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Dynamics for the energy critical nonlinear Schrödinger equation in high dimensions

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Abstract

In [T. Duyckaerts, F. Merle, Dynamic of threshold solutions for energy-critical NLS, preprint, arXiv:0710.5915 [math.AP]], T. Duyckaerts and F. Merle studied the variational structure near the ground state solution W of the energy critical NLS and classified the solutions with the threshold energy $E(W)$ in dimensions $d = 3, 4, 5$ under the radial assumption. In this paper, we extend the results to all dimensions $d \geqslant 6$. The main issue in high dimensions is the non-Lipschitz continuity of the nonlinearity which we get around by making full use of the decay property of W.

Keywords: Energy critical; Schrödinger equation; Variational structure; Ground state

1. Introduction

We consider the Cauchy problem of the focusing energy critical nonlinear Schrödinger equation:

$$\begin{cases} iu_t + \Delta u + |u|^{\frac{4}{d-2}} u = 0, \\ u(0, x) = u_0(x), \end{cases} \tag{1.1}$$

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where $u(t,x)$ is a complex function on $\mathbb{R}\times\mathbb{R}^d$, $d\geqslant 3$ and $u_0\in\dot{H}^1_x(\mathbb{R}^d)$. The name "energy critical" refers to the fact that the scaling

$$u(t,x)\to u_\lambda(t,x)=\lambda^{-\frac{d-2}{2}}u\big(\lambda^{-2}t,\lambda^{-1}x\big). \tag{1.2}$$

leaves both the equation and the energy invariant. Here, the energy is defined by

$$E\big(u(t)\big)=\frac{1}{2}\big\|\nabla u(t)\big\|_2^2-\frac{d-2}{2d}\big\|u(t)\big\|_{\frac{2d}{d-2}}^{\frac{2d}{d-2}}, \tag{1.3}$$

and is conserved in time. We refer to the first part as "kinetic energy" and the second part as "potential energy."

From the classical local theory [4], for any $u_0\in\dot{H}^1_x(\mathbb{R}^d)$, there exists a unique maximal-lifespan solution of (1.1) on a time interval $(-T_-,T_+)$ such that the local scattering size

$$S_I(u)=\|u\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(I\times\mathbb{R}^d)}<\infty,$$

for any compact interval $I\subset(-T_-,T_+)$. If $S_{[0,T^+)}(u)=\infty$, we say u blows up forward in time. Likewise u blows up backward in time if $S_{(-T_-,0]}(u)=\infty$. We also recall the fact that the non-blowup of u in one direction implies scattering in that direction.

For the defocusing energy critical NLS, the global well-posedness and scattering was established in [2,5,16,18,21]. In the focusing case, depending on the size of the kinetic energy of the initial data, both scattering and blowup may occur. One can refer to [3] for scattering of small kinetic energy solutions and [7] for the existence of finite time blowup solutions. The threshold between blowup and scattering is believed to be determined by the ground state solution of Eq. (1.1):

$$W(x)=\left(1+\frac{|x|^2}{d(d-2)}\right)^{-\frac{d-2}{2}},$$

which solves the static NLS

$$\Delta W+W^{\frac{d+2}{d-2}}=0.$$

This was verified by Kenig and Merle [9] in dimensions $d=3,4,5$ in the spherically symmetric case and by Killip and Visan [13] in all dimensions $d\geqslant 5$ *without the radial assumption*. To summarize, we have the following

Theorem 1.1 *(Global well-posedness and scattering [9,13]). Let $u=u(t,x)$ be the maximal-lifespan solution of* (1.1) *on $I\times\mathbb{R}^d$ in dimension $d\geqslant 3$, in the case when $d=3,4$, we also require that u is spherically symmetric. If*

$$E_*:=\sup_{t\in I}\big\|\nabla u(t)\big\|_2<\|\nabla W\|_2,$$

then $I=\mathbb{R}$ and the scattering size of u is finite,

$$S_I(u)=\|u\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(I\times\mathbb{R}^d)}<C(E_*).$$

As a consequence of this theorem and the coercive property of W [9], they also proved

Corollary 1.2. *(See [9,13].) Let $d \geqslant 3$ and $u_0 \in \dot{H}^1_x(\mathbb{R}^d)$. In dimension $d = 3, 4$ we also require u_0 is spherically symmetric. If*

$$E(u_0) < E(W),$$
$$\|\nabla u_0\|_2 \leqslant \|\nabla W\|_2,$$

then the corresponding solution $u = u(t, x)$ exists globally and scatters in both time directions.

Theorem 1.1 and Corollary 1.2 confirmed that the threshold between blowup and scattering is given by the ground state W. Our purpose of this paper is not to investigate the global well-posedness and scattering theory blow the threshold. Instead, we aim to continue the study in [6] on what will happen if the solution has the threshold energy $E(W)$. In that paper, T. Duyckaerts and F. Merle carried out a very detailed study of the dynamical structure around the ground solution W. They were able to give the characterization of solutions with the threshold energy in dimensions $d = 3, 4, 5$ under the radial assumption. Note that the energy-critical problem here can be compared with the focusing mass critical problem

$$iu_t + \Delta u = -|u|^{\frac{4}{d}} u.$$

There the ground state solution Q satisfies the equation

$$\Delta Q - Q + Q^{1+\frac{4}{d}} = 0.$$

And the mass of Q turns out to be the threshold between blowup and scattering. The characterization of the minimal mass blowup solution was established in [10,11,15,22].

In this paper, we aim to extend the results in [6] to all dimensions $d \geqslant 6$. Although the whole framework designed for low dimensions can also be used for the high dimensional setting, there are a couple of places where the arguments break down in high dimensions. Roughly speaking, this is caused by the non-smoothness of the nonlinearity; more precisely, in high dimensions, the nonlinearity $|u|^{4/(d-2)}u$ is no longer Lipschitz continuous in the usual Strichartz space $\dot{S}^1$ (see Section 2 for the definition). This reminds us of the similar problem one encountered in establishing the stability theory for high dimensional energy critical problem where this was gotten around by using exotic Strichartz estimates (see, for example, [19]).[1]

However, the exotic Strichartz trick will inevitably cause the loss of derivatives and one cannot go back to the natural energy space H^1_x. On the other hand, the H^1_x regularity is heavily used in the spectral analysis around the ground state W (see for example the proof of Proposition 5.9 in [6]). To solve this problem, we will use a different technique where the decay property of W is fully considered. When constructing the threshold solutions $W^{\pm}$ (see Theorem 1.3 below), we transform the problem into solving a perturbation equation with respect to W using the fixed point argument. Although the nonlinearity of the perturbed equation is not Lipschitz

[1] The main ingredient of exotic Strichartz trick is as follows: instead of using spaces $\dot{S}^1$, we use the space which has the same scaling but lower regularity. The nonlinearity can be shown to be Lipshitz continuous in such spaces. (See lecture notes [12], Section 3 for more details.)

continuous for general functions, it is for perturbations which are much smaller than W. The reason is that if we restrict ourselves to the regime $|z| \ll 1$, we can expand the real analytic function $|1+z|^{\frac{4}{d-2}}(1+z)$ (which corresponds to the form of energy critical nonlinearity) and get the Lipschitz continuity. This consideration leads us to working in the space of functions which have much better decay than W. The weighted Sobolev space $H^{m,m}$ (see (3.4) for the definition) turns out to be a good candidate for this purpose. By doing this, besides proving the existence of the threshold solutions $W^{\pm}$, we can actually show the difference $W^{\pm} - W$ has very high regularity and good decay properties.

This property also helps us in the next step where we have to show after extracting the linear term, the perturbed nonlinearity is superlinear with respect to the perturbations. The superlinearity is needed to show the rigidity of the threshold solutions $W^{\pm}$. This time again we make use of the decay estimate of W. We split $\mathbb{R}^d$ into regimes where the solution dominates W and the complement. In the first regime, we can transform some portion of W to increase the power of the solution, and get the superlinearity (cf. Lemma 2.3). In the regime where the solution is dominated by W, we simply use the real analytic expansion. The fact that the difference $W^{\pm} - W$ has enough decay in space and time plays a crucial role in the whole analysis.

In all, the material in this paper allows us to extend the argument in [6] to all dimensions $d \geqslant 6$. With some suitable modifications, the same technique can be used to treat the high dimensional energy critical nonlinear wave equation and we will address this problem elsewhere [14]. For NLS we have the following

Theorem 1.3. *Let $d \geqslant 6$. There exists a spherically symmetric global solution W^- of* (1.1) *with $E(W^-) = E(W)$ such that*

$$\left\|\nabla W^-(t)\right\|_2 < \|\nabla W\|_2, \quad \forall t \in \mathbb{R}.$$

Moreover, W^- scatters in the negative time direction and blows up in the positive time direction, in which W^- is asymptotically close to W:

$$\lim_{t\to+\infty}\left\|W^-(t) - W\right\|_{\dot{H}^1_x} = 0.$$

There also exists a spherically symmetric solution W^+ with $E(W^+) = E(W)$ such that

$$\left\|\nabla W^+(t)\right\|_2 > \|\nabla W\|_2, \quad \forall t \in \mathbb{R}.$$

Moreover in the positive time direction, W^+ blows up at infinite time and is asymptotically close to W

$$\lim_{t\to+\infty}\left\|W^+(t) - W\right\|_{\dot{H}^1_x} = 0.$$

In the negative time direction, W^+ blows up at finite time.

