\) and \(K_1\) are characteristic hypersurfaces of \(a(x, D)\) and that \(K_0 \cap K_1 = T\).

Define a function \(h(x)\) by \(h(x) = -x_0^{q}/x_1\) for \(x \notin T\). If \(x_1 = 0\), we set \(h(x) = \infty\) by convention. Then it is easy to see that

\[
S = \{x; h(x) = 0\} \cup T,
K_0 = \{x; h(x) = \infty\} \cup T \supset S \cup K_0 \cup K_1,
K_1 = \{x; h(x) = -1\} \cup T \supset S \cup K_0 \cup K_1.
\]

We consider the following Cauchy problem:

\[
(1) \quad a(x, D)u(x) = x_0v(x), \quad D_0^ju(x)|_{x_0=0} = 0 \quad (j = 0, 1).
\]

Here we assume that there exists an open connected neighborhood \(\Omega\) of the origin such that \(v(x)\) is holomorphic in the universal covering space of \(\Omega \setminus (K_0 \cup K_1)\). In a neighborhood of \(y \in \Omega \cap (S \setminus T)\), the Cauchy problem (1) admits a unique holomorphic solution.

Our main result is:

**Theorem 1.** – There exists an open connected neighborhood \(\mathcal{O}\) of \(0 \in \mathbb{C}^{n+1}\) such that for \(j = 0, 1\) the solution \(u(x)\) to (1) extends holomorphically to the universal covering space of \(\mathcal{O} \setminus A_j\).

Notice that the point \(y\) can be assumed to be arbitrarily close to the origin by [1]. Moreover in Section 7, we assume, without loss of generality by [1] again, that \(y \in \{x_0 = 0, x_1 < 0\}\).
Remark. – The above theorem means that \( u(x) \) extends holomorphically along any path \( \gamma : I \to \mathcal{O} \) with \( \gamma(0) = y \) and \( \gamma(t) \not\in A_j \) for \( t > 0 \). Here \( I \) is the closed interval \([0, 1]\).

Consider, for example, a path \( \gamma_0 : I \to \mathcal{O} \) with \( \gamma_0(0) = y \) such that \( h \circ \gamma_0(t) = 4t (0 \leq t \leq 1/2) \) and that \( h \circ \gamma_0(t) \in \mathbb{C} \) rotates many times along the circle \( |z| = 2 \) as \( t \) increases from 1/2 to 1. This is a situation where \( \gamma_0(t) \) moves around \( K_0 \setminus T \). So the theorem \((j = 0)\) implies the ramification of the solution \( u(x) \) around \( K_0 \setminus T \).

Next, let \( \varepsilon > 0 \) be sufficiently small and consider a path \( \gamma_1 : I \to \mathcal{O} \) with \( \gamma_1(0) = y \) such that \( h \circ \gamma_1(t) = -2(1 - \varepsilon)t (0 \leq t \leq 1/2) \) and that \( h \circ \gamma_1(t) \) rotates many times along \( |z + 1| = \varepsilon \) for \( t \geq 1/2 \). This is a situation where \( \gamma_1(t) \) moves around \( K_1 \setminus T \). So the theorem \((j = 1)\) implies the ramification of \( u(x) \) around \( K_1 \setminus T \).

2. Integral representation

Following [9], we will give an integral representation of the solution \( u(x) \). It will be given in the form of a series whose \( m \)-th term is defined by using an integral on a singular \( m \)-simplex.

Let \( \Delta_m \) \((m \geq 1)\) be the standard \( m \)-dimensional simplex \( \subset \mathbb{R}^m \):

\[
\Delta_m = \left\{ t \in \mathbb{R}^m; 0 \leq t_1 \leq \cdots \leq t_m \leq 1 \right\}, \quad t = (t_1, \ldots, t_m).
\]

The system of coordinates of \( \mathbb{C}^m \) is \( \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_m) \) and the singular \( m \)-simplex \( S_m = S_m(x_0) \), depending on the parameter \( x_0 \in \mathbb{C} \), is defined by:

\[
S_m(x_0) : t \in \Delta_m \mapsto x_0 t \in \mathbb{C}^m.\]

If \( f = f(\sigma, x) \) is a holomorphic function near \((0, 0) \in \mathbb{C}^m \times \mathbb{C}^{n+1} \), we define the integral of the \( m \)-form \( f \, d\sigma_{(m)} \), \( d\sigma_{(m)} = d\sigma_1 \wedge \cdots \wedge d\sigma_m \), on \( S_m \) by:

\[
I(x) = \int_{S_m} f \, d\sigma_{(m)} = \int_{S_m} f(\sigma, x) \, d\sigma_1 \wedge \cdots \wedge d\sigma_m = \int_{\Delta_m} f(\sigma_0, x) t_1 \wedge \cdots \wedge dt_m
\]

(2)

Later we will extend this definition to some meromorphic functions.

We have

\[
D_0 I(x) = \int_{S_m} D_0 f \, d\sigma_{(m)} + \int_{S_m} f \big|_{\sigma_m = 0} \, d\sigma_{(m-1)},
\]

\[
D_j I(x) = \int_{S_m} D_j f \, d\sigma_{(m)} \quad 1 \leq j \leq n.
\]

This is valid for any \( m \in \mathbb{Z} \) if we set by convention:

\[
\int_{S_m} f(x) \, d\sigma_{(m)} = \begin{cases} 
\int_0^1 f(x) \, dt \quad & (m = 0), \\
0 & (m < 0).
\end{cases}
\]
We introduce multiphase functions following [9]. Put $k_0(x) = \varphi_0(x) = x_1$, $k_1(x) = x_1 - x_0^q$ and for $m \geq 1$,
\[
\varphi_m(\sigma, x) = k_m(x) + \sum_{j=1}^{m} (-1)^{j+1} \sigma_j^q,
\]
where $k_m = k_0$ if $m$ is even and $k_m = k_1$ if $m$ is odd. They satisfy the eikonal equation for $a(x, D)$ and we have:
\[
\varphi_{m+2}\big|_{\sigma_{m+2}=x_0} = \varphi_m + 1\big|_{\sigma_{m+1}=x_0} = \varphi_m \quad (m \geq 0).
\]
We will also use the functions:
\[
\psi_{k, l}(\sigma_k, \ldots, \sigma_l) = \sum_{j=k}^{l} (-1)^{j+1} \sigma_j^q, \quad 1 \leq k \leq l.
\]

We fix some notation. The space $C_{x_0}^{n-1}$, $x'' = (x_2, \ldots, x_n)$, is equipped with the norm $||x''|| = \max_{2 \leq j \leq n} |x_j|$ and its subset
\[
D_{a}^{n-1} = \{x'' \in C_{x_0}^{n-1}; ||x''|| < a\}, \quad a > 0
\]
is a polydisk. In $C_{\sigma_0}^{m-2}$, $\sigma' = (\sigma_2, \ldots, \sigma_{m-1})$, the open neighborhood of the origin $\Omega_{a}^{m-2}$ ($a > 0$) is defined by:
\[
\Omega_{a}^{m-2} = \{\sigma' \in C_{\sigma_0}^{m-2}; \max_{2 \leq j \leq m-1} |\sigma_j| < a, \quad \max_{2 \leq j \leq m-1} |\psi_{2, l}(\sigma')| < a\}.
\]
Moreover we will need $C_{\xi}^2$, $\xi = (\xi_0, \xi_1)$. We put $||\xi|| = \max(|\xi_0|, |\xi_1|)$ and $D_{a}^2 = \{\xi \in C_{\xi}^2; ||\xi|| < a\}, a > 0$. We consider the hypersurfaces:
\[
\mathcal{K}_0: \xi_1 = 0, \quad \mathcal{K}_1: \xi_1 = \xi_0^q
\]
and set
\[
\mathcal{X} = C_{\xi}^2 \setminus (\mathcal{K}_0 \cup \mathcal{K}_1), \quad \mathcal{X}_a = \mathcal{X} \cap D_{a}^2.
\]
Let $\tilde{\mathcal{X}}$ and $\tilde{\mathcal{X}}_a$ be their universal covering space respectively.

We will give the solution $u(x)$ to (1) near the point $y$ in the form of a series of the type
\[
(3) \quad u(x) = \sum_{m=2}^{\infty} I_m(x),
\]
with
\[
I_m(x) = \int_{S_m} u_m(\sigma_1, \varphi_m(\sigma, x), \sigma', x'') \, d\sigma(m)
\]
\[
= \int_{\Delta_m} u_m \left( x_0 t_1, \varphi_m(x_0 t_1, x), x_0 t_2, \ldots, x_0 t_{m-1}, x'' \right) x_0^m \, dt_1 \wedge \cdots \wedge dt_m.
\]
where \( u_m = u_m(\zeta, \sigma', x'') \) are meromorphic functions which will be constructed later. As a matter of fact, if \( m \) is odd, then \( u_m \equiv 0 \) and \( I_m(x) \equiv 0 \).