Next, we classify solutions with the threshold energy. Since the equation is invariant under several symmetries, we can determine the solution only modulo these symmetries. In the spherically symmetric setting, when we say $u = v$ up to symmetries, we mean there exist θ_0, $t_0 \in \mathbb{R}$, $\lambda_0 > 0$ such that

$$u(t,x) = e^{i\theta_0}\lambda_0^{-\frac{d-2}{2}} v\left(\frac{t+t_0}{\lambda_0^2}, \frac{x}{\lambda_0}\right).$$

With this convention we have

Theorem 1.4. *Let $d \geqslant 6$, $u_0 \in \dot{H}_x^1(\mathbb{R}^d)$ be spherically symmetric and such that $E(u_0) = E(W)$. Let u be the corresponding maximal-lifespan solution of* (1.1) *on $I \times \mathbb{R}^d$. We have*

(1) *If $\|\nabla u_0\|_2 < \|\nabla W\|_2$, then either $u = W^-$ up to symmetries or u scatters in both time directions.*
(2) *If $\|\nabla u_0\|_2 = \|\nabla W\|_2$, then $u = W$ up to symmetries.*
(3) *If $\|\nabla u_0\|_2 > \|\nabla W\|_2$ and $u_0 \in L_x^2(\mathbb{R}^d)$, then either $|I|$ is finite or $u = W^+$ up to symmetries.*

The proof of Theorems 1.3 and 1.4 will follow roughly the same strategy as in [6]. Here we make a remark about the proof of Theorem 1.4. The second point is a direct application of variational characterization of W (see the last section for more details). To prove (1) and (3), in [6], a large portion of the work was devoted to showing the exponential convergence of the solution to W, which after several minor changes, also works for higher dimensions. For this reason, we do not repeat that part of the argument and build our starting point on the following

Proposition 1.5 *(Exponential convergence to W [6]). Suppose u_0, u satisfy the same conditions as in Theorem* 1.4 *and u blows up on I forward in time. If $\|\nabla u_0\|_2 > \|\nabla W\|_2$, we assume* $[0, \sup I) = [0, \infty)$. *If $\|\nabla u_0\|_2 \leqslant \|\nabla W\|_2$, then the solution exists globally and $I = \mathbb{R}$. In all cases there exist $\theta_0 \in \mathbb{R}$, $\gamma_0 > 0$, $\mu_0 > 0$ such that*

$$\left\| u(t) - W_{[\theta_0,\mu_0]} \right\|_{\dot{H}^1} \leqslant Ce^{-\gamma_0 t}, \quad \forall t \geqslant 0, \tag{1.4}$$

where

$$W_{[\theta_0,\mu_0]}(x) = e^{i\theta_0}\mu_0^{-\frac{d-2}{2}} W\left(\frac{x}{\mu_0}\right).$$

This paper is organized as follows. In Section 2, we introduce some notations and collect some basic estimates. Section 3 is devoted to proving Theorem 1.3. In Section 4, we give the proof of Theorem 1.4 by assuming Proposition 1.5.

2. Preliminaries

We use $X \lesssim Y$ or $Y \gtrsim X$ whenever $X \leqslant CY$ for some constant $C > 0$. We use $O(Y)$ to denote any quantity X such that $|X| \lesssim Y$. We use the notation $X \sim Y$ whenever $X \lesssim Y \lesssim X$. We will add subscripts to C to indicate the dependence of C on the parameters. For example, $C_{i,j}$ means that the constant C depends on i, j. The dependence of C upon dimension will be suppressed.

We use the 'Japanese bracket' convention $\langle x\rangle := (1 + |x|^2)^{1/2}$.

Throughout this paper, we will use p_c to denote the total power of nonlinearity:

$$p_c = \frac{d+2}{d-2}.$$

We write $L_t^q L_x^r$ to denote the Banach space with norm

$$\|u\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^d)} := \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^d} \left|u(t,x)\right|^r dx \right)^{q/r} dt \right)^{1/q},$$

with the usual modifications when q or r are equal to infinity, or when the domain $\mathbb{R}\times\mathbb{R}^d$ is replaced by a smaller region of spacetime such as $I \times \Omega$. When $q = r$ we abbreviate $L_t^q L_x^q$ as $L_{t,x}^q$.

For a positive integer k, we use $W^{k,p}$ to denote the space with the norm

$$\|u\|_{W^{k,p}} = \sum_{0\leqslant j\leqslant k} \left\|\nabla^j u\right\|_{L_x^p},$$

when $p = 2$, we write $W^{k,2}$ as H^k.

2.1. Strichartz estimates

Let the dimension $d \geqslant 6$. We say a couple (q,r) is admissible if $2 \leqslant q \leqslant \infty$ and

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}.$$

Let I be a time slab. We denote $\dot{S}^0(I) = \bigcap_{(q,r)\text{ admissible}} L_t^q L_x^r(I \times \mathbb{R}^d)$ and $\dot{N}^0(I)$ as its dual space. We will use $\dot{S}^1(I)$ and $\dot{N}^1(I)$ to denote the space of functions u such that $\nabla u \in \dot{S}^0(I)$ and $\nabla u \in \dot{N}^0(I)$ respectively. By Sobolev embedding, it is easy to verify that

$$\|u\|_{L_t^q L_x^r} \lesssim \|u\|_{\dot{S}^1}, \tag{2.1}$$

for all $\dot{H}^1$ admissible pairs (q,r) in the sense that $2 \leqslant q \leqslant \infty$, and $\frac{2}{q} + \frac{d}{r} = \frac{d}{2} - 1$. Two typical $\dot{H}^1$ admissible pairs are $(\infty, \frac{2d}{d-2})$, $(2, \frac{2d}{d-4})$. Other pairs will also be used in this paper without mentioning this embedding.

With the notations above, we record the standard Strichartz estimates as follows.

Lemma 2.1. *(See Strichartz estimates [8,17].) Let $k = 0, 1$. Let I be an interval, $t_0 \in I$, $u_0 \in \dot{H}^k$ and $f \in \dot{N}^k(I)$. Then, the function u defined by*

$$u(t) := e^{i(t-t_0)\Delta} u_0 - i \int_{t_0}^{t} e^{i(t-t')\Delta} f(t')\, dt'$$

obeys the estimate

$$\|u\|_{\dot{S}^k(I)} \lesssim \|u_0\|_{\dot{H}^k} + \|f\|_{\dot{N}^k(I)}.$$

2.2. Derivation of the perturbation equation near W

Let u be the solution of (1.1) and $v = u - W$, then v satisfies the equation

$$i\partial_t v + \Delta v + \Gamma(v) + iR(v) = 0,$$

where,

$$\Gamma(v) = \frac{p_c + 1}{2} W^{p_c - 1} v + \frac{p_c - 1}{2} W^{p_c - 1} \bar{v},$$
$$iR(v) = |v + W|^{p_c - 1}(v + W) - W^{p_c} - \frac{p_c + 1}{2} W^{p_c - 1} v - \frac{p_c - 1}{2} W^{p_c - 1} \bar{v}.$$

Define the linear operator $\mathcal{L}$ by

$$\mathcal{L}(v) = -i\Delta v - i\frac{p_c + 1}{2} W^{p_c - 1} v - i\frac{p_c - 1}{2} W^{p_c - 1} \bar{v}.$$

We write the equation for v equivalently as

$$\partial_t v + \mathcal{L}(v) + R(v) = 0.$$

For the spectral properties of $\mathcal{L}$, we need the following lemma from [6].

Lemma 2.2. *(See [6].) The operator $\mathcal{L}$ admits two eigenfunctions $\mathcal{Y}_+$, $\mathcal{Y}_- \in \mathcal{S}(\mathbb{R}^d)$ with real eigenvalues*

$$\mathcal{L}Y_+ = e_0 \mathcal{Y}_+, \qquad \mathcal{L}Y_- = -e_0 \mathcal{Y}_-,$$

and

$$\mathcal{Y}_+ = \bar{\mathcal{Y}}_-, \quad e_0 > 0.$$

Proof. See Lemma 5.1 of [6]. □

2.3. Basic estimates

We will use the following lemma many times throughout this paper.

Lemma 2.3. *Let I be a time slab. We have*

$$\big\| |u|^{p_c - 1} \nabla v \big\|_{\dot{N}^0(I)} \lesssim \|u\|_{\dot{S}^1(I)}^{p_c - 1} \|v\|_{\dot{S}^1(I)},$$

$$\big\| \nabla W |v|^{p_c - 1} \big\|_{\dot{N}^0(I; |v| > \frac{1}{4} W)} \lesssim \|v\|_{\dot{S}^1(I)}^{\frac{d + \frac{3}{2}}{d - 2}},$$

$$\big\| W^{p_c - 2} \nabla W v \big\|_{\dot{N}^0(I; |v| > \frac{1}{4} W)} \lesssim \|v\|_{\dot{S}^1(I)}^{\frac{d + \frac{3}{2}}{d - 2}},$$

$$\big\| W^{p_c-3}\nabla W v^2\big\|_{\dot N^0(I;|v|\leqslant\frac14 W)} \lesssim \|v\|_{\dot S^1(I)}^{\frac{d+\frac32}{d-2}},$$

$$\big\| W^{-1}\nabla W|v|^{p_c}\big\|_{\dot N^0(I;|v|\leqslant\frac14 W)} \lesssim \|v\|_{\dot S^1(I)}^{\frac{d+\frac32}{d-2}}.$$

Here $\dot N^0(I;|v|>\frac14 W)$ denotes $\dot N^0(I\times\Omega)$, where $\Omega=\{x\colon |v(x)|>\frac14 W(x)\}$. Similar conventions apply to $\dot N^0(I;|v|<\frac14 W)$.

Proof. The first one follows directly from Hölder's inequality, we have

$$\begin{aligned}\big\||u|^{p_c-1}\nabla v\big\|_{\dot N^0(I)} &\leqslant \big\||u|^{p_c-1}\nabla v\big\|_{L_t^2L_x^{\frac{2d}{d+2}}(I\times\mathbb{R}^d)}\\ &\leqslant \|u\|^{p_c-1}_{L_t^\infty L_x^{\frac{2d}{d-2}}(I\times\mathbb{R}^d)}\|\nabla v\|_{L_t^2L_x^{\frac{2d}{d-2}}(I\times\mathbb{R}^d)}\\ &\leqslant \|u\|^{p_c-1}_{\dot S^1(I)}\|v\|_{\dot S^1(I)}.\end{aligned}$$

Now we verify the second one. Noting $|\nabla W(x)|\lesssim\langle x\rangle^{-(d-1)}$, we have

$$\begin{aligned}\big\|\nabla W|v|^{p_c-1}\big\|_{\dot N^0(I;|v|>\frac14 W)} &\lesssim \big\||v|^{p_c-1}\langle x\rangle^{-(d-\frac52)}\langle x\rangle^{-\frac32}\big\|_{\dot N^0(I;|v|\geqslant\frac14 W)}\\ &\lesssim \big\||v|^{\frac{d+\frac32}{d-2}}\langle x\rangle^{-\frac32}\big\|_{L_t^2L_x^{\frac{2d}{d+2}}(I\times\mathbb{R}^d)}\\ &\leqslant \big\|\langle x\rangle^{-\frac32}\big\|_{L_x^{\frac{4d}{5}}}\big\||v|^{\frac{d+\frac32}{d-2}}\big\|_{L_t^2L_x^{\frac{4d}{2d-1}}(I\times\mathbb{R}^d)}\\ &\lesssim \|v\|^{\frac{d+\frac32}{d-2}}_{L_t^{\frac{2d+3}{d-2}}L_x^{\frac{4d(d+\frac32)}{(d-2)(2d-1)}}(I\times\mathbb{R}^d)}\\ &\leqslant \|v\|^{\frac{d+\frac32}{d-2}}_{\dot S^1(I)}.\end{aligned}$$

To see the third one, we use the bound $W^{p_c-2}|\nabla W|\lesssim\langle x\rangle^{-5}$ to control

$$W^{p_c-2}|\nabla W|\lesssim\langle x\rangle^{-\frac32}|v|^{\frac{d+\frac32}{d-2}},\quad |v|>\frac14 W,$$

the same argument in proving the second one yields the desired estimate.

We verify the fourth inequality:

$$\begin{aligned}\big\|W^{p_c-3}\nabla W v^2\big\|_{\dot N^0(I;|v|\leqslant\frac14 W)} &\lesssim \big\|W^{\frac{d+3}{d-2}-2}v^2\big\|_{\dot N^0(I;|v|\leqslant\frac14 W)}\\ &\lesssim \big\|W^{\frac{3}{2(d-2)}}|v|^{\frac{d+\frac32}{d-2}}\big\|_{L_t^2L_x^{\frac{2d}{d+2}}(I\times\mathbb{R}^d)}\end{aligned}$$

$$\lesssim \left\| W^{\frac{3}{2(d-2)}} \right\|_{L_x^{\frac{4d}{5}}} \|v\|^{\frac{d+\frac{3}{2}}{d-2}}_{L_t^{\frac{2d+3}{d-2}} L_x^{\frac{2d(2d+3)}{(d-2)(2d-1)}}(I\times\mathbb{R}^d)}$$

$$\lesssim \|v\|^{\frac{d+\frac{3}{2}}{d-2}}_{\dot{S}^1(I\times\mathbb{R}^d)}.$$

The last one follows from the bound

$$W|\nabla W||v|^{p_c} \lesssim \langle x\rangle^{-\frac{3}{2}}|v|^{\frac{d+\frac{3}{2}}{d-2}}$$

for $|v| \leqslant \frac{1}{4}W$ and Hölder inequality, as in the second one. □

3. The existence of W^-, W^+

As in [6], we will construct the threshold solutions W^-, W^+ as the limit of a sequence of near solutions $W_k^a(t,x)$ in the positive time direction. It follows from this construction that both W^- and W^+ approach to the ground state W exponentially fast as $t \to +\infty$. On the other hand, the asymptotic behaviors of W^- and W^+ are quite different in the negative time direction (see Remark 3.10). We begin with the following result:

Lemma 3.1. *(See [6].) Let $a \in \mathbb{R}$. There exist functions $\{\Phi_j^a\}_{j\geqslant 1}$ in $\mathcal{S}(\mathbb{R}^d)$ such that $\Phi_1^a = a\mathcal{Y}_+$ (see Lemma 2.2 for the definition of $\mathcal{Y}_+$) and the function*

$$W_k^a(t,x) = W(x) + \sum_{j=1}^{k} e^{-je_0t}\Phi_j^a(x),$$

is a near solution of Eq. (1.1) in the sense that

$$(i\partial_t + \Delta)W_k^a + \left|W_k^a\right|^{\frac{4}{d-2}} W_k^a = \varepsilon_k^a,$$

where the error ε_k^a is exponentially small in $\mathcal{S}(\mathbb{R}^d)$. More precisely, $\forall J, M \geqslant 0$, J, M are integers, there exists a constant $C_{J,M}$ such that

$$\langle x\rangle^M \left|\nabla^J \varepsilon_k^a(x)\right| \leqslant C_{J,M} e^{-(k+1)e_0t}.$$

Remark 3.2. Since all Φ_j are Schwartz functions, we have the following properties for the difference

$$v_k = W_k^a - W = \sum_{j=1}^{k} e^{-je_0t}\Phi_j^a(x). \tag{3.1}$$

For any $j, m \geqslant 0$, there exists $C_{k,j,m} > 0$ such that

$$\left|\langle x\rangle^j \nabla^m v_k(t,x)\right| \leqslant C_{k,j,m} e^{-e_0t}. \tag{3.2}$$

Next we show that there exists a unique genuine solution $W^a(t,x)$ of (1.1) which can be approximated by the above constructed near solutions $W_k^a(t,x)$. The existence and uniqueness of the solution W^a can be transformed to that of $h := W^a - W_k^a$ which satisfies the equation

$$i\partial_t h + \Delta h = -\Gamma(h) - iR(v_k + h) + iR(v_k) + i\varepsilon_k^a. \tag{3.3}$$

Remark that this is the first place where the proof in [6] breaks down in higher dimensions. In [6] for dimensions $d = 3, 4, 5$, they made use of the fact that the nonlinearity $G(h) := -\Gamma(h) - iR(v_k + h) + iR(v_k)$ is Lipschitz[2] in $\dot{S}^1$ to construct the solution to (3.3) by the fixed point argument. In higher dimensions $d \geqslant 6$, the Lipschitz continuity does not hold anymore. However, since v_k is small compared with W, we can use real analytic expansion for the complex function $|1+z|^{p_c-1}(1+z)$ to show that $R(v_k + h) - R(v_k)$ is actually Lipschitz in h once h is small. This observation motivates us to construct the solution in a certain space consisting of functions which decay much faster than W. It turns out that the weighted Sobolev space $H^{m,m}$ with the norm

$$\|f\|_{H^{m,m}} = \sum_{0 \leqslant j \leqslant m} \left\| \langle x \rangle^{m-j} \nabla^j f \right\|_2 \tag{3.4}$$

for large m serves this purpose.

We have several properties for $H^{m,m}$.

Lemma 3.3 *(Linear estimate in $H^{m,m}$). For any $m \geqslant 1$, there exists a constant C depending on m such that*[3]

$$\left\| e^{it\Delta} u_0 \right\|_{H^{m,m}} \leqslant e^{C|t|} \|u_0\|_{H^{m,m}}, \quad \forall t \in \mathbb{R}. \tag{3.5}$$

Let $t_0 > 0$, $\alpha > 2C$ and Σ_{t_0} be the space with the norm

$$\|u\|_{\Sigma_{t_0}} = \sup_{t \geqslant t_0} e^{\alpha t} \left\| u(t) \right\|_{H^{m,m}}, \tag{3.6}$$

then the following holds:

$$\left\| \int_t^\infty e^{i(t-\tau)\Delta} F(\tau)\, d\tau \right\|_{\Sigma_{t_0}} \leqslant \frac{1}{\alpha - C} \|F\|_{\Sigma_{t_0}}. \tag{3.7}$$

Proof. (3.5) follows directly from the standard energy method. To obtain (3.7), we use (3.5) to estimate

[2] More precisely $\|G(h_1) - G(h_2)\|_{\dot{N}^1} \lesssim \|h_1 - h_2\|_{\dot{S}^1}$.

[3] Certainly the estimate (3.5) is not optimal. For example, one can improve it to: $\|e^{it\Delta} u_0\|_{H^{m,m}} \lesssim (1+|t|)^m \|u_0\|_{H^{m,m}}$. However, the rough estimate (3.5) is enough for our use.

$$\begin{aligned}\left\| \int_t^\infty e^{i(t-\tau)\Delta} F(\tau)\,d\tau \right\|_{H^{m,m}} &\leqslant \int_t^\infty \left\| e^{i(t-\tau)\Delta} F(\tau) \right\|_{H^{m,m}} d\tau \\ &\leqslant \int_t^\infty e^{C(\tau-t)} \left\| F(\tau) \right\|_{H^{m,m}} d\tau \\ &\leqslant \int_t^\infty e^{C(\tau-t)} e^{-\alpha\tau}\,d\tau \|F\|_{\Sigma_{t_0}} \\ &\leqslant \frac{1}{\alpha - C} e^{-\alpha t} \|F\|_{\Sigma_{t_0}}.\end{aligned}$$

(3.7) now follows immediately. □

Lemma 3.4 *(Embedding in $H^{m,m}$). Let k_1, k_2 be non-negative integers, then for any $m \geqslant k_1 + k_2 + \frac{d}{2} + 1$, we have*

$$\left\| \langle x \rangle^{k_1} \nabla^{k_2} f \right\|_\infty \lesssim \|f\|_{H^{m,m}},$$

where the implicit constant depends only on k_1, k_2.

Proof. Denote $[\frac{d}{2}+]$ as the smallest integer strictly bigger than $\frac{d}{2}$. By Sobolev embedding

$$\|f\|_\infty \lesssim \|f\|_{H^{[\frac{d}{2}+]}},$$

we have

$$\begin{aligned}\left\| \langle x \rangle^{k_1} \nabla^{k_2} f \right\|_\infty &\lesssim \left\| \langle x \rangle^{k_1} \nabla^{k_2} f \right\|_2 + \left\| \nabla^{[\frac{d}{2}+]} \big(\langle x \rangle^{k_1} \nabla^{k_2} f \big) \right\|_2 \\ &\lesssim \|f\|_{H^{m,m}} + \sum_{0 \leqslant j \leqslant [\frac{d}{2}+]} \left\| \nabla^j \big(\langle x \rangle^{k_1} \big) \nabla^{[\frac{d}{2}+]-j+k_2} f \right\|_2 \\ &\lesssim \|f\|_{H^{m,m}} + \sum_{0 \leqslant j \leqslant [\frac{d}{2}+]} \left\| \langle x \rangle^{k_1-j} \nabla^{[\frac{d}{2}+]-j+k_2} f \right\|_2 \\ &\lesssim \|f\|_{H^{m,m}}.\end{aligned}$$

The lemma is proved. □

Lemma 3.5 *(Bilinear estimate in $H^{m,m}$). We have*

$$\|fg\|_{H^{m,m}} \lesssim \|f\|_{W^{m,\infty}} \|g\|_{H^{m,m}}, \tag{3.8}$$

with the implicit constant depending only on m.

Proof.

$$\begin{aligned}
\|fg\|_{H^{m,m}} &\leqslant \sum_{0\leqslant j\leqslant m} \big\|\langle x\rangle^{m-j}\nabla^j(fg)\big\|_2 \\
&\lesssim \sum_{0\leqslant j\leqslant m}\sum_{0\leqslant k\leqslant j} \big\|\langle x\rangle^{m-j}\nabla^{j-k}f\nabla^k g\big\|_2 \\
&\lesssim \sum_{0\leqslant j\leqslant m}\sum_{0\leqslant k\leqslant j} \big\|\big(\langle x\rangle^{m-k}\nabla^k g\big)\big(\langle x\rangle^{k-j}\nabla^{j-k}f\big)\big\|_2 \\
&\lesssim \sum_{0\leqslant j\leqslant m}\sum_{0\leqslant k\leqslant j} \big\|\langle x\rangle^{m-k}\nabla^k g\big\|_2\big\|\nabla^{j-k}f\big\|_\infty \\
&\lesssim \|g\|_{H^{m,m}}\|f\|_{W^{m,\infty}}. \qquad \square
\end{aligned}$$

Lemma 3.6. *Let $C>0$, $j\geqslant 2$ and $m\geqslant \frac{d}{2}+1+\frac{Cj}{j-1}$, then*

$$\big\|\langle x\rangle^{Cj}h^j\big\|_{H^{m,m}} \lesssim j^m\|h\|^j_{H^{m,m}},$$

where the implicit constant depends only on m.