In the next section, we will prove:

**Theorem 2.** There exist \( a > 0, r > 0 \) and holomorphic functions

\[
u_m : \hat{\mathcal{X}}_a \times \{ \sigma' \in \Omega^{m-2}; \sigma_3 \sigma_5 \mathcal{S} \cdots \mathcal{S}_{m-3} \mathcal{S}_{m-1} \neq 0 \} \times D_r^{n-1} \to C,
\]

for \( m = 2, 4, 6, \ldots \), such that the series (3) convergent near \( y \), is the solution to (1). Here \( u_m \equiv 0 \) if \( m \) is odd. Moreover \( \sigma_3 \sigma_5 \mathcal{S} \cdots \mathcal{S}_{m-3} \mathcal{S}_{m-1} u_m(\zeta, \sigma', x'') \) is holomorphic in \( \hat{\mathcal{X}}_a \times \Omega^{m-2} \times D_r^{n-1} \) and for any compact set \( \mathcal{K} \subset \hat{\mathcal{X}}_a \), there exists a constant \( c_K > 0 \) such that:

\[
|\sigma_3 \sigma_5 \mathcal{S} \cdots \mathcal{S}_{m-3} \mathcal{S}_{m-1} u_m| \leq c_K^{m+1} \cdot \frac{m}{2} \cdot \left( \frac{m}{2} \right)!,
\]

for all \((\zeta, \sigma', x'') \in \mathcal{K} \times \Omega_r^{m-2} \times D_r^{n-1}, m = 4, 6, 8, \ldots \).

Let us prove that the series in (3) is uniformly and absolutely convergent in a neighborhood of \( y \). As is shown in [9], there exist a neighborhood \( V \) of \( y \) and a compact set \( \mathcal{K} \subset \hat{\mathcal{X}}_a \) such that for all \( x \in V \), all \( m \) and all \( t \in \Delta_m \), we have:

\[
(x_0 t_1', \varphi_m(x_0 t_1, x)) \in \mathcal{K}, \quad (x_0 t_2', \ldots, x_0 t_m) \in \Omega_r^{m-2}, \quad x'' \in D_r^{n-1}.
\]

Here \( \mathcal{K} \) can be made arbitrarily small and can be regarded as a compact subset of \( \hat{\mathcal{X}}_a \). This implies that

\[
|\sigma_3 \sigma_5 \mathcal{S} \cdots \mathcal{S}_{m-1} u_m(\sigma_1, \varphi_m(\sigma, x), \sigma', x'')| \leq c_K^{m+1} \cdot \frac{m}{2} \cdot \left( \frac{m}{2} \right)!,
\]

for \( \sigma = \mathcal{S}_m(\zeta_0)(t), \quad x \in V \). Hence we have:

(4) \[
|u_m(x_0 t_1', \varphi_m(x_0 t_1, x), x_0 t_2', \ldots, x_0 t_m, x'')| \leq \frac{|x_0|^{1+m/2}}{t_1 t_3 t_5 t_7 \cdots t_{m-1} t_m} \cdot c_K^{m+1} \cdot \frac{m}{2} \cdot \left( \frac{m}{2} \right)!. \]

**Lemma 1.** For \( m = 4, 6, 8, \ldots \), we have the following equalities:

(5) \[
i_m = \frac{1}{\Delta_m} \int \frac{dt_1 dt_2 \cdots dt_{m-1}}{(t_1 t_3 t_5 t_7 \cdots t_{m-3} t_{m-1})^{1/2}} = \frac{1}{2} \left( \frac{m}{2} \right) ! \cdot \frac{m}{2} \cdot \left( \frac{m}{2} \right) !.
\]

(6) \[
j_m = \frac{1}{\Delta_m} \int \frac{dt_1 dt_2 \cdots dt_{m-1}}{(t_1 t_3 t_5 t_7 \cdots t_{m-3} t_{m-1})^{1/2}} = \left( \frac{m}{2} \right) ! \cdot \frac{m}{2} \cdot \left( \frac{m}{2} \right) !.
\]

**Proof.** We perform the change of variables \( t_1 = t_3 s_1, \quad t_2 = t_3 s_2 \) (here \( t_3 \) is fixed) to obtain:

\[
i_4 = \int \frac{dt_1 dt_2 dt_3 dt_4}{t_3^2} = \int \frac{dt_3 dt_4}{t_3^2} \int \frac{dt_1 dt_2}{t_3^2} = \int \frac{dt_3 dt_4}{t_3^2} \int \frac{dt_1 dt_2}{t_3^2} = \int 1 \frac{dt_3 dt_4}{t_3^2} = \left( \frac{1}{2} \right)^2.
\]

Moreover we have the recurrence relation \( i_m = \frac{1}{2} i_{m-2} \) for \( m = 6, 8, 10, \ldots \), because by the same change of variables:
Moreover we have:

Therefore the series in (3) converges and determines a holomorphic function in $V$ for $m$ for all $t_j = t_j S_j, t_2 = t_3 S_2$ again and get

Moreover we have:

for $m = 6, 8, 10, \ldots$, because by the change of variables $t_j = t_{m-1} S_j (j = 1, 2, \ldots, m - 2)$ (here $t_{m-1}$ is fixed) we get:

By using (4) and (6), we find that $I_m(x)$ is holomorphic in $V$ and that

Therefore the series in (3) converges and determines a holomorphic function in $V$.

In the proof of (32), we will need the following:

**Lemma 2.** Let $C$ be a positive constant and set $\Delta_{m,\varepsilon} = \Delta_m \cap \{t_3 \leq C \varepsilon\}$ for $m = 4, 6, \ldots, \varepsilon > 0$. Then there exists a positive constant $M_m$ independent of $\varepsilon$ such that we have:

for all $\varepsilon > 0$. 

\[
I_m(x) \leq |x_0|^{1+\frac{m}{2}} \varepsilon^{m+1} \left\{ \left( \frac{m}{2} \right)^2 \right\}^{-1}.
\]
Proof. – We prove this estimate by induction on \( m \). First we have:

\[
j_4(\varepsilon) = \int_{\{0 \leq t_1 \leq t_2 \leq 1 \mid t_3 \leq C\varepsilon\}} dt_3 \, dt_4 = \int_{\{0 \leq t_1 \leq t_2 \leq 1 \mid t_3 \leq C\varepsilon\}} \frac{dt_1 \, dt_2}{t_3}
\]

Next for \( m \geq 6 \), assume that we have \( j_{m-2}(\varepsilon) \leq M_{m-2} \varepsilon^2 \); then

\[
j_m(\varepsilon) = \int_{0 \leq t_{m-1} \leq t_m \leq 1} \frac{I}{t_{m-1}} \, dt_{m-1} \, dt_m.
\]

Here \( I \) is defined by:

\[
I = \int_{\{0 \leq t_1 \leq \cdots \leq t_{m-1} \mid t_3 \leq C\varepsilon\}} \frac{dt_1 \, dt_2 \cdots dt_{m-2}}{t_3 \, t_5 \cdots t_{m-3}}
\]

By putting \( t_j = t_{m-1} s_j \) (\( 1 \leq j \leq m-2 \)), we get

\[
I = t_{m-1}^{m/2} \int_{\{0 \leq s_1 \leq \cdots \leq s_{m-2} \leq 1 \mid C\varepsilon \leq t_{m-1}\}} \frac{ds_1 \, ds_2 \cdots ds_{m-2}}{s_3 \, s_5 \cdots s_{m-3}}
\]

\[
= t_{m-1}^{m/2} j_{m-2}(\varepsilon/T_{m-1}) \leq t_{m-1}^{m/2} \cdot M_{m-2}(\varepsilon/T_{m-1})^2.
\]

Therefore we obtain:

\[
j_m(\varepsilon) \leq M_{m-2} \varepsilon^2 \int_{0 \leq t_{m-1} \leq t_m \leq 1} \frac{t_{m-1}^{m-3}}{t_{m-1}^{m-1} \, dt_{m-1} \, dt_m} \leq \frac{M_{m-2} \varepsilon^2}{2}.
\]

3. Proof of Theorem 2

We will present sufficient conditions on the functions \( u_m \) for the series (3) to give the solution \( u(x) \) to (1).

We can prove the following lemma easily:

**Lemma 3.** – For the function \( I(x) \) defined by (2) we have:

\[
a(x, D)I(x) = \mathop{\sum}_{S_w} A^m \mathop{\sum}_{S_{m-1}} A^m f|_{\sigma_m = \sigma_0} \, d\sigma_{(m-1)} + \mathop{\sum}_{S_w} x_0 f|_{\sigma_m = \sigma_0 = \sigma_{(m-2)}} \, d\sigma_{(m-2)},
\]

where \( A^m = A^m_1(x, D_{\sigma_0}, D_0, D_1) = x_0(D_{\sigma_0} + 2D_0 + q x_0^{-1} D_1) \).

Next consider functions \( u_{\delta}(\xi, \sigma', x''') \) meromorphic in a neighborhood of \( (\xi, \sigma', x''') = ((0, 0), 0, 0) \) and \( u_{\delta} \circ \varphi_m = u_{\delta}(\sigma', \varphi_m(\sigma, x'), \sigma'', x''') \). Differentiation with respect to \( \xi_1 \) is denoted by \( \partial_1 \). Then we can show the following two lemmas:
LEMMMA 4. – We have
\[ a(x, D)(u_x \circ \varphi_m) = \left\{ P_1(x, D^{\prime \prime})u_x \right\} \circ \varphi_m + \left\{ P_0^m(x) \partial_1 u_x \right\} \circ \varphi_m, \]
where
\[ P_1(x, D^{\prime \prime}) = \sum_{j=2}^{n} a_j(x) D_j + b(x), \]
\[ P_0^m(x) = \left\{ x_0 D_0^2 + a_1(x) D_1 \right\} k_m(x). \]
The function \( P_0^m \) depends only on the parity of \( m \).

LEMMMA 5. – We have
\[ A^m_m(x, D_{\sigma_m}, D_0, D_1)(u_x \circ \varphi_m) = 0 \quad \text{for} \quad \sigma_m = x_0. \]
The function \( u(x) \) defined by (3) is the solution to (1) if we have, for \( m \geq 0, \sigma_{m+1} = x_0, \zeta_0 = \sigma_1, \zeta_1 = \varphi_m(\sigma, x) \).