Proof. From the definition and the chain rule, we estimate

$$\begin{aligned}
&\big\|\langle x\rangle^{Cj}h^j\big\|_{H^{m,m}} \\
&\quad\leqslant \sum_{0\leqslant l\leqslant m} \big\|\langle x\rangle^{m-l}\nabla^l\big(\langle x\rangle^{Cj}h^j\big)\big\|_2 \\
&\quad\lesssim \sum_{\substack{0\leqslant l\leqslant m\\ l_0+\cdots+l_\alpha=l}} j^l\big\|\langle x\rangle^{m-l+Cj-l_0}\nabla^{l_1}h\nabla^{l_2}h\cdots\nabla^{l_\alpha}hh^{j-\alpha}\big\|_2 \\
&\quad\lesssim j^m \sum_{\substack{0\leqslant l\leqslant m\\ l_0+\cdots+l_\alpha=l}} \big\|\langle x\rangle^{m-l+Cj-l_0}\langle x\rangle^{-(m-l_1+m-l_2-\frac{d}{2}-1+\cdots+m-l_\alpha-\frac{d}{2}-1+(j-\alpha)(m-\frac{d}{2}-1))}\big\|_\infty \\
&\qquad\cdot\big\|\langle x\rangle^{m-l_1}\nabla^{l_1}h\big\|_2\cdot\big\|\langle x\rangle^{m-l_2-\frac{d}{2}-1}\nabla^{l_2}h\big\|_\infty\cdots \\
&\qquad\cdot\big\|\langle x\rangle^{m-l_\alpha-\frac{d}{2}-1}\nabla^{l_\alpha}h\big\|_\infty\big\|\langle x\rangle^{m-\frac{d}{2}-1}h\big\|^{j-\alpha}_\infty.
\end{aligned}$$

Since $m>\frac{d}{2}+1+\frac{Cj}{j-1}$, it is not difficult to verify that the exponent of $\langle x\rangle$ is non-positive in the first factor of the last expression. This combined with Lemma 3.4 shows that

$$\big\|\langle x\rangle^{Cj}h^j\big\|_{H^{m,m}} \lesssim j^m\|h\|^j_{H^{m,m}}. \qquad \square \tag{3.9}$$

We will prove the following:

Proposition 3.7. *Let $a \in \mathbb{R}$. Let $\mathcal{Y}_+$ and $W_k^a = W_k^a(t,x)$ be the same as in Lemma* 3.1. *Assume $m \geqslant 3d$ is fixed. Then there exists $k_0 > 0$ and a unique solution $W^a(t,x)$ for the equation in* (1.1) *which satisfies the following: for any $k \geqslant k_0$, there exists $t_k \geqslant 0$ such that $\forall t \geqslant t_k$,*

$$\left\| W^a(t) - W_k^a(t) \right\|_{H^{m,m}} \leqslant e^{-\alpha t}, \quad \alpha = \left(k + \frac{1}{2}\right) e_0. \tag{3.10}$$

Moreover, we have

$$\left\| W^a(t) - W - a e^{-e_0 t} \mathcal{Y}_+ \right\|_{H^{m,m}} \leqslant e^{-\frac{3}{2} e_0 t}. \tag{3.11}$$

Proof. Let $h = W^a - W_k^a$, then W^a is the solution of (1.1) as long as h is a solution of Eq. (3.3) which tends to 0 as $t \to \infty$. From Duhamel's formula, it is equivalent to solve the following integral equation

$$\begin{aligned} h(t) &= i \int_t^\infty e^{i(t-s)\Delta} \left[-\Gamma(h) - i R(h + v_k) + i R(v_k) + i \varepsilon_k^a \right](s)\, ds \\ &=: \Phi\big(h(t)\big). \end{aligned} \tag{3.12}$$

Define the space Σ_{t_k} by $\|f\|_{\Sigma_{t_k}} = \sup_{t \geqslant t_k} e^{\alpha t} \|f(t)\|_{H^{m,m}}$ and introduce the unit ball

$$B_k = \left\{ f = f(t,x) \colon \|f\|_{\Sigma_{t_k}} \leqslant 1 \right\}.$$

We shall show that Φ is a contraction on B_k. Taking $h \in B_k$, we compute the $H^{m,m}$ norm of $\Phi(h(t))$:

$$\left\| \Phi\big(h(t)\big) \right\|_{H^{m,m}} \leqslant \int_t^\infty \left\| e^{i(t-s)\Delta} \Gamma\big(h(s)\big) \right\|_{H^{m,m}} ds \tag{3.13}$$

$$+ \int_t^\infty \left\| e^{i(t-s)\Delta} \big(R(h + v_k) - R(v_k) \big)(s) \right\|_{H^{m,m}} ds \tag{3.14}$$

$$+ \int_t^\infty \left\| e^{i(t-s)\Delta} \varepsilon_k^a(s) \right\|_{H^{m,m}} ds. \tag{3.15}$$

To estimate (3.13), we use Lemmas 3.5, 3.3 to get

$$\begin{aligned} (3.13) &\leqslant \int_t^\infty e^{C|t-s|} \left\| \Gamma\big(h(s)\big) \right\|_{H^{m,m}} ds \\ &\lesssim \int_t^\infty e^{C|t-s|} \left\| W^{p_c - 1} \right\|_{W^{m,\infty}} \left\| h(s) \right\|_{H^{m,m}} ds \end{aligned}$$

$$\lesssim \int_t^\infty e^{C|t-s|} e^{-\alpha s} \|h\|_{\Sigma_{t_k}}\, ds$$

$$\lesssim e^{-\alpha t} \|h\|_{\Sigma_{t_k}} \int_t^\infty e^{-(\alpha-C)(s-t)}\, ds$$

$$\lesssim \frac{1}{\alpha - C} e^{-\alpha t} \|h\|_{\Sigma_{t_k}}. \tag{3.16}$$

Since $\alpha = (k + \frac{1}{2})e_0$, by taking k_0 sufficient large, we have

$$(3.13) \leqslant \frac{1}{100} e^{-\alpha t} \|h\|_{\Sigma_{t_k}} \leqslant \frac{1}{100} e^{-\alpha t} \tag{3.17}$$

for all $k \geqslant k_0$.

Now we deal with (3.15). Note that by Lemma 3.1, $\varepsilon_k^a(t) = O(e^{-(k+1)e_0 t})$ in $\mathcal{S}(\mathbb{R}^d)$. This implies

$$\left\| \varepsilon_k^a(t) \right\|_{H^{m,m}} \leqslant C_k e^{-(k+1)e_0 t}.$$

Thus,

$$(3.15) \leqslant \int_t^\infty e^{C|t-s|} \left\| \varepsilon_k^a(s) \right\|_{H^{m,m}} ds$$

$$\leqslant C_k \int_t^\infty e^{C|t-s|} e^{-(k+1)e_0 s}\, ds$$

$$\leqslant C_k \frac{e^{-\frac{1}{2}e_0 t}}{(k+1)e_0 - C} e^{-(k+\frac{1}{2})e_0 t} \leqslant \frac{1}{100} e^{-\alpha t} \tag{3.18}$$

if $t \geqslant t_k$ and t_k is sufficiently large.

It remains to estimate (3.14). The reason that we can take m derivatives is that both v_k and h are small compared to W. Indeed by Remark 3.2, we have

$$\left| v_k(t,x) \right| < \frac{1}{2} W(x), \quad \forall t \geqslant t_k,\ x \in \mathbb{R}^d. \tag{3.19}$$

Moreover, since $h \in \Sigma_{t_k}$ and $m \geqslant 3d$, by Lemma 3.4 we have

$$\left\| \langle x \rangle^{d-2} h(t) \right\|_\infty \lesssim \left\| h(t) \right\|_{H^{m,m}} \leqslant e^{-\alpha t} \|h\|_{\Sigma_{t_k}}.$$

As a consequence, we have

$$\left| h(t,x) \right| \lesssim e^{-\alpha t} \langle x \rangle^{-(d-2)} \|h\|_{\Sigma_{t_k}} \leqslant \frac{1}{4} W(x). \tag{3.20}$$

Using (3.19) and (3.20) together with the expansion for the real analytic function $P(z) = |1+z|^{p_c-1}(1+z)$ for $|z| \leqslant \frac{3}{4}$ which takes the form

$$P(z) = 1 + \frac{p_c+1}{2}z + \frac{p_c-1}{2}\bar{z} + \sum_{j_1+j_2\geqslant 2} a_{j_1,j_2} z^{j_1}\bar{z}^{j_2}, \tag{3.21}$$

we write

$$\begin{aligned} i\big(R(v_k+h) - R(v_k)\big) &= W^{p_c}\Bigg[\left|1+\frac{v_k+h}{W}\right|^{p_c-1}\left(1+\frac{v_k+h}{W}\right) - \left|1+\frac{v_k}{W}\right|^{p_c-1}\left(1+\frac{v_k}{W}\right) \\ &\quad - \frac{p_c+1}{2}\frac{h}{W} - \frac{p_c-1}{2}\frac{\bar{h}}{W}\Bigg] \\ &= W^{p_c}\Bigg[\sum_{j_1+j_2\geqslant 2} a_{j_1,j_2}\Bigg(\left(\frac{v_k+h}{W}\right)^{j_1}\left(\frac{\bar{v}_k+\bar{h}}{W}\right)^{j_2} - \left(\frac{v_k}{W}\right)^{j_1}\left(\frac{\bar{v}_k}{W}\right)^{j_2}\Bigg)\Bigg] \\ &= O\Bigg(\sum_{j\geqslant 2,\, 1\leqslant i\leqslant j} a_j C_{i,j} W^{p_c-j}\mathcal{O}\big(v_k^{j-i}h^i\big)\Bigg), \end{aligned} \tag{3.22}$$

where the last equality following from using binomial expansion and regrouping coefficients, and we have the bound

$$a_j = O\left(\frac{p_c(p_c-1)\cdots(p_c-j+1)}{j!}\right) \quad \text{and} \quad |a_j| \leqslant 1,$$

$$C_{i,j} = O\left(\frac{j!}{i!(j-i)!}\right) \quad \text{and} \quad C_{i,j} \leqslant 2^j.$$

The notation $\mathcal{O}(v_k^{j-i}h^i)$ denotes terms of the form

$$v_k^{\alpha_1}\bar{v}_k^{\alpha_2}h^{\beta_1}\bar{h}^{\beta_2}$$

with $\alpha_1+\alpha_2 = j-i$, $\beta_1+\beta_2 = i$.

Now we use this expression to estimate $\|R(v_k+h) - R(v_k)\|_{H^{m,m}}$. Using Lemmas 3.5 and 3.6, we have

$$\begin{aligned} \big\|R(v_k+h) - R(v_k)\big\|_{H^{m,m}} &\lesssim \sum_{j\geqslant 2,\, 1\leqslant i\leqslant j} a_j C_{i,j}\big\|W^{p_c-j}v_k^{j-i}h^i\big\|_{H^{m,m}} \\ &\lesssim \sum_{j\geqslant 2} 2^j\big\|W^{-j}v_k^{j-1}h\big\|_{H^{m,m}} \\ &\quad + \sum_{j\geqslant 2,\, 2\leqslant i\leqslant j} 2^j\big\|\big(W^{-1}v_k\big)^{j-i}W^{-i}h^i\big\|_{H^{m,m}} \\ &\lesssim \sum_{j\geqslant 2} 2^j\big\|W^{-j}v_k^{j-1}\big\|_{W^{m,\infty}}\|h\|_{H^{m,m}} \\ &\quad + \sum_{j\geqslant 2,\, 2\leqslant i\leqslant j} 2^j i^m\big\|\big(W^{-1}v_k\big)^{j-i}\big\|_{W^{m,\infty}}\|h\|_{H^{m,m}}^i. \end{aligned}$$

Applying Remark 3.2 and in view of $h \in \Sigma_{t_k}$, we have

$$\left\| W^{-j} v_k^{j-1} \right\|_{W^{m,\infty}} \leqslant j^m C_{k,m} e^{-(j-1)e_0 t},$$
$$\left\| \left(W^{-1} v_k\right)^{j-i} \right\|_{W^{m,\infty}} \leqslant j^m C_{k,m} e^{-(j-i)e_0 t}.$$

Noting moreover that

$$\|h\|_{H^{m,m}} \leqslant e^{-\alpha t} \|h\|_{\Sigma_{t_k}},$$

we estimate

$$\begin{aligned}
\left\| R(v_k + h)(t) - R(v_k)(t) \right\|_{H^{m,m}} &\leqslant \sum_{j \geqslant 2,\, 1 \leqslant i \leqslant j} 2^j j^{2m} C_{k,m} \left(e^{-\alpha t} \|h\|_{\Sigma_{t_k}}\right)^i \left(e^{-e_0 t}\right)^{j-i} \\
&\leqslant e^{-\alpha t} \|h\|_{\Sigma_{t_k}} \sum_{j \geqslant 2,\, 1 \leqslant i \leqslant j} 2^j j^{2m} C_{k,m} e^{-(\alpha(i-1)+(j-i)e_0)t} \\
&\leqslant e^{-\alpha t} \|h\|_{\Sigma_{t_k}} \sum_{j \geqslant 2} 2^j j^{2m} C_{m,k} e^{-(j-1)e_0 t_k} \\
&\leqslant \frac{1}{100} e^{-\alpha t} \|h\|_{\Sigma_{t_k}}.
\end{aligned}$$

The last inequality comes from the fact we can choose t_k large enough such that the series converges. Now we are ready to estimate (3.14). Using Lemma 3.3, we have

$$\begin{aligned}
(3.14) &\leqslant \int_t^\infty e^{C|t-s|} \left\| R(v_k + h)(s) - R(v_k)(s) \right\|_{H^{m,m}} ds \\
&\leqslant \frac{1}{100} \|h\|_{\Sigma_{t_k}} \int_t^\infty e^{C(s-t)} e^{-\alpha s}\, ds \leqslant \frac{1}{100} e^{-\alpha t} \|h\|_{\Sigma_{t_k}} \\
&\leqslant \frac{1}{100} e^{-\alpha t}.
\end{aligned} \tag{3.23}$$

Collecting the estimates (3.17), (3.18) and (3.23), we obtain

$$\left\| \Phi\left(h(t)\right) \right\|_{H^{m,m}} \leqslant \frac{1}{10} e^{-\alpha t} \tag{3.24}$$

for all $k \geqslant k_0$ and $t \geqslant t_k$. Therefore

$$\left\| \Phi(h) \right\|_{\Sigma_{t_k}} \leqslant \frac{1}{10},$$

which shows that Φ maps B_k to itself. Next we show that Φ is a contraction. Taking h_1 and h_2 in Σ_{t_k}, we compute

$$\begin{aligned}
&\left\|\Phi\big(h_1(t)\big)-\Phi\big(h_2(t)\big)\right\|_{H^{m,m}}\\
&\quad\leqslant\int_t^\infty\left\|e^{i(t-s)\Delta}\Gamma\big(h_1(s)-h_2(s)\big)\right\|_{H^{m,m}}ds \qquad (3.25)\\
&\qquad+\int_t^\infty\left\|e^{i(t-s)\Delta}\big(R(v_k+h_1)-R(v_k+h_2)\big)(s)\right\|_{H^{m,m}}ds. \qquad (3.26)
\end{aligned}$$

The estimate of (3.25) is the same as (3.16), we omit the details. To estimate (3.26), we write

$$\begin{aligned}
&-i\big(R(v_k+h_1)-R(v_k+h_2)\big)\\
&\quad=\sum_{j\geqslant 2}a_{j_1,j_2}W^{p_c-j}\left[\left(\frac{v_k+h_1}{W}\right)^{j_1}\left(\frac{\bar v_k+\bar h_1}{W}\right)^{j_2}-\left(\frac{v_k+h_2}{W}\right)^{j_1}\left(\frac{\bar v_k+\bar h_2}{W}\right)^{j_2}\right]\\
&\quad=O\left(\sum_{j\geqslant 2,\,1\leqslant i\leqslant j-1}a_jC_{i,j}W^{p_c-j}\mathcal{O}\big((h_1-h_2)v_k^{j-i}h^i\big)\right),
\end{aligned}$$

where the constants a_j, $C_{i,j}$ are the same as in (3.22). We are in the same situation as before. Therefore, we obtain

$$\left\|\Phi\big(h_1(t)\big)-\Phi\big(h_2(t)\big)\right\|_{H^{m,m}}\leqslant\frac{1}{10}e^{-\alpha t}\|h_1-h_2\|_{\Sigma_{t_k}},\quad\forall k\geqslant k_0,\ t\geqslant t_k, \qquad (3.27)$$

which shows that Φ is a contraction in B_k. This proves the existence and uniqueness of the solution to the equation in (1.1) such that (3.10) holds. It only remains to show that $W^a(t,x)$ is independent of k. Indeed, let $k_1<k_2$ and W^a, $\widetilde{W^a}$ be the corresponding solutions such that

$$\begin{aligned}
&\left\|W^a(t)-W^a_{k_1}(t)\right\|_{H^{m,m}}\leqslant e^{-(k_1+\frac12)e_0t},\quad\forall t\geqslant t_{k_1},\\
&\left\|\widetilde{W^a}(t)-W^a_{k_2}(t)\right\|_{H^{m,m}}\leqslant e^{-(k_2+\frac12)e_0t},\quad\forall t\geqslant t_{k_2}.
\end{aligned}$$

Without lose of generality we also assume $t_{k_1}\leqslant t_{k_2}$, then the triangle inequality gives that

$$\begin{aligned}
\left\|\widetilde{W^a}(t)-W^a_{k_1}(t)\right\|_{H^{m,m}}&\leqslant\left\|\tilde W^a(t)-W^a_{k_2}(t)\right\|_{H^{m,m}}+\left\|\sum_{k_1\leqslant j<k_2}e^{-je_0t}\Phi_j\right\|_{H^{m,m}}\\
&\leqslant e^{-(k_1+\frac12)e_0t},\quad\forall t\geqslant t_{k_2}.
\end{aligned}$$

Therefore $W^a(t)=\widetilde{W^a}(t)$ on $[t_{k_2},\infty)$ and we conclude $W^a\equiv\widetilde{W^a}$ by uniqueness of solutions to (1.1). This shows that W^a does not depend on k. The existence and uniqueness of the solution to (3.12) is proved.

We finally verify (3.11). Let k_0 be the constant specified above, then by the triangle inequality and Lemma 3.1 we have

$$\begin{aligned}\big\|W^a(t) - W - ae^{-e_0 t}\mathcal{Y}_+\big\|_{H^{m,m}} &\leqslant \big\|W^a(t) - W^a_{k_0}(t)\big\|_{H^{m,m}} + \big\|v_{k_0}(t) - ae^{-e_0 t}\mathcal{Y}_+\big\|_{H^{m,m}} \\ &\leqslant e^{-(k_0+\frac{1}{2})e_0 t} + \bigg\|\sum_{2\leqslant j\leqslant k_0} e^{-je_0 t}\Phi^a_j\bigg\|_{H^{m,m}} \\ &\leqslant e^{-\frac{3}{2}e_0 t}\end{aligned}$$

for all sufficiently large t. Proposition 3.7 is proved. □

Corollary 3.8. *Let $w^a = W^a - W$, then there exists $t_0 > 0$ such that for all $t \geqslant t_0$ and all $2 \leqslant p \leqslant \infty$, we have*

$$\big\|\langle x\rangle^{l_1}\nabla^{l_2} w^a(t)\big\|_{L^p_x} \leqslant e^{-\frac{1}{2}e_0 t}, \tag{3.28}$$

as long as $l_1 + l_2 + \frac{d}{2} + 1 \leqslant m$. In particular,

$$\big\|w^a\big\|_{\dot{S}^1([t,\infty))} \leqslant e^{-\frac{1}{2}e_0 t}.$$

Proof. Let k_0 be the same as in Proposition 3.7, then by Remark 3.2 we have

$$\begin{aligned}\big\|w^a(t)\big\|_{H^{m,m}} &\leqslant \big\|w^a(t) - v_{k_0}(t)\big\|_{H^{m,m}} + \big\|v_{k_0}(t)\big\|_{H^{m,m}} \\ &\leqslant e^{-(k_0+\frac{1}{2})e_0 t} + \bigg\|\sum_{1\leqslant j\leqslant k_0} e^{-je_0 t}\Phi^a_j\bigg\|_{H^{m,m}}.\end{aligned}$$

Thus for t_0 sufficiently large and $t \geqslant t_0$, we obtain

$$\big\|w^a(t)\big\|_{H^{m,m}} \leqslant e^{-\frac{2}{3}e_0 t}.$$

An application of Sobolev embedding gives that for any p with $2 \leqslant p \leqslant \infty$,

$$\big\|\langle x\rangle^{l_1}\nabla^{l_2} w^a(t)\big\|_{L^p_x} \lesssim \big\|w^a\big\|_{H^{m,m}} \leqslant e^{-\frac{1}{2}e_0 t},$$

provided $l_1 + l_2 + \frac{d}{2} + 1 < m$. The corollary is proved. □

Before finishing this section, we make the following two remarks.