(7) \[ \left\{ P_1(x, D^{\prime \prime}) + P_0^m(x) \partial_1 \right\} u_m + x_0 u_{m+2} = \delta_m^0 x_0 v(x). \]
Here \( \delta_0^0 = 1 \) and \( \delta_m^0 = 0 \) for \( m > 0 \). Moreover we set \( u_m \equiv 0 \) if \( m \leq 0 \) or \( m \) is odd.

From now on we only consider the cases where \( m \) is even. Then \( P_0^m = a_1(x) \) is independent of \( m \).

For \( m = 0 \), we obtain
\[ u_2(\zeta, x^{\prime \prime}) = u(\zeta, x^{\prime \prime}). \]
Therefore there exists \( b > 0 \) such that \( u_2 \) is holomorphic in \( \mathbb{C}_b \times D^{n-1}_{\beta}. \)

For \( m = 4, 6, 8, \ldots \), the recurrence relation (7) can be written in the form
\[ x_0 u_m = Q(x, \partial_1, D^{\prime \prime})u_{m-2} \]
for \( \sigma_{m-1} = x_0, \zeta_0 = \sigma_1, \zeta_1 = \varphi_2(\sigma_1, \ldots, \sigma_{m-2}, x) \), where \( Q \) is a first order operator independent of \( m \).

Notice that the following conditions (a) and (b) are equivalent:
(a) \( \sigma_{m-1} = x_0, \zeta_1 = \varphi_m(\sigma_1, \ldots, \sigma_{m-2}) \)
(b) \( x_0 = \sigma_{m-1}, x_1 = \zeta_1 = \sigma_1 = \varphi_{m-2}(\sigma_2, \ldots, \sigma_{m-2}) \).

Hence \( u_m(m = 4, 6, 8, \ldots) \) is given by:
\[ \sigma_{m-1} u_m(\xi, \sigma_2, \ldots, \sigma_{m-1}, x^{\prime \prime}) = R(\alpha, \beta, \xi, x^{\prime \prime}, \partial_1, D^{\prime \prime})u_{m-2}, \]
for \( \alpha = \sigma_{m-1}, \beta = \varphi_{m-2}(\sigma_2, \ldots, \sigma_{m-2}) \), where \( R(\alpha, \beta, \xi, x^{\prime \prime}, \partial_1, D^{\prime \prime}) \) is a first order operator with holomorphic coefficients in a neighborhood of the origin of \( \mathbb{C}^2_{\alpha, \beta} \times \mathbb{C}^2_\xi \times \mathbb{C}^{n-1}_{x^{\prime \prime}} \).

Choose positive constants \( R' \) and \( R'' \) with \( 0 < R' < R'' \) such that all the coefficients of \( R(\alpha, \beta, \xi, x^{\prime \prime}, \partial_1, D^{\prime \prime}) \) are holomorphic and bounded in \( \Delta^2_R \times D^2_{R'} \times D^{n-1}_{R''} \), where \( \Delta^2_R \) is the polydisk defined by:
\[ \Delta^2_R = \{ (x, \beta) \in \mathbb{C}^2; \max \{|x|, |\beta|\} < R' \}. \]
There exists a constant $c_0 > 0$ such that if $r(\alpha, \beta, \xi, x'')$ is anyone of these coefficients, we have

$$\partial^q r(\alpha, \beta, \xi, x'') \ll c_0^{q+1} q! \frac{1}{R'' - \xi}, \quad \xi = \sum_{j=2}^n x_j,$$

for all $q \geq 0$ and all $(\alpha, \beta, \xi) \in \Delta_{R''}^2 \times D_{R''}^2$. This estimate means that the right-hand side is a majorant power series in $x'' = (x_2, \ldots, x_n)$: that is, the variables $\alpha, \beta$ and $\xi$ are parameters.

Let $R' > 0$ be so small that $0 < R' \leq b$ and put $a = R'$. Then we have

**LEMMA 6.** The function $\sigma_3 \sigma_5 \sigma_7 \cdots \sigma_{m-3} \sigma_{m-1} u_m (\xi, \sigma', x'')$ is holomorphic in $\hat{X}_a \times \Omega_{R''}^{m-2} \times D_{R''}^{m-1}$ for $m = 4, 6, 8, \ldots$.

Choose a constant $R$ with $0 < R < R'$ and set

$$\Phi(\xi) = \frac{1}{R - \xi}.$$

It is known that for all $l \in \mathbb{N} = \{0, 1, 2, \ldots\}$, we have

$$\frac{1}{R'' - \xi} D^l \Phi(\xi) \ll \frac{1}{R'' - R} D^l \Phi(\xi).$$

Then we obtain the following lemma:

**LEMMA 7.** For any compact subset $K \subset \hat{X}_a$, there exists a constant $c > 0$ such that for all $p \in \mathbb{N}$, $\xi \in K$, $m = 4, 6, 8, \ldots$ and $\sigma' \in \Omega_{R''}^{m-2}$, we have:

$$\partial^p \left[ \sigma_3 \sigma_5 \cdots \sigma_{m-1} u_m (\xi, \sigma', x'') \right] \ll \sum_{i+j \leq (m-2)/2} c^{m+p} (p+i)! D^i \Phi(\xi), \quad \xi = \sum_{j=2}^n x_j.$$

**Proof.** Use the method of the proof of [9], Lemme 3.5. \(\square\)

Choose a constant $r_0$ with $0 < r_0 < R$. Then for all $(\xi, \sigma') \in K \times \Omega_{R''}^{m-2}$ and $x'' \in \{x''; \sum_{j=2}^n |x_j| < r_0\}$, we have

$$|\sigma_3 \sigma_5 \cdots \sigma_{m-1} u_m| \leq \sum_{i+j \leq (m-2)/2} c^{m+i} \frac{j!}{(R - r_0)^{j+1}}.$$

Since $l!j! \leq (m/2)!$, the proof of Theorem 2 is completed if we set $r = r_0/(n-1)$.

### 4. Preliminary construction

Let $f : \mathbb{C} \to \mathbb{C}$ be the function $f(z) = \varepsilon^l + 1$ and set $K = f^{-1}(2S^1) = \{z; |\varepsilon^l + 1| = 2\} \subset \mathbb{C}^*$. The curve $K$ is a smooth simple closed curve because $f$ is locally a diffeomorphism in $\mathbb{C}^* \supset K$.

Define a smooth curve $\alpha_p : I \to K (0 \leq p \leq q - 1)$ by:

$$f \circ \alpha_p(t) = 2\beta(t), \quad \alpha_p(0) = \beta(p/q), \quad \alpha_p(1) = \beta((p+1)/q), \quad 2\pi p/q \leq \arg \alpha_p(t) \leq 2\pi (p+1)/q.$$

Here we set $\beta(t) = \exp(2\pi it)$. We see that $K = \bigcup_{p=0}^{q-1} \alpha_p$. 

In addition, we set $\alpha_p^{\pm q} = \alpha_p (n \in \mathbb{Z})$ and hence $\alpha_p$ is defined for any integer $p$. We put $\alpha_p^{-1}(t) = \alpha_p(1-t)$.

Let $\omega_0, \ldots, \omega_{q-1}$ be the $q$-th roots of $-1$ defined by $\omega_p = \beta((2p+1)/2q)$, $0 \leq p < q-1$, and $L_p$ be the (closed) segment joining $0$ and $\omega_p$: that is,

$$L_p = \{z = i\omega_p \in \mathbb{C}; \ i \in I\}, \ I = [0, 1].$$

Set $L = \bigcup_{p=0}^{q-1} L_p$. Obviously we have $L = \{z; -1 \leq z \leq 0\}$ and $f : \mathbb{C} \setminus L \to \mathbb{C} \setminus I$.

Let $R : (\mathbb{C} \setminus I) \times I \to \mathbb{C} \setminus I$ be the deformation retraction of $\mathbb{C} \setminus I$ onto $2S^1$ defined by:

$$R(z, s) = (1 - s)z + 2s \frac{z}{|z|}, \quad (z, s) \in (\mathbb{C} \setminus I) \times I.$$

In polar coordinates, it is written in the form

$$R(r \exp(i\theta), s) = ((1 - s)r + s) \exp(i\theta), \quad (r \exp(i\theta), s) \in (\mathbb{C} \setminus I) \times I,$$

and $|R(z, \cdot)|$ is monotone.

There exists a continuous mapping $R' : (\mathbb{C} \setminus L) \times I \to \mathbb{C} \setminus L$ with $R'(z, 0) = z$ such that for $(z, s) \in (\mathbb{C} \setminus L) \times I$, we have:

$$(f \circ R')(z, s) = R(f(z), s) = (1 - s)f(z) + 2s \frac{f(z)}{|f(z)|}.$$

(The mappings $R$ and $R'$ are defined in a different way in [9].) If $z \in \mathbb{C} \setminus L$ is in the locally closed sector

$$S(p) = \{z; 2\pi p/q \leq \arg z < 2\pi (p+1)/q\} \subset \mathbb{C}^*$$

with $0 \leq p < q-1$, then $R'(z, s)$ is also in the same sector $S(p)$. The mapping $R'$ is a deformation retraction of $\mathbb{C} \setminus L$ onto $K$. More precisely, it induces a deformation retraction of $(\mathbb{C} \setminus L) \cap S[p] = S[p] \setminus L_p$ onto $\alpha_p$ for $0 \leq p < q-1$, where:

$$S[p] = \{z; 2\pi p/q \leq \arg z < 2\pi (p+1)/q\} \subset \mathbb{C}^*.$$

Since $f$ is locally a diffeomorphism, we can write, locally:

$$R'(z, s) = f^{-1} \circ R(f(z), s)$$

That is, $R'$ is obtained from $R$ by a change of coordinates. This fact implies the smoothness of $R'$. It is clear that $|f(R'(z, \cdot))| = |R(f(z), \cdot)|$ is monotone.