Remark 3.9. In next section we shall show for a, b such that $ab > 0$, W^a is just a time translation of W^b. This will allow us to define $W^{\pm}$ as $W^{\pm 1}$ and to classify the solutions with threshold energy.

The second remark concerns the behavior of $W^{\pm}$ in the negative time direction.

Remark 3.10. From the construction of $W^{\pm}(t)$, it is clear that they both approaches to the ground state W exponentially fast as $t \to +\infty$. For the behavior of $W^{\pm}$ in negative time direction, we can apply the same argument in [6] (see Corollaries 3.2, 4.2 for instance) to conclude that W^-

scatters when $t \to -\infty$ and W^+ blows up at finite time. To get the blowup of W^+, we need the crucial property $W^+ \in L_x^2$ which is now available as we are in dimensions $d \geqslant 6$.

4. Classification of the solution

Our purpose of this section is to prove Theorem 1.4. Following the argument in [6], the key step is to establish the following

Theorem 4.1. *Let $\gamma_0 > 0$. Assume u is the solution of the equation in (1.1) satisfying $E(u) = E(W)$ and*

$$\left\| u(t) - W \right\|_{\dot{H}^1} \leqslant Ce^{-\gamma_0 t}, \quad \forall t \geqslant 0, \tag{4.1}$$

then there exists $a \in \mathbb{R}$ such that

$$u = W^a.$$

As a corollary of Theorem 4.1, we see that modulo time translation, all the $\{W^a,\ a > 0\}$ and $\{W^a,\ a < 0\}$ are same.

Corollary 4.2. *For any $a \neq 0$, there exists $T_a \in \mathbb{R}$ such that*

$$\begin{cases} W^a(t) = W^+(t + T_a), & \text{if } a > 0, \\ W^a(t) = W^-(t + T_a), & \text{if } a < 0. \end{cases} \tag{4.2}$$

We now prove Theorem 4.1. The strategy is the following: we first prove that there exists $a \in \mathbb{R}$ such that $u(t) - W^a(t)$ has enough decay, then using the decay estimate to show that $u(t) - W^a(t)$ is actually identically zero. To this end, we have to input the condition (4.1) and upgrade it to the desired decay estimate. At this point, we need the following crucial result from [6].

Lemma 4.3. *Let h be the solution of the equation*

$$\partial_t h + \mathcal{L}h = \varepsilon. \tag{4.3}$$

And for $t \geqslant 0$,

$$\|\varepsilon\|_{\dot{N}^1([t,\infty))} \leqslant Ce^{-c_1 t}, \qquad \|\varepsilon\|_{L_x^{\frac{2d}{d+2}}} \leqslant Ce^{-c_1 t},$$
$$\left\| h(t) \right\|_{\dot{H}^1} \leqslant Ce^{-c_0 t},$$

where $c_0 < c_1$. Then the following statements hold true,

- *If $c_0 < c_1$ or $e_0 < c_0 < c_1$, then*

$$\|h\|_{\dot{S}^1([t,\infty))} \leqslant C_\eta e^{-(c_1 - \eta)t}. \tag{4.4}$$

- *If $c_0 \leqslant e_0 < c_1$, then there exists $a \in \mathbb{R}$ such that*

$$\left\| h - ae^{-e_0 t}\mathcal{Y}_+ \right\|_{\dot S^1([t,\infty))} \leqslant C_\eta e^{-(c_1-\eta)t}. \tag{4.5}$$

Let $v = u - W$, then (4.1) gives that

$$\left\| v(t) \right\|_{\dot H^1} \leqslant e^{-\gamma_0 t}, \quad t \geqslant 0. \tag{4.6}$$

Without loss of generality we assume $\gamma_0 < e_0$. We first show that this decay rate can be upgraded to $e^{-e_0 t}$. More precisely, we have

Proposition 4.4. *Let $v = u - W$, then there exists $t_0 > 0$ such that for all $t \geqslant t_0$,*

$$\| v \|_{\dot S^1([t,\infty))} \leqslant Ce^{-e_0 t}, \tag{4.7}$$

$$\left\| R(v) \right\|_{\dot N^1([t,\infty))} \leqslant Ce^{-\frac{d+\frac{3}{2}}{d-2} e_0 t}, \qquad \left\| R(v)(t) \right\|_{L_x^{\frac{2d}{d+2}}} \leqslant Ce^{-p_c \gamma_0 t}. \tag{4.8}$$

In particular, there exists $a \in \mathbb{R}$ such that

$$\left\| v - ae^{-e_0 t}\mathcal{Y}_+ \right\|_{\dot S^1([t,\infty))} \leqslant C_\eta e^{-(\frac{d+\frac{3}{2}}{d-2} - \eta) e_0 t}. \tag{4.9}$$

Proof. First we show that (4.9) is a consequence of (4.7), (4.8). To see this, note that v satisfies the equation

$$i\partial_t v + \Delta v + \Gamma(v) + iR(v) = 0, \tag{4.10}$$

or equivalently,

$$\partial_t v + \mathcal{L} v = -R(v). \tag{4.11}$$

Applying Lemma 4.3 with $h = v$, $\varepsilon = -R(v)$, $c_0 = e_0$, $c_1 = \frac{d+\frac{3}{2}}{d-2} e_0$ and using the estimates (4.7) and (4.8), we obtain (4.9). So we only need to establish (4.7) and (4.8). This will be done in two steps. At the first step, we prove that the Strichartz norm of v decays like $e^{-\gamma_0 t}$ and the dual Strichartz norm of $R(v)$ decays even faster. Secondly, we iterate this process and upgrade the decay estimate by using Lemma 4.3 finitely many times.

Step 1. We prove there exists $t_0 > 0$ such that for $t \geqslant t_0$,

$$\| v \|_{\dot S^1([t,\infty))} \lesssim e^{-\gamma_0 t}, \qquad \left\| R(v) \right\|_{\dot N^1([t,\infty))} \lesssim e^{-\frac{d+\frac{3}{2}}{d-2} \gamma_0 t}. \tag{4.12}$$

Let τ be a small constant to be chosen later. Using Strichartz estimate on the time interval $[t, t+\tau]$, we have

$$\| v \|_{\dot S^1([t,t+\tau])} \leqslant \left\| v(t) \right\|_{\dot H^1} + \left\| \Gamma(v) \right\|_{\dot N^1([t,t+\tau])} + \left\| R(v) \right\|_{\dot N^1([t,t+\tau])}. \tag{4.13}$$

For the linear term, we have

$$\begin{aligned}
&\bigl\|\Gamma(v)\bigr\|_{\dot N^1([t,t+\tau])}\\
&\quad\lesssim \bigl\|W^{p_c-1}\nabla v\bigr\|_{L_t^1L_x^2([t,t+\tau]\times\mathbb{R}^d)}+\bigl\|W^{p_c-2}\nabla W v\bigr\|_{L_t^1L_x^2([t,t+\tau]\times\mathbb{R}^d)}\\
&\quad\lesssim \tau\bigl\|W^{p_c-1}\bigr\|_{L_x^\infty}\|\nabla v\|_{L_t^\infty L_x^2([t,t+\tau]\times\mathbb{R}^d)}\\
&\qquad+\tau\bigl\|W^{p_c-2}\nabla W\bigr\|_{L_x^d}\|v\|_{L_t^\infty L_x^{\frac{2d}{d-2}}([t,t+\tau]\times\mathbb{R}^d)}\\
&\quad\lesssim \tau\|v\|_{\dot S^1([t,t+\tau])}.
\end{aligned}\tag{4.14}$$

This is good for us. Now we deal with the term $R(v)$. In lower dimensions, it is easy to see that $R(v)$ is superlinear in v. In higher dimensions ($d\geqslant 6$), this is trickier. Here we will rely heavily on the fact that W has nice decay to show that $R(v)$ is essentially superlinear in v. We claim for any time interval I, that

$$\bigl\|R(v)\bigr\|_{\dot N^1(I)}\lesssim \|v\|_{\dot S^1(I)}^{\frac{d+\frac32}{d-2}}+\|v\|_{\dot S^1(I)}^{p_c}.\tag{4.15}$$

Assume the claim is true for the moment. By (4.6), (4.13)–(4.15), we have

$$\|v\|_{\dot S^1([t,t+\tau])}\lesssim e^{-\gamma_0 t}+\tau\|v\|_{\dot S^1([t,t+\tau])}+\|v\|_{\dot S^1([t,t+\tau])}^{p_c}+\|v\|_{\dot S^1([t,t+\tau])}^{\frac{d+\frac32}{d-2}}\tag{4.16}$$

Taking τ small enough, a continuity argument shows that there exists $t_0>0$, such that for all $t\geqslant t_0$,

$$\|v\|_{\dot S^1([t,t+\tau])}\lesssim e^{-\gamma_0 t}.\tag{4.17}$$

Therefore, we have

$$\begin{aligned}
\|v\|_{\dot S^1([t,\infty))}&\leqslant\sum_{j\geqslant 0}\|v\|_{\dot S^1([t+\tau j,t+\tau(j+1)])}\\
&\lesssim\sum_{j\geqslant 0}e^{-\gamma_0(t+\tau j)}\leqslant e^{-\gamma_0 t}\frac{1}{1-e^{-\gamma_0\tau}}\\
&\lesssim e^{-\gamma_0 t}.
\end{aligned}$$

Plugging this estimate into (4.15), we have proved (4.12).

Now it remains to prove the claim (4.15). Recall that

$$\begin{aligned}
iR(v)&=|v+W|^{p_c-1}(v+W)-W^{p_c}-\frac{p_c+1}{2}W^{p_c-1}v-\frac{p_c-1}{2}W^{p_c-1}\bar v,\\
&=W^{p_c}J\biggl(\frac{v}{W}\biggr),
\end{aligned}$$

where

$$J(z) = |1+z|^{p_c-1}(1+z) - 1 - \frac{p_c+1}{2}z - \frac{p_c-1}{2}\bar{z}.$$

We write $\nabla R(v)$ as,

$$\begin{aligned}
i\nabla R(v) &= p_c W^{p_c-1}\nabla W J\left(\frac{v}{W}\right) + W^{p_c} J_z\left(\frac{v}{W}\right)\nabla\left(\frac{v}{W}\right) + W^{p_c} J_{\bar{z}}\left(\frac{v}{W}\right)\nabla\left(\frac{\bar{v}}{W}\right) \\
&= \frac{p_c+1}{2}\big[|v+W|^{p_c-1}\nabla(v+W) - W^{p_c-1}\nabla W - (p_c-1)W^{p_c-2}\nabla W v - W^{p_c-1}\nabla v\big] \\
&\quad + \frac{p_c-1}{2}\big[|v+W|^{p_c-3}(v+W)^2\nabla(\bar{v}+W) - W^{p_c-1}\nabla W \\
&\quad - (p_c-1)W^{p_c-2}\nabla W\bar{v} - W^{p_c-1}\nabla\bar{v}\big]. \qquad (4.18)
\end{aligned}$$

Note moreover for $|z| < 1$, $J(z)$ is real analytic in z. Thus for $|z| \leqslant \frac{3}{4}$, we have

$$\begin{aligned}
\big|J(z)\big| &\lesssim |z|^2, \\
\big|J_z(z)\big|, \big|J_{\bar{z}}(z)\big| &\lesssim |z|. \qquad (4.19)
\end{aligned}$$

To estimate $\|\nabla R(v)\|_{\dot{N}^0}$, we split $\mathbb{R}^d$ into regimes $\{x\colon |v(x)| \leqslant \frac{1}{4}W(x)\}$ and $\{x\colon |v(x)| > \frac{1}{4}W(x)\}$. In the first regime, we use the expression (4.18), the estimate (4.19) and Lemma 2.3 to get

$$\begin{aligned}
\big\|\nabla R(v)\big\|_{\dot{N}^0(I,|v|\leqslant\frac{1}{4}W)} &\lesssim \left\|W^{p_c-1}\nabla W\frac{v^2}{W^2}\right\|_{\dot{N}^0(I,|v|\leqslant\frac{1}{4}W)} + \left\|W^{p_c}\frac{v}{W}\nabla\left(\frac{v}{W}\right)\right\|_{\dot{N}^0(I;|v|\leqslant\frac{1}{4}W)} \\
&\lesssim \big\|W^{p_c-3}\nabla W v^2\big\|_{\dot{N}^0(I;|v|\leqslant\frac{1}{4}W)} + \big\|W^{p_c-2}v\nabla v\big\|_{\dot{N}^0(I;|v|\leqslant\frac{1}{4}W)} \\
&\leqslant \|v\|_{\dot{S}^1(I)}^{\frac{d+\frac{3}{2}}{d-2}} + \|v\|_{\dot{S}^1(I)}^{p_c}.
\end{aligned}$$

In the second regime, we use the second equality in (4.18). Using the triangle inequality, Lemma 2.3 and noting that the conjugate counterpart will give the same contribution we obtain

$$\begin{aligned}
\big\|\nabla R(v)\big\|_{\dot{N}^1(I;|v|>\frac{1}{4}W)} &\lesssim \big\|\big(|v+W|^{p_c-1} - W^{p_c-1}\big)\nabla v\big\|_{\dot{N}^0(I;|v|>\frac{1}{4}W)} \\
&\quad + \big\|\big(|v+W|^{p_c-1} - W^{p_c-1}\big)\nabla W\big\|_{\dot{N}^0(I;|v|>\frac{1}{4}W)} \\
&\quad + \big\|W^{p_c-2}\nabla W v\big\|_{\dot{N}^0(I;|v|>\frac{1}{4}W)} \\
&\lesssim \big\||v|^{p_c-1}\nabla v\big\|_{\dot{N}^0(I)} + \big\|\nabla W|v|^{p_c-1}\big\|_{\dot{N}^0(I;|v|>\frac{1}{4}W)} \\
&\quad + \big\|W^{p_c-2}\nabla W v\big\|_{\dot{N}^0(|v|>\frac{1}{4}W)} \\
&\lesssim \|v\|_{\dot{S}^1(I)}^{p_c} + \|v\|_{\dot{S}^1(I)}^{\frac{d+\frac{3}{2}}{d-2}}.
\end{aligned}$$

Combining the two estimates together, (4.15) is verified. Finally, we quickly show that

$$\left\| R(v) \right\|_{L_x^{\frac{2d}{d+2}}} \lesssim e^{-p_c \gamma_0 t}.$$

Indeed, note $R(v) = -i W^{p_c} J(\frac{v}{W})$ and $J(z) \lesssim |z|^{p_c}$, we estimate

$$\begin{aligned} \left\| R(v)(t) \right\|_{L_x^{\frac{2d}{d+2}}} &\lesssim \left\| |v|^{p_c} \right\|_{L_x^{\frac{2d}{d+2}}} \\ &\lesssim \left\| v(t) \right\|_{L_x^{\frac{2d}{d-2}}} \lesssim \left\| v(t) \right\|_{\dot H_x^1} \\ &\lesssim e^{-p_c \gamma_0 t}. \end{aligned}$$

Step 2. Iteration. Since v satisfies the equation (4.11), we can use Lemma 4.3 with $h = v$ and $\varepsilon = -R(v)$ to get

$$\left\| v(t) \right\|_{\dot H^1} \leqslant C\left(e^{-e_0 t} + e^{-\frac{d+1}{d-2}\gamma_0 t}\right).$$

If $\frac{d+1}{d-2}\gamma_0 \geqslant e_0$, then by repeating the same arguments as above, we have

$$\|v\|_{\dot S^1([t,\infty))} \lesssim e^{-e_0 t},$$

and the proposition is proved. Otherwise, we are at the same situation as the first step with γ_0 now replaced by $\frac{d+1}{d-2}\gamma_0$. Iterating this process finitely many times, we obtain the proposition. □

Based on this result, we now show that $u - W^a$ decays arbitrarily fast.

Proposition 4.5. *For any $m > 0$, there exists $t_m > 0$ such that*

$$\left\| u - W^a \right\|_{\dot S^1([t,\infty))} \leqslant e^{-mt}, \quad \forall t \geqslant t_m. \tag{4.20}$$

Proof. *Step 1*. We first remark that as a consequence of Proposition 4.4, we can prove

$$\left\| u - W^a \right\|_{\dot S^1([t,\infty))} \leqslant e^{-\frac{d}{d-2} e_0 t}, \quad \forall t \geqslant t_0.$$

Indeed by the triangle inequality and recalling that $v = u - W$, we estimate

$$\begin{aligned} \left\| u - W^a \right\|_{\dot S^1([t,\infty))} &\leqslant \left\| v - a e^{-e_0 t} \mathcal{Y}_+ \right\|_{\dot S^1([t,\infty))} \\ &\quad + \left\| w^a - v_{k_0} \right\|_{\dot S^1([t,\infty))} + \left\| v_{k_0} - a e^{-e_0 t} \mathcal{Y}_+ \right\|_{\dot S^1([t,\infty))}. \end{aligned}$$

For the first term, we use (4.9) to get

$$\left\| v - a e^{-e_0 t} \mathcal{Y}_+ \right\|_{\dot S^1([t,\infty))} \leqslant \frac{1}{2} e^{-\frac{d}{d-2} e_0 t}.$$

For the last two terms, we use the definition of v_k (see (3.1)) and Proposition 3.7 to obtain for any $2 \leqslant p \leqslant \infty$:

$$\left\|\nabla\left(v_{k_0}-ae^{-e_0t}\mathcal{Y}_+\right)\right\|_{L_x^p}\leqslant e^{-\frac{5}{4}e_0t},$$

$$\begin{aligned}\left\|\nabla\left(w^a-v_{k_0}\right)\right\|_{L_x^p}&\lesssim\left\|w^a-v_{k_0}\right\|_{H^{m,m}}\\&\lesssim\left\|W^a-W^a_{k_0}\right\|_{H^{m,m}}\\&\lesssim e^{-(k_0+\frac{1}{2})e_0t}.\end{aligned}$$

Integrating in the time variable over $[t,\infty)$, we have

$$\left\|w^a-v_{k_0}\right\|_{\dot S^1([t,\infty))}+\left\|v_{k_0}-ae^{-e_0t}\mathcal{Y}_+\right\|_{\dot S^1([t,\infty))}\leqslant\frac{1}{2}e^{-\frac{3}{2}e_0t}\leqslant\frac{1}{2}e^{-\frac{d}{d-2}e_0t}.$$

Hence

$$\left\|u-W^a\right\|_{\dot S^1([t,\infty))}\leqslant e^{-\frac{d}{d-2}e_0t}.$$

Step 2. We will prove (4.20) by induction. More precisely, suppose there exists $t_{m_1}>0$ such that

$$\left\|u-W^a\right\|_{\dot S^1([t,\infty))}\leqslant e^{-m_1t},\quad\forall t\geqslant t_{m_1},\tag{4.21}$$

we aim to prove for t large enough that

$$\left\|u-W^a\right\|_{\dot S^1([t,\infty))}\leqslant e^{-\frac{e_0}{2(d-2)}t}e^{-m_1t}.\tag{4.22}$$

From Step 1, we can assume that (4.21) holds with $m_1\geqslant\frac{d}{d-2}e_0$. Let $h=u-W^a$, then h solves the equation

$$\partial_th+\mathcal{L}h=-R\left(h+w^a\right)+R\left(w^a\right),\tag{4.23}$$

with

$$\|h\|_{\dot S^1([t,\infty))}\leqslant e^{-m_1t},\quad m_1>e_0.$$

An application of Lemma 4.3 gives immediately (4.22) if we establish the following

$$\left\|R\left(h+w^a\right)(t)-R\left(w^a\right)(t)\right\|_{L_x^{\frac{2d}{d+2}}}\leqslant e^{-\frac{e_0}{d-2}t-m_1t},\tag{4.24}$$

$$\begin{aligned}\left\|R\left(h+w^a\right)-R\left(w^a\right)\right\|_{\dot N^1([t,\infty))}&\lesssim e^{-\frac{7}{4(d-2)}e_0t}\|h\|_{\dot S^1([t,\infty))}\\&\leqslant e^{-\frac{e_0}{d-2}t-m_1t},\end{aligned}\tag{4.25}$$

for t large enough.

The remaining part of the proof is devoted to showing (4.25), (4.24). The idea is similar to the proof of (4.15). We split the space into two regimes. In the regime where h is large, we use the decay estimate of W to show that $R(h+w^a)-R(w^a)$ is superlinear in h. In the regime where h is small, we simply use the real analytic expansion of the complex function $P(z) = |1+z|^{p_c-1}(1+z)$. However, the argument here is more involved than the proof of (4.15).