Let us consider a path $\gamma : I \to \mathbb{C}$ with

\[
\begin{align*}
\gamma(0) &= 0, & \gamma(t) &\not\in L \quad &\text{if } t \neq 0, \\
\frac{-1}{q} \pi &< \arg\gamma(t) < \frac{1}{q} \pi & \text{if } 0 \leq t < \epsilon,
\end{align*}
\]

for some $\epsilon > 0$ and some branch of arg.

The latter condition means that $\gamma([0, \epsilon])$ is in the sector between $L_{-1}$ and $L_0$. (We put $L_{-1} = L_{q-1}$ by convention.)

We say that two paths $\gamma$ and $\gamma'$ satisfying $(\ast)$ with $\gamma(1) = \gamma'(1)$ are $(\ast)$-homotopic and write $\gamma \sim \gamma'$ if there exists a continuous mapping $H(s, t) : I \times I \to \mathbb{C}$ such that:
\[ H(s, t) \notin L \quad \text{if } t \neq 0, \]
\[ H(0, t) = \gamma(t), \quad t \in I, \]
\[ H(1, t) = \gamma'(t), \quad t \in I. \]

It is easy to see that \( \sim \) is an equivalence relation.

A path \( \gamma \) satisfying (*) is \((\ast)\)-homotopic to the path \( \gamma_1 \gamma_2 \gamma_3 \), where

\[ \gamma_1(t) = t, \]
\[ \gamma_2(t) = \begin{cases} 
\lambda_j(J t - j + 1), & (j - 1)/J \leq t \leq j/J, \quad 1 \leq j \leq [J], \\
\lambda_{[J]+1}(J t - [J]), & [J]/J \leq t \leq 1,
\end{cases} \]
\[ \gamma_3(t) = R'(\gamma(1), 1 - t), \]
\[ (\gamma_1 \gamma_2 \gamma_3)(t) = \begin{cases} 
\gamma_1(4t), & 0 \leq t \leq 1/4, \\
\gamma_2(4t - 1), & 1/4 \leq t \leq 1/2, \\
\gamma_3(2t - 1), & 1/2 \leq t \leq 1.
\end{cases} \]

Here \( \lambda_k = \alpha_{k-1} \) \((1 \leq k \leq [J] + 1)\) or \( \lambda_k = \alpha_k^{-1} \) \((1 \leq k \leq [J] + 1)\).

It is easy to see that \( |f \circ \gamma_1(t)| \) increases from 1 to 2 and that \( |f \circ \gamma_2(t)| \equiv 2 \). In addition, \( |f \circ \gamma_3(t)| \) is monotone and its value changes from 2 to \( |f(\gamma(1))| \).

Consider the set of all paths satisfying (*) and let \( \mathcal{R}(C \setminus L) \) be its quotient space by the relation \( \sim \). Let \( \pi_L : \mathcal{R}(C \setminus L) \to C \setminus L \), \( \hat{z} = [\gamma] \mapsto z = \gamma(1) \) be the canonical projection.

We can identify \( \mathcal{R}(C \setminus L) \) with the universal covering space of \( C \setminus L \) as is defined in the usual way. One way to construct the universal covering space is taking the quotient by homotopy of the set of all paths \( C \setminus L \) issuing from the point 1. Let \( (C \setminus L)^\sim \) be the one defined in this way. Then \( \mathcal{R}(C \setminus L) \) is identified with \( (C \setminus L)^\sim \) by the following correspondence: set \( l(t) = 1 - t, \quad t \in I \), and if \( \gamma \) satisfies (*), then associate to it a suitable deformation of \( l \gamma \), where:

\[
\left\{ \begin{array}{ll}
(l \gamma)(t) = l(2t), & 0 \leq t \leq 1/2, \\
(l \gamma)(t) = \gamma(2t - 1), & 1/2 \leq t \leq 1.
\end{array} \right.
\]

Since \( \gamma_1 \gamma_2 \gamma_3 \) depends only on the \((\ast)\)-homotopy class of \( \gamma \), we can define a continuous function

\[ F : \mathcal{R}(C \setminus L) \times I \to Z, \]
\[ Z = \{ z \in C; f(z) = z^q + 1 \neq 0 \} \subset C, \]

by using the above process. Obviously we have:

\[ F : \mathcal{R}(C \setminus L) \times (I \setminus \{0\}) \to C \setminus L. \]

We see that for \( \hat{z} \in \mathcal{R}(C \setminus L) \),

\[ F(\hat{z}, 0) = 0, \quad F(\hat{z}, 1) = \pi_L(\hat{z}) = z \in C \setminus L. \]

We define a continuous function \( G : \mathcal{R}(C \setminus L) \times I \to C^* \) by:

\[ G^q = f \circ F = F^q + 1, \quad G(\hat{z}, 0) = 1. \]
Then we have $G(\hat{z},1)^q = f(z)$, $z = \pi_L(\hat{z})$.

We set $r = |G| : \mathcal{R}(C \setminus L) \times I \to \mathbb{R}_+^\ast$. It is easy to see that

- $r(\hat{z}, \cdot)$ is non-decreasing in $0 \leq t \leq 1/2$ and its value changes from 1 to $2^{1/q}$,
- $r(\hat{z}, \cdot)$ is monotone in $1/2 < t \leq 1$ and its value changes from $2^{1/q}$ to $|f(z)|^{1/q}$.

Let $\theta : \mathcal{R}(C \setminus L) \times I \to \mathbb{R}$ be a continuous function with $G = re^{i\theta}$. The function $\theta(\hat{z}, \cdot)$ is piecewise affine and its total variation $\Theta : \mathcal{R}(C \setminus L) \to \mathbb{R}_+$ is continuous in $\mathcal{R}(C \setminus L)$.

Later we will need the continuous function $H = F/G : \mathcal{R}(C \setminus L) \times I \to \mathbb{C}$.

5. The simplex $S_m(\hat{z}, \cdot)$

Let $m$ be an even positive integer. We define the mapping $S_m = (\sigma_1, \ldots, \sigma_m) : \mathcal{R}(C \setminus L) \times \Delta_m \to \mathbb{C}_m$ by

$$
\xi_j = F(\hat{z}, t_j), \quad \eta_j = G(\hat{z}, t_j) \quad \text{if } j \text{ is even},
\xi_j = H(\hat{z}, t_j), \quad \eta_j = G(\hat{z}, t_j)^{-1} \quad \text{if } j \text{ is odd},
$$

$$
\sigma_j = \sigma_j(\hat{z}, t_j) = \xi_j \prod_{i=j+1}^m \eta_i
$$

for $1 \leq j \leq m$. Here $\prod_{i=m+1}^m \eta_i = 1$ by convention. These functions are continuous and piecewise smooth.

We denote the faces of $\Delta_m$ by:

$$
\Delta_m^j = \{ t \in \Delta_m ; t_j = t_{j+1} \} \quad (0 \leq j \leq m), \quad t_0 = 0, \quad t_{m+1} = 1
$$

and introduce a family of hyperplanes in $\mathbb{C}_m^m$ defined by

$$
H^j(z) = \{ \sigma \in \mathbb{C}_m^m ; \sigma_j = \sigma_{j+1} \} \quad (0 \leq j \leq m), \quad \sigma_0 = 0, \quad \sigma_{m+1} = z.
$$

**Lemma 8.** We have for $m = 2, 4, 6, \ldots$,

$$
\psi_{k,l} \circ S_m = - \left( \sum_{i=k}^m \eta_i^q - \prod_{i=k+1}^m \eta_i^q \right) \quad \text{for } 1 \leq k \leq l \leq m,
$$

$$
1 - \psi_{k,m}(S_m(\hat{z}, t)) = \prod_{i=k}^m \eta_i^q \neq 0 \quad \text{for } 1 \leq k \leq m,
$$

$$
S_m(\hat{z}, \Delta_m^j) \subset H^j(z) \quad \text{for } 0 \leq j \leq m, \quad z = \pi_L(\hat{z}).
$$

**Proof.** It can be proved in the same way as [9], (6.3), (6.7) and (6.8). \qed

Next we will establish some estimates.

Set $r_j = r(\hat{z}, t_j) = |G(\hat{z}, t_j)|$, $s_j = |\eta_j|$ for $1 \leq j \leq m$. It is trivial that $s_j = r_j$ if $j$ is even and $s_j = 1/r_j$ if $j$ is odd.

**Lemma 9.** Let $k$ be an integer with $1 \leq k \leq m$. Then

$$
\min \left( \frac{1}{2}, \frac{|f(z)|}{2} \right) \leq \left( \prod_{j=k}^m s_j \right)^q \leq \max \{ 4, |f(z)| \}.
$$
Proof. – We may assume that $k = 1$, because we can get the general case by putting $t_1 = \cdots = t_{k-1} = 0$.

(i) If $|f(z)| \geq 2$, then $r$ is not smaller than 1 and non-decreasing in $[0, 1]$. So by virtue of the same argument as in [9], Lemma 6.1, we obtain

$$1 \leq \left( \prod_{j=1}^{m} s_j \right)^q \leq |f(z)|.$$

(ii) Assume that $|f(z)| < 2$. Then $r$ is non-decreasing in $[0, 1/2]$ and decreasing in $[1/2, 1]$. Let $l \geq 2$ be such that $0 \leq t_1 \leq \cdots \leq t_{l-1} < 1/2 \leq t_l$. Then it follows that $1 \leq r_1 \leq \cdots \leq r_{l-1} \leq 2^{1/q}$ and we can deal with $(\prod_{j=1}^{l-1} s_j)^q$ and $(\prod_{j=1}^{l-2} s_j)^q$ by the method of [9], Lemma 6.1. We obtain

$$1 \leq \left( \prod_{j=1}^{l-1} s_j \right)^q \leq 2, \quad 1 \leq \left( \prod_{j=1}^{l-2} s_j \right)^q \leq 2. \quad (15)$$

Next we have $2^{1/q} \geq r_1 \geq r_{l+1} \geq \cdots \geq r_m \geq |f(z)|^{1/q}$ and for $l \leq j \leq m-1$,

$$s_j = \begin{cases} \frac{r_j^{-1} r_{j+1} - 1}{r_j r_{j+1} - 1} & \text{if } j \text{ is even,} \\ \frac{r_{j-1} r_{j+1}}{r_j} & \text{if } j \text{ is odd.} \end{cases}$$

(ii-a) Assume that $l$ is odd. First the estimate from above is proved by using (15) in the following way:

$$\left( \prod_{j=1}^{m} s_j \right)^q = \left( \prod_{j=1}^{l-1} s_j \right)^q \cdot (s_l s_{l+1})^q (s_{l+2} s_{l+3})^q \cdots (s_{m-1} s_m)^q \leq 2.$$

Next we show the estimate from below. We have $s_{l+1} s_l = r_{l+1}/r_1 \geq 1/2^{1/q} = 2^{-1/q}$ and

$$\prod_{j=l+1}^{m} s_j = (s_{l+1} s_{l+2}) \cdots (s_{m-2} s_{m-1}) s_m \geq s_m = m \geq |f(z)|^{1/q}.$$

Therefore

$$\left( \prod_{j=1}^{m} s_j \right)^q = \left( \prod_{j=1}^{l-2} s_j \right)^q (s_{l-1} s_l)^q \left( \prod_{j=l+1}^{m} s_j \right)^q \geq 1 \cdot \frac{1}{2} \cdot |f(z)| = \frac{|f(z)|}{2}.$$  

(ii-b) Assume that $l$ is even. We have $s_{l+1} s_l = r_l/r_{l+1} \leq 2^{1/q}/1 = 2^{1/q}$ and

$$\prod_{j=l+1}^{m} s_j = (s_{l+1} s_{l+2}) \cdots (s_{m-1} s_m) \leq 1.$$  

Therefore by combining these estimates and (15), we obtain:

$$\left( \prod_{j=1}^{m} s_j \right)^q = \left( \prod_{j=1}^{l-2} s_j \right)^q \cdot (s_{l-1} s_l)^q \cdot \left( \prod_{j=l+1}^{m} s_j \right)^q \leq 2 \cdot 2 \cdot 1 = 4.$$
Next we show the estimate from below. We have
\[
\prod_{j=1}^{m} s_j = (s_l s_{l+1}) \cdots (s_m - 2s_{m-1})s_m \geq s_m = r_m \geq |f(z)|^{1/q}.
\]

Hence
\[
\left( \prod_{j=1}^{m} s_j \right)^{q} \geq \left( \prod_{j=1}^{l-1} s_j \right)^{q} \cdot \left( \prod_{j=l}^{m} s_j \right)^{q} \geq 1 \cdot |f(z)| = |f(z)|.
\]

**Lemma 10.** Let \( c \) be a non-negative constant; then
\[
\left( \prod_{j=1}^{m} s_j^{j+c} \right)^{q} \leq \max \left( \left| f(z) \right|^{m+c}, 2^{4m+3c-1}, 2^{3m+2c-1} \right) \left| f(z) \right|^{-(m+c)}.
\]

**Proof.** (i) If \( |f(z)| \geq 2 \) then \( r \) is non-decreasing in \([0, 1]\) and we have by the method of [9], Lemma 6.1,
\[
\left( \prod_{j=1}^{m} s_j^{j+c} \right)^{q} \leq |f(z)|^{m+c}.
\]

(ii) If \( |f(z)| < 2 \) then \( r \) is non-decreasing in \([0, 1/2]\) and decreasing in \([1/2, 1]\). Let \( l \geq 2 \) be such that \( 0 \leq t_1 \leq \cdots \leq t_{n-1} < 1/2 < t_n \). Then \( 1 \leq r_1 \leq \cdots \leq r_{n-1} \leq 2^{1/q} \) and we can conclude that
\[
\left( \prod_{j=1}^{l-1} s_j^{j+c} \right)^{q} \leq 2^{l+c-1} \leq 2^{m+c-1}.
\]

Next we have \( 2^{1/q} \geq r_1 \geq r_{l+1} \geq \cdots \geq r_{m-1} \geq r_m \geq |f(z)|^{1/q} \) and if \( j \) is an odd integer with \( l \leq j \leq m - 1 \), then
\[
s_j^{j+c} s_{j+1}^{j+1+c} = r_j^{-j-c} r_{j+1}^{j+1+c} \leq r_j^{-j-c} r_{j+1}^{j+1+c} = r_j \leq 2^{1/q}.
\]

Therefore
\[
\prod_{j=l}^{m} s_j^{j+c} \leq 2^{m/2q} \max \left( 1, s_l^{l+c}, s_m^{m+c}, s_l^{l+c}, s_m^{m+c} \right)
\]

Since \( |f(z)|^{1/q} \leq s_m = r_m \leq 2^{1/q} \), we have
\[
\prod_{j=l}^{m} s_j^{j+c} \leq 2^{m/2q} \max \left( 1, s_l^{l+c}, 2^{(m+c)/q}, 2^{(m+c)/q} s_l^{l+c} \right).
\]

Combining this estimate with
\[
s_l \leq \max \left( 2^{1/q}, |f(z)|^{-1/q} \right),
\]
we obtain
\[
\prod_{j=l}^{m} s_j^{j+c} \leq 2^{m/2q} \max \left( 2^{2(m+c)/q}, 2^{(m+c)/q} |f(z)|^{-(m+c)/q} \right). \qed
\]
Remark. – We can improve Lemma 10 by the method used in the proof of Lemma 9. We do not present such an improvement because Lemma 10 is good enough to prove Proposition 1 below.

Lemma 11. – We have

\[ |\sigma_j(\hat{z}, t)|^q \leq 2 \max(4, |f(z)|). \] (17)

Proof. – Since \( \xi_j^q = (-1)^{m-j}(\eta_j^q - 1) \) holds,

\[
|\sigma_j| = |\xi_j^q \prod_{i=j+1}^m \eta_i^q| = |\eta_j^q - 1| \prod_{i=j+1}^m |\eta_i^q|
\]

\[
\leq \prod_{i=j}^m |\eta_i|^q + \prod_{i=j+1}^m |\eta_i|^q = \prod_{i=j}^m s_i^q + \prod_{i=j+1}^m s_i^q.
\]

Then apply Lemma 9. \( \Box \)

Lemma 12. – For any compact subset \( \mathcal{L} \) of \( \mathbb{R}(C \setminus L) \), there exists a positive constant \( C = C_{\mathcal{L}} \) such that for all \( \hat{z} \in \mathcal{L}, m = 2, 4, 6, \ldots, t \in \Delta_m \) and \( j \), we have

\[ |\sigma_j(\hat{z}, t)| \geq C t^j. \]

Proof. – By Lemma 9, we have

\[ \prod_{i=j+1}^m |\eta_i| \geq \left( \min(1, |f(z)|/2) \right)^{1/q}. \] (18)

Hence

\[ |\sigma_j| \geq |\xi_j| \left( \min(1, |f(z)|/2) \right)^{1/q}. \] (19)

On the other hand, since we have \( \xi_j(\hat{z}, t) = F(\hat{z}, t_j) \) or \( (F/G)(\hat{z}, t_j) \) and \( |G(\hat{z}, t_j)| = r_j \leq \max(2^{1/q}, |f(z)|^{1/q}) \), we obtain

\[ |\xi_j(t)| \geq |F(\hat{z}, t_j)| \min(2^{-1/q}, |f(z)|^{-1/q}). \] (20)

By using (19) and (20), we get:

\[ |\sigma_j| \geq |F(\hat{z}, t_j)| \min(2^{-1/q}, |f(z)|^{-1/q}) \left( \min(1, |f(z)|/2) \right)^{1/q}. \]

Then the lemma follows from the fact that there exists a positive constant \( C' = C'_{\mathcal{L}} \) such that \( |F(\hat{z}, t)| \geq C't \) for all \( \hat{z}, t \in \mathcal{L} \times I \). \( \Box \)

We will derive an estimate related to \( S_m \).

The Jacobian matrix \( \partial S_m / \partial t = (\partial \sigma_j / \partial t)_{1 \leq i, j \leq m} \) is upper-triangular and its determinant is:

\[ \det \frac{\partial S_m}{\partial t} = \prod_{j=1}^m \frac{d\xi_j(t_j)}{dt_j} \cdot \prod_{j=2}^m \eta_j(t_j)^{j-1}. \]

It is clear that:
\[ d\sigma_{(m)} = d\sigma_1 \wedge \cdots \wedge d\sigma_m \]
\[ = \prod_{j=1}^{m} \frac{d\xi_j(t_j)}{dt_j} \cdot \prod_{j=2}^{m} \eta_j(t_j) \cdot \, dt_1 \wedge dt_2 \cdots \wedge dt_m. \]

Set
\[ \text{Mes } S_m(\hat{z}, \cdot) = \int_A \frac{|d\sigma_{(m)}|}{|\sigma_3 \sigma_5 \cdots \sigma_{m-1}|} \]
\[ = \int_A \frac{1}{|\sigma_3 \sigma_5 \cdots \sigma_{m-1}|} \left| \det \frac{\partial S_m(\hat{z}, t)}{\partial t} \right| \, dt_1 \, dt_2 \cdots \, dt_m. \]

**Proposition 1.** – For any compact subset \( L \) of \( \mathcal{R}(C \setminus L) \), there exists a positive constant \( C_2 = C_{2,L} \) such that for all \( m = 2, 4, 6, \ldots \), and \( \hat{z} \in L \), we have:
\[ \text{Mes } S_m(\hat{z}, \cdot) \leq C_2^{m+1} \left( \left( \frac{m}{2} \right)^{q-1} \right) \left( \frac{m}{2} + 1 \right)^{-1}. \]

**Proof.** – By the same calculation as in [9], we obtain
\[ d\sigma_{(m)} = \prod_{j=1}^{m} \eta_j^{j+(q-1)/2} G(\hat{z}, t_j)^{-(q+1)/2} \, dF(\hat{z}, t_j). \]

Lemma 10 implies that there exists a positive constant \( C_3 = C_{3,L} \) such that for all \( (\hat{z}, t) \in L \times \Delta_m \), we have:
\[ \prod_{j=1}^{m} |\eta_j^{j+(q-1)/2}| \leq C_3^{m+1}, \quad |G(\hat{z}, t_j)|^{-1} \leq C_3, \quad |dF(\hat{z}, t_j)/dt_j| \leq C_3. \]

So the proposition follows from Lemma 12 and Lemma 1, (6). \( \square \)

### 6. The simplex \( T_m(\hat{x}, \cdot) \)

Let \( \mathcal{O} \) be the open neighborhood of the origin of \( C^{n+1} \) defined by:
\[ \mathcal{O} = \left\{ x \in C^{n+1} : \max_{0 \leq j \leq n} |x_j| < \delta \right\}, \quad \delta > 0. \]

Set \( X = \mathcal{O} \setminus (K_0 \cup K_1) \) and let \( \hat{X} \to X, \hat{x} \mapsto x = \pi_X(\hat{x}) \) be its universal covering space. It is the quotient by homotopy of the set of all paths \( \gamma : I \to X \) with \( \gamma(0) = y \). Let \( \hat{y} \in \hat{X} \) be the class of \( \gamma(t) = y \) (\( t \in I \)). Then we have \( \pi_X(\hat{y}) = y \).

Let us choose a branch of \( (-k_0(x))^{-1/q} \) on \( \hat{X} \). The choice will be specified later. Its value at \( \hat{x} \in \hat{X} \) is denoted by \( (-k_0(\hat{x}))^{-1/q} \). Recall that \( Z = \{ z \in C ; z^q + 1 \neq 0 \} \) and let \( \hat{Z} \to Z, \hat{z} \mapsto z = \pi_Z(\hat{z}) \) be its universal covering space. The space \( \hat{Z} \) is the quotient by homotopy of the set of all paths \( \gamma : I \to Z \) with \( \gamma(0) = 0 \in Z \). We have a well-defined injective mapping \( \mathcal{R}(C \setminus L) \to \hat{Z} \). Let \( 0 \in \hat{Z} \) be the class of the path \( \gamma_0 \) defined by \( \gamma_0(t) = t, t \in I \).

We introduce a holomorphic function
\[ g : \hat{X} \to \hat{Z}, \quad \hat{x} \mapsto x_0(-k_0(\hat{x}))^{-1/q}. \]
Obviously we have \( h(x) = g(\tilde{x})^q \).

Let \( \hat{\gamma} : \hat{X} \to \hat{Z} \) be the holomorphic function satisfying

\[
\pi_Z \circ \hat{\gamma} = g, \quad \hat{\gamma}(\tilde{y}) = \hat{0}.
\]

Let us consider paths \( \gamma : I \to \mathcal{O} \) with

\[
(**) \quad \gamma(0) = y \in A_0, \quad \gamma(t) \not\in A_0 \text{ if } t \neq 0.
\]

We say that two paths \( \gamma \) and \( \gamma' \) satisfying \((**\)) with \( \gamma(1) = \gamma'(1) \) are \((**)-homotopic\) and write \( \gamma \simeq \gamma' \) if there exists a continuous mapping \( H(s, t) : I \times I \to \mathcal{O} \) such that:

\[
\begin{align*}
H(s, t) &\not\in A_0 \quad \text{if } t \neq 0, \\
H(0, t) &= \gamma(t), \quad t \in I, \\
H(1, t) &= \gamma'(t), \quad t \in I.
\end{align*}
\]

Consider the set of all paths satisfying \((**\)). Let \( \mathcal{U} \) be its quotient space by the relation \( \simeq \) and \( \hat{\mathcal{O}} \) be the canonical projection. We have a well-defined mapping \( \hat{\mathcal{O}} : \mathcal{U} \to \mathcal{O} \) by:

\[
\begin{align*}
\hat{\mathcal{O}}(\hat{x}) &= \pi_z(\hat{\gamma}(\tilde{x})) \quad \text{if the branch of } \gamma(1) \text{ is suitably chosen.}
\end{align*}
\]

Holomorphic functions \( g(\tilde{x}) = g(\tilde{y}) : \mathcal{U} \to Z \) and \( \gamma(1) = \gamma'(1) \) are similarly induced. We have:

\[
\pi_L(\hat{\gamma}(\tilde{x})) = \pi_Z(\hat{\gamma}(\tilde{x})) = g(\tilde{x}) = g(\tilde{y}).
\]

We find that \( \mathcal{U} \) is the universal covering space of \( \mathcal{O} \setminus A_0 \). To see this, let \( (\mathcal{O} \setminus A_0)^- \) be the universal covering space constructed in the usual way with the base point \( \tilde{x} = [\gamma] \mapsto x = \pi_U(\tilde{x}) = \gamma(1) \) be the canonical projection.

We have a well-defined mapping \( \mathcal{U} \to \hat{X}, \hat{x} \mapsto \hat{x} \). It induces a holomorphic mapping

\[
\hat{\gamma} : \mathcal{U} \to \mathbb{R}(\mathbb{C} \setminus L) \subset \hat{Z}, \quad \hat{x} \mapsto \hat{\gamma}(\tilde{x}) = \hat{g}(\tilde{x}),
\]

if the branch of \((-k_0)^{-1/q} \) is suitably chosen. Holomorphic functions \( g(\tilde{x}) = g(\tilde{y}) : \mathcal{U} \to \mathbb{Z} \) are similarly induced. We have:

\[
\pi_L(\hat{g}(\tilde{x})) = \pi_Z(\hat{g}(\tilde{x})) = g(\tilde{x}) = g(\tilde{y}).
\]

We define a mapping \( T_m : \mathcal{U} \times \Delta_m \to \mathbb{C}_m^m (m = 2, 4, 6, \ldots) \) by:

\[
T_m(\hat{x}, t) = (-k_0(\tilde{x}))^{1/q} S_m(\hat{g}(\tilde{x}), t).
\]

Its components are denoted by \( \sigma_j(\hat{x}, t) \) (\( 1 \leq j \leq m \)). (Caution: It differs from \( \sigma_j(\tilde{x}, t) \) by the factor \((-k_0)^{1/q}\).) We define \( C^1_m : \mathcal{U} \times \Delta_m \to \mathbb{C}^2 \) by:

\[
C^1_m(\hat{x}, t) = (\sigma_1(\hat{x}, t), \varphi_m(T_m(\hat{x}, t), x)).
\]

Notice that we have

\[
(21) \quad \varphi_m(T_m(\hat{x}, t), x) = k_0(x) \prod_{i=1}^{m} \eta_i^q,
\]

as in [9], (7.2).

**Proposition 2.** – For all \((\hat{x}, t) \in \mathcal{U} \times \Delta_m\), we have:
Consider a path $y\in \mathcal{X}$. Then we have
\begin{align}
C_m^1(\hat{x}, t) &\in \mathcal{X} = \{ (\xi_0, \xi_1) : \xi_1(\xi_1 - \xi_0) \neq 0 \}, \\
T_m(\hat{x}, \Delta_m^j) &\subset H^j(x_0), \quad 0 \leq j \leq m.
\end{align}

\textbf{Proof.} – The argument in [9], §7 shows that (22) and (23) follow from (13) and (14) respectively. \qed

On the other hand, we have by (21), (17), (12) and Lemma 9,
\begin{align}
|\varphi_m(T_m(\hat{x}, t), x)| &\leq \max \{4|k_0(x)|, |k_1(x)|\}, \\
|\sigma_j(\hat{x}, t)| &\leq 2 \max \{4|k_0(x)|, |k_1(x)|\}, \\
|\psi_{2j} \circ T_m(\hat{x}, t)| &\leq 2 \max \{4|k_0(x)|, |k_1(x)|\}.
\end{align}

If $\delta > 0$ is sufficiently small, then we have for all $(\hat{x}, t) \in \mathcal{U} \times \Delta_m$,
\begin{align}
C_m^1(\hat{x}, t) &\in \mathcal{X}_b \subset \mathcal{X}_a, \\
0 &< b < a, \text{ and} \\
|\sigma'(\hat{x}, t)| &\in \Omega_{\rho}^{*}, \\
x &\in \mathcal{O} \Rightarrow x'' \in D_{\rho}^{n-1}
\end{align}

Hence we can use Theorem 2.

Next we will define a lift of $C_m^1$ up to the universal covering space of $\mathcal{X}$. It is obvious that
\begin{align}
\hat{\gamma}^{-1}(t) &= t\hat{x} + (1-t)y
\end{align}

where $y$ is a path satisfying $(**)$ with $\hat{\gamma}^{-1}(0) = y$ and $\hat{\gamma}^{-1}(1) = \hat{x}$. Let $\hat{\gamma} \in \mathcal{U}$ be its $(**)$-homotopy class. Then we have
\begin{align}
C_m^1(\hat{x}, 0) &= (0, y_1) \in \mathcal{X}.
\end{align}

We construct the universal covering space $\pi_{X} : \hat{\mathcal{X}} \to \mathcal{X}$ with the base point $(0, y_1)$. Consider a path $\gamma_1(t) \equiv (0, y_1)$ and denote by $\hat{\gamma}_1$ its homotopy class in $\hat{\mathcal{X}}$. Obviously we have
\begin{align}
\pi_{X}(\hat{\gamma}_1) &= (0, y_1).
\end{align}

Since $\mathcal{U} \times \Delta_m$ is simply connected, there exists a unique continuous mapping
\begin{align}
\hat{C}_m^1 : \mathcal{U} \times \Delta_m \to \hat{\mathcal{X}}
\end{align}

with $C_m^1 = \pi_{X} \circ \hat{C}_m$, \quad $\hat{C}_m^1(\hat{x}, 0) = \hat{\gamma}_1$.

\textbf{Lemma 13}. – For any compact subset $K \subset \mathcal{U}$, there exists a compact subset $\hat{K} \subset \hat{\mathcal{X}}$ such that
\begin{align}
\hat{C}_m^1(K \times \Delta_m) \subset \hat{K} \quad \text{for all } m = 2, 4, 6, \ldots .
\end{align}

\textbf{Proof}. – We introduce the following three mappings:
\begin{align}
Q_m : \mathcal{U} \times \Delta_m &\to \mathbb{C}^*, \quad Q_m(\hat{x}, t) = (-k_0(\hat{x}))^{1/q} \prod_{i=2}^m \eta_i, \\
B : \mathcal{U} \times \Delta_1 &\to \mathcal{X}, \quad B(\hat{x}, t_1) = (\xi_1(\hat{x}, t_1) - q_0(\hat{x}, t_1), \hat{\zeta} = \hat{q}(\hat{x}), \\
\Lambda : \mathcal{X} \times \mathbb{C}^* &\to \mathcal{X}, \quad \Lambda(\zeta_0, \zeta_1, \lambda) = (\lambda \zeta_0, \lambda^q \zeta_1).
\end{align}

Then we have $C_m^1 = \Lambda \circ (B, Q_m)$. Notice that $B$ is independent of $m$.

Let $\hat{\mathcal{R}}(\mathbb{C}^*)$ be the universal covering space of $\mathbb{C}^*$ and $Q_m : \mathcal{U} \times \Delta_m \to \hat{\mathcal{R}}(\mathbb{C}^*)$ be a lift of $Q_m$. 

It is enough to prove that $Q_m(K \times \Delta_m)$ is included in a compact subset of $R(C^*)$ independent of $m$.

By using Lemma 9 and the fact that
\[ |f(g(x))| = |k_1(x)/k_0(x)| = |k_1(x)/k_0(x)|, \]
we can show that:
\[ (0 <) \min \left( |k_0(x)|, \frac{1}{2} |k_1(x)| \right) \leq |Q_m(x, t)|^q \leq \max \left( 4 |k_0(x)|, |k_1(x)| \right). \]

Next, let $\arg Q_m: \mathcal{U} \times \Delta_m \to \mathbb{R}$ and $\vartheta: \mathcal{U} \to \mathbb{R}$ be a continuous determination of the argument of $Q_m$ and $(-k_0)^{1/q}$ respectively. Then we have
\[ |\arg Q_m(x, t) - \vartheta(x)| \leq \Theta(\hat{g}(x)). \]

Set:
\[ \text{Mes } T_m(x, \cdot) = \int_{\Delta_m} \frac{|d\sigma(m)|}{|\sigma_3 \sigma_5 \cdots \sigma_{m-1}|} \]
\[ = \int_{\Delta_m} |\sigma_3(x, t) \cdots \sigma_{m-1}(x, t)| \left| \det \frac{\partial T_m(x, \cdot)}{\partial t} \right| dt_1 dt_2 \cdots dt_m. \]

The following proposition is a consequence of Lemma 12 and Proposition 1.

**Proposition 3.** For any compact subset $K \subset \mathcal{U}$, there exist positive constants $c(K)$ and $C(K)$ such that for all $m = 2, 4, 6, \ldots$, and all $\hat{x} \in K$, we have:
\[ |\sigma_3(\hat{x}, t)| \geq c(K) t_j, \]
\[ \text{Mes } T_m(\hat{x}, \cdot) \leq C(K)^{m+1} \left( \frac{m}{2} \right) \left( \frac{m}{2} + 1 \right)^{-1}. \]

### 7. Analytic continuation

For a fixed $m = 2, 4, 6, \ldots$, put
\[ F(\sigma, x) = u_m(\sigma_1, \varphi_m(\sigma, x), \sigma', \sigma''), \]
\[ \omega(\sigma, x) = F(\sigma, x) d\sigma(m). \]

Recall that $I_m(x) = \int_{S_m(x)} \omega(\sigma, x)$ exists and is holomorphic in $x$. We can show that $\int_{T_m(\hat{x}, \cdot)} \omega(\sigma, x)$ exists by using (30), Lemma 6 and (6).

We may assume by [1] that $y \in \{x_0 = 0, x_1 < 0\}$. If $\hat{x} \in \mathcal{U}$ is represented by a path which is in a sufficiently small neighborhood of $y$, then $\hat{x}$ is identified with $x = \pi_\mathcal{U}(\hat{x}) \in \mathcal{O} \setminus A_0$ near $y$.

Assume that $x_0 > 0, x_1 < 0$ for $x = \pi_\mathcal{U}(\hat{x})$. (This assumption is a technical one and we will discuss its removal later.) It implies that $g(x) > 0$ and that any component of $S_m$ and $T_m$ is non-negative. In this situation we have:

**Lemma 14.** We have
\[ I_m(x) = \int_{T_m(\hat{x}, \cdot)} \omega(\sigma, x). \]
Proof. – We assume that \( m \geq 4 \). If \( m = 2 \), the proof is easier because \( u_2 \) has no singularity.

For \((\hat{x}, t, r) \in \mathcal{U} \times \Delta_m \times I\), set

\[
R(\hat{x}, t, r) = (1 - r)S_m(x_0)(t) + rT_m(\hat{x}, t) \in \mathbb{C}_m^m.
\]

By the assumption there exists a positive constant \( C \) which can be taken locally uniformly with respect to \( x \) such that we have \( Ct_j \leq \sigma_j \) for any \((r, t)\), where \( R = (\sigma_1, \ldots, \sigma_m) \). In particular \( \sigma_j = 0 \) if and only if \( t_j = 0 \).

For \( \varepsilon > 0 \), let \( \varphi(s) \) be a smooth function on \( \mathbb{R}_+ = \{s \in \mathbb{R} : s \geq 0\} \) satisfying:

\[
\begin{cases}
\varphi(s) \geq 0, \\
\varphi(s) \equiv 0 & \text{in } 0 \leq s \leq 1, \\
\varphi(s) \equiv 1 & \text{in } 2 \leq s,
\end{cases}
\]

and set \( \Phi_\varepsilon = \varphi(1/\varepsilon) \). We find that \( \Phi_\varepsilon \equiv 0 \) near \( \{\sigma_1 \sigma_3 \cdots \sigma_{m-1} = 0\} \) because if any \( \sigma_j (j = 3, 5, \ldots, m - 1) \) vanishes, then \( t_j = 0 \) and we have \( t_3 < t_j \) for \( t \in \Delta_m \).

We have

\[
\int_{\partial R} \Phi_\varepsilon \omega = \int_{T_m(\hat{x})} \Phi_\varepsilon \omega - \int_{S_m(x_0)} \Phi_\varepsilon \omega
\]

because the pullback on the other faces vanishes. The existence of the limit \( I_m(x) = \lim_{\varepsilon \to 0} \int_{T_m(\hat{x})} \Phi_\varepsilon \omega \) is proved in Section 2. In a similar manner we can deal with \( \lim_{\varepsilon \to 0} \int_{T_m(\hat{x})} \Phi_\varepsilon \omega = \int_{T_m(\hat{x})} \omega \) by means of (30), Lemma 6 and (6). The proof will end as soon as we show \( \lim_{\varepsilon \to 0} \int_{\partial R} \Phi_\varepsilon \omega = 0 \).

By Stokes’ formula, we have (\( x \) is fixed)

\[
\int_{\partial R} \Phi_\varepsilon \omega = \int_R \Phi_\varepsilon \wedge \omega + \int_R \Phi_\varepsilon \omega = \int_R \Phi_\varepsilon \wedge \omega.
\]

The second equality holds because \( \omega = 0 \) on supp \( \Phi_\varepsilon \). In terms of \((r, t)\), the support of \( \Phi_\varepsilon \) is contained in \( L \times (\Delta_m \cap \{Ct_j \leq 2\varepsilon\}) \). Therefore by (30), there exists a positive constant \( C' \) which can be taken locally uniformly with respect to \( x \) such that:

\[
\int_R \Phi_\varepsilon \wedge \omega \leq C' \frac{1}{\varepsilon} \int_0^1 dr \int_{\Delta_m \cap \{Ct_j \leq 2\varepsilon\}} \frac{dt_1 dt_2 \cdots dt_m}{t_1 t_2 \cdots t_{m-1}}.
\]

Lemma 2 shows that the right-hand side tends to 0 as \( \varepsilon \to +0 \). Hence we obtain \( \lim_{\varepsilon \to +0} \int_{\partial R} \Phi_\varepsilon \omega = 0 \). \( \square \)

In the preceding lemma, we imposed a technical assumption that \( x_0 > 0, x_1 < 0 \) and (32) is proved to hold only in a (partially real) domain of \( \mathbb{C}^2 \times \mathbb{C}^{m-1} \). As a matter of fact, however, if the right-hand side is holomorphic in \( \mathcal{U} \), then (32) holds there by analytic continuation. In particular this means that \( I_m(x) \) extends holomorphically to \( \mathcal{U} \). In Lemma 15 below, we will prove that this is exactly the case.

So (32) holds without the assumption that \( x_0 > 0, x_1 < 0 \). Direct proof not relying on analytic continuation seems possible, but the definition of \( R(\hat{x}, t, r) \) must be replaced by a much more complicated one.
LEMMA 15. – The function $I_m(x)$ extends holomorphically to $U$ and it is represented by

$$I_m(\hat{x}) = \int_{\Delta_m} \omega(\sigma, x) = \int_{\Delta_m} u_m(C^1_m(\hat{x}, t), \sigma'(\hat{x}, t), x') \frac{\partial T_m}{\partial t} dt.$$ 

Proof. – By the method of [4], we show that the right-hand side is holomorphic in $U$. Since this claim is of local nature, we use $x$ as a system of local coordinates of $U$ and denote $O_x$ by $x$ for simplicity.

Set

$$J(x) = \int_{T_m(x, \cdot)} \omega(\sigma, x),$$

for $(r, t) \in I \times \Delta_m$. Here $x$ is fixed and $z \in \mathbb{C}$ is a constant with sufficiently small modulus. It is trivial that $U(0, t) = \{x, T_m(x, t)\}$ and that $U(r, t)$ converges to $\{x, T_m(x, t)\}$ uniformly by the order $O(|z|)$ as $|z| \to 0$. Stokes’ formula and (23) imply that:

$$\int_U d\omega = \int_{\Delta_m} \omega = J(x_0 + z, x') - J(x) + \int_V \omega,$$

where we set for $(r, t_1, \ldots, t_{m-1}) \in I \times \Delta_{m-1},$

$$V(r, t_1, \ldots, t_{m-1}) = U(r, t_1, \ldots, t_{m-1}, 1) \in \{\sigma_m = x_0 + rz\}.$$

Our integrands are singular at $[\sigma_3 \sigma_5 \cdots \sigma_{m-1} = 0]$ and the above calculation requires justification. The integral over $I \times \Delta_m$ is the limit as $\epsilon \to 0$ of that over $[t_2 \geq \epsilon] \subset I \times \Delta_m$. Remark that the image of $[t_2 \geq \epsilon]$ under $U(r, t)$ does not intersect $[\sigma_3 \sigma_5 \cdots \sigma_{m-1} = 0]$ because of (30) and $t_2 \leq t_3 \leq \cdots \leq t_{m-1}$. The convergence of

$$\int_U d\omega = \lim_{\epsilon \to 0} \int_{U/[t_2 \geq \epsilon]} d\omega$$

follows from Lemma 6, (30) and (5). Indeed, $|\int_U d\omega|$ is estimated by means of integrals of the form

$$k_{m, j} = \int_{\Delta_m} \frac{dt_1 dt_2 \cdots dt_{m-1} dt_m}{f_3 \cdots f_{j-2} f_{j-1} f_{j+1} \cdots f_{m-1} (2 \leq j \leq m)}$$

and we have $k_{m, j} \leq \epsilon_m < \infty$.

To show that

$$\lim_{\epsilon \to 0} \int_{U/[t_2 \geq \epsilon]} \omega = J(x_0 + z, x') - J(x) + \int_V \omega,$$

we have only to remark that the integral over $[t_2 = \epsilon] \subset I \times \Delta_m$ tends to 0 as $\epsilon \to 0$. We have

$$\int_{0 \leq r \leq 1} \frac{dr dt_1 dt_2 dt_3 \cdots dt_m}{f_3 f_5 \cdots f_{m-1}}$$
\[
= \epsilon \int_{\mathbb{R}^{n-1}} \frac{dt_3 \cdots dt_m}{t_3 t_5 \cdots t_{m-1}} < \epsilon \int_{\mathbb{R}^{n-1}} \frac{dt_3 \cdots dt_m}{t_3 t_5 \cdots t_{m-1}},
\]

where \( A = \{(t_3, \ldots, t_m); \epsilon \leq t_j \leq 1 \ (j: \text{odd}), 0 \leq t_j \leq 1 \ (j: \text{even})\} \).

This quantity is bounded by
\[
\epsilon \left( \int_{\epsilon}^{1} \frac{dt}{t} \right)^{(m-2)/2} \leq \epsilon (-\log \epsilon)^{(m-2)/2}.
\]

Hence our version of Stokes' formula has finally been justified.

We calculate the left-hand side of (33). We have:
\[
d\omega = \frac{\partial F}{\partial x_0} \, dx_0 \wedge d\sigma(m) + \sum_{j=1}^{n} \frac{\partial F}{\partial x_j} \, dx_j \wedge d\sigma(m).
\]

By using \( U^* dx_0 = z \, dr \) and \( U^* dx_j = 0 \ (j \geq 1) \), we obtain
\[
U^* d\omega = \frac{\partial F}{\partial x_0} (U(r, t)) z \, dr \wedge U^* \sigma(m).
\]

Therefore
\[
\int_U d\omega = z \int_{I \times \Delta_n} \frac{\partial F}{\partial x_0} (U(r, t)) \, dr \wedge U^* \sigma(m) = z \int_{I \times \Delta_n} \frac{\partial F}{\partial x_0} \, d\sigma(m) + O(|z|).
\]

Next let us calculate the right-hand side of (33). By using \( V^* d\sigma_m = z \, dr \), we get
\[
V^* \omega = V^* (F \, d\sigma_{m-1}) \wedge z \, dr,
\]

(34) \( \int_V \omega = -z \int_{I \times \Delta_{m-1}} dr \wedge V^* (F \, d\sigma_{m-1}) = -z \int_{V \setminus \{0, 1\}} F \, d\sigma_{m-1} + O(|z|) \).

Combination of (33), (34) and (35) implies that \( J(x) \) is holomorphic in \( x_0 \) and that
\[
\frac{\partial J}{\partial x_0} = \int_{I \times \Delta_n} \frac{\partial F}{\partial x_0} \, d\sigma(m) + \int_{V \setminus \{0, 1\}} F \, d\sigma(m-1).
\]

In the same way we can prove that \( J(x) \) is holomorphic in \( x_j \ (j \geq 1) \) and that
\[
\frac{\partial J}{\partial x_j} = \int_{I \times \Delta_n} \frac{\partial F}{\partial x_j} \, d\sigma(m), \quad j \geq 1.
\]
8. Proof of Theorem 1

First we prove the case $j = 0$. Lemma 13 and (27) show that for any compact subset $K \subset U$, $\check{C}^1_u(K \times \Delta_m)$ is contained in a common compact subset $K \subset \tilde{X}_m$ for all $m = 2, 4, 6, \ldots$. By Theorem 2, (28), (29) and (31), we find that for all $x \in K$:

$$|I_m(\tilde{x})| \leq c^{m+1}_K \cdot \frac{m}{2} \cdot \left(\frac{m}{2}\right)! \cdot C(K)^{m+1} \left(\frac{m}{2}\right)^{m+1} \left(\frac{m}{2} + 1\right)^{-2} \leq \left\{c_K C(K)\right\}^{m+1} \left(\frac{m}{2}\right)!^{-1}.$$ 

Therefore $\sum_{m=2,4,\ldots} I_m(\tilde{x})$ is convergent in $U$. This proves the case $j = 0$.

Next we prove the case $j = 1$. As a matter of fact, it is a corollary of the case $j = 0$.

We perform the change of coordinates $y = \varphi(x)$ defined by:

$$\begin{cases}
y_0 = x_0, \\
y_1 = -x_1 + x_0^q, \\
y_j = x_j \quad (j = 2, 3, \ldots, n).
\end{cases}$$

The hypersurfaces $K_0$ and $K_1$ are exchanged and $T$ is preserved by $\varphi$. It is easy to see that $\varphi^2 = \text{id}$ and that:

$$\begin{cases}
D_{y_0} = D_{x_0} + qy_0^{q-1}D_{x_1}, \\
D_{x_1} = -D_{y_1}, \\
D_{x_j} = D_{y_j} \quad (j = 2, 3, \ldots, n).
\end{cases}$$

So we have

$$a(x, D) = y_0 \left(D_{y_0} + qy_0^{q-1}D_{y_1}\right)D_{y_0} + \sum_{j=1}^{n} \tilde{a}_j(y)D_{y_j} + \tilde{b}(y)$$

for some $\tilde{a}_j$ and $\tilde{b}$ and we can employ the case $j = 0$ with respect to $y$. We have

$$\tilde{h}(y) = \left.\frac{y_0}{y_1}\right| = \frac{x_0^q}{x_1 - x_0^q} = \frac{-h(x)}{1 + h(x)}.$$

Therefore

$$-1 \leq \tilde{h}(y) \leq 0 \text{ or } \tilde{h}(y) = \infty \iff h(x) \geq 0 \text{ or } h(x) = \infty \text{ or } h(x) = -1.$$ 

In other words, the transformation $\varphi$ exchanges $A_0$ and $A_1$. The proof of Theorem 1 is now complete.

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