We first show (4.24). To begin with, we recall the exact form of $R(h+w^a) - R(w^a)$. We have

$$\begin{aligned} &i\big(R\big(h+w^a\big) - R\big(w^a\big)\big) \\ &\quad = \big|w^a + h + W\big|^{p_c-1}\big(w^a + h + W\big) - \big|w^a + W\big|^{p_c-1}\big(w^a + W\big) \\ &\qquad - \frac{p_c+1}{2} W^{p_c-1} h - \frac{p_c-1}{2} W^{p_c-1} \bar{h}. \end{aligned} \tag{4.26}$$

By triangle inequality, we estimate

$$\begin{aligned} &\big\|R\big(h+w^a\big)(t) - R\big(w^a\big)(t)\big\|_{L_x^{\frac{2d}{d+2}}} \\ &\quad \leqslant \big\|R\big(h+w^a\big)(t) - R\big(w^a\big)(t)\big\|_{L_x^{\frac{2d}{d+2}}(|h|>\frac{1}{4}W)} &(4.27)\\ &\qquad + \big\|R\big(h+w^a\big)(t) - R\big(w^a\big)(t)\big\|_{L_x^{\frac{2d}{d+2}}(|h|\leqslant\frac{1}{4}W)}. &(4.28) \end{aligned}$$

For (4.27), we use the fact that $|w^a(t,x)| \leqslant \frac{1}{2}W(x)$ which follows from Corollary 3.8 to estimate

$$\begin{aligned} &\big\|R\big(h+w^a\big)(t) - R\big(w^a\big)(t)\big\|_{L_x^{\frac{2d}{d+2}}(|h|>\frac{1}{4}W)} \\ &\quad \lesssim \big\||h|^{p_c-1}\big(w^a + W\big)\big\|_{L_x^{\frac{2d}{d+2}}(|h|>\frac{1}{4}W)} + \big\|\big|w^a + W + h\big|^{p_c-1} h\big\|_{L_x^{\frac{2d}{d+2}}(|h|>\frac{1}{4}W)} \\ &\quad \lesssim \big\|\big|h(t)\big|^{p_c-1} h(t)\big\|_{L_x^{\frac{2d}{d-2}}} \\ &\quad \lesssim \big\|h(t)\big\|^{p_c}_{L_x^{\frac{2d}{d+2}}} \\ &\quad \lesssim e^{-m_1 p_c t}. \end{aligned} \tag{4.29}$$

For (4.28), we use $P(z) = |1+z|^{p_c-1}(1+z)$ to rewrite (4.26) into

$$\begin{aligned} &i\big(R\big(h+w^a\big) - R\big(w^a\big)\big) &(4.30)\\ &\quad = W^{p_c}\left(P\left(\frac{h+w^a}{W}\right) - P\left(\frac{w^a}{W}\right) - \frac{p_c+1}{2}\frac{h}{W} - \frac{p_c-1}{2}\frac{\bar{h}}{W}\right). &(4.31) \end{aligned}$$

Note that

$$\frac{|w^a + h|}{W} \leqslant \frac{3}{4}, \qquad \frac{|w^a|}{W} \leqslant \frac{1}{2}. \tag{4.32}$$

We use the expansion for $P(z)$ (see (3.21)) to write

$$i\big(R\big(h+w^a\big)-R\big(w^a\big)\big)=\sum_{j_1+j_2\geqslant 2}a_{j_1,j_2}\bigg[\bigg(\frac{h+w^a}{W}\bigg)^{j_1}\bigg(\frac{\bar h+\bar w^a}{W}\bigg)^{j_2}-\bigg(\frac{w^a}{W}\bigg)^{j_1}\bigg(\frac{\bar w^a}{W}\bigg)^{j_2}\bigg]$$
$$=O\bigg(\sum_{j\geqslant 2,\,1\leqslant i\leqslant j}a_jC_{i,j}W^{p_c-1-j}\nabla W\big(w^a\big)^{j-i}h^i\bigg),$$

where $|a_j|\lesssim 1$ and $C_{i,j}\lesssim 2^j$. Therefore by triangle inequality we have

$$\big\|R\big(h+w^a\big)(t)-R\big(w^a\big)(t)\big\|_{L_x^{\frac{2d}{d+2}}(|h|\leqslant\frac14 W)} \tag{4.33}$$
$$\lesssim\sum_{j\geqslant 2,\,1\leqslant i\leqslant j}2^j\big\|W^{p_c-j}\big(w^a(t)\big)^{j-i}h(t)^i\big\|_{L_x^{\frac{2d}{d+2}}(|h(t)|\leqslant\frac14 W)}$$
$$\lesssim\sum_{j\geqslant 2}2^j\big\|h(t)\big\|_{L_x^{\frac{2d}{d-2}}}\big\|W^{p_c-j}\big(w^a(t)\big)^{j-1}\big\|_{L_x^{\frac d2}} \tag{4.34}$$
$$+\sum_{j\geqslant 2,\,2\leqslant i\leqslant j}2^j\big\|h(t)\big\|^{p_c}_{L_x^{\frac{2d}{d-2}}}\big\|W^{p_c-j}\big(w^a(t)\big)^{j-i}h(t)^{i-p_c}\big\|_{L_x^\infty(|h|\leqslant\frac14 W)}$$
$$\lesssim\sum_{j\geqslant 2}\big\|h(t)\big\|_{L_x^{\frac{2d}{d-2}}}\big\|W^{-1}w^a(t)\big\|^{j-1}_{L_x^{\frac d2(j-1)}}$$
$$+\sum_{j\geqslant 2,\,2\leqslant i\leqslant j}2^j\big\|h(t)\big\|^{p_c}_{L_x^{\frac{2d}{d-2}}}\big\|hW^{-1}\big\|^{i-p_c}_{L_x^\infty}\big\|w^aW^{-1}\big\|^{j-i}_{L_x^\infty}$$
$$\lesssim\big\|h(t)\big\|_{\dot H_x^1}\sum_{j\geqslant 2}2^je^{-e_0(j-1)t}+\big\|h(t)\big\|^{p_c}_{\dot H_x^1}\sum_{j\geqslant 2,\,2\leqslant i\leqslant j}2^j\bigg(\frac14\bigg)^{i-p_c}e^{-e_0(j-i)t}$$
$$\lesssim e^{-(e_0+m_1)t}.$$

Collecting estimates (4.29) and (4.33) we obtain (4.24).

Next we prove (4.25). To this end, we take the gradient and regroup the term, we have

$$i\nabla\big(R\big(h+w^a\big)-R\big(w^a\big)\big)$$
$$=\frac{p_c+1}{2}\Big[\big(\big|w^a+h+W\big|^{p_c-1}-W^{p_c-1}\big)\nabla h$$
$$+\big(\big|w^a+h+W\big|^{p_c-1}-\big|w^a+W\big|^{p_c-1}\big)\nabla\big(w^a+W\big)$$
$$+(p_c-1)W^{p_c-2}\nabla Wh\Big]$$
$$+\frac{p_c-1}{2}\Big[\big(\big|w^a+h+W\big|^{p_c-3}\big(w^a+h+W\big)^2-W^{p_c-1}\big)\nabla\bar h$$
$$+\big(\big|w^a+h+W\big|^{p_c-3}\big(w^a+h+W\big)^2$$
$$-\big|w^a+W\big|^{p_c-3}\big(w^a+W\big)^2\big)\nabla\big(\bar w^a+W\big)$$
$$+(p_c-1)W^{p_c-2}\nabla W\bar h\Big].$$

By Lemma 2.3, Corollary 3.8 and the triangle inequality we have

$$
\begin{aligned}
&\big\|\nabla\big(R\big(h+w^a\big)-R\big(w^a\big)\big)\big\|_{\dot N^0([t,\infty);|h|>\frac14 W)} \\
&\lesssim \big\|\big|h+w^a\big|^{p_c-1}\nabla h\big\|_{\dot N^0([t,\infty);|h|>\frac14 W)} \\
&\quad+\big\||h|^{p_c-1}\nabla\big(w^a+W\big)\big\|_{\dot N^0([t,\infty);|h|>\frac14 W)}+\big\|W^{p_c-2}\nabla W h\big\|_{\dot N^0([t,\infty);|h|>\frac14 W)} \\
&\lesssim \big\|w^a\big\|^{p_c-1}_{\dot S^1([t,\infty))}\|h\|_{\dot S^1([t,\infty))}+\|h\|^{p_c}_{\dot S^1([t,\infty))}+\|h\|^{\frac{d+\frac32}{d-2}}_{\dot S^1([t,\infty))} \\
&\lesssim e^{-\frac{e_0}{2}(p_c-1)t}\|h\|_{\dot S^1([t,\infty))}.
\end{aligned}
\tag{4.35}
$$

To get the estimate in the regime where $|h|$ is small, we adopt the form (4.30). By chain rule we have

$$
\begin{aligned}
&i\nabla\big(R\big(h+w^a\big)-R\big(w^a\big)\big) \\
&\quad= p_c W^{p_c-1}\nabla W\bigg[P\bigg(\frac{h+w^a}{W}\bigg)-P\bigg(\frac{w^a}{W}\bigg)-\frac{p_c+1}{2}\frac{h}{W}-\frac{p_c-1}{2}\frac{\bar h}{W}\bigg] \qquad (4.36)\\
&\qquad+W^{p_c}\nabla\bigg[P\bigg(\frac{h+w^a}{W}\bigg)-P\bigg(\frac{w^a}{W}\bigg)-\frac{p_c+1}{2}\frac{h}{W}-\frac{p_c-1}{2}\frac{\bar h}{W}\bigg]. \qquad (4.37)
\end{aligned}
$$

In view of (4.32),we can use the expansion for $P(z)$ (see (3.21)) to write (4.36) as

$$
\begin{aligned}
(4.36) &= p_c W^{p_c-1}\nabla W\sum_{j_1+j_2\geqslant 2} a_{j_1,j_2}\bigg[\bigg(\frac{h+w^a}{W}\bigg)^{j_1}\bigg(\frac{\bar h+\bar w^a}{W}\bigg)^{j_2}-\bigg(\frac{w^a}{W}\bigg)^{j_1}\bigg(\frac{\bar w^a}{W}\bigg)^{j_2}\bigg] \\
&= O\bigg(\sum_{j\geqslant 2,\,1\leqslant i\leqslant j} a_j C_{i,j} W^{p_c-1-j}\nabla W\big(w^a\big)^{j-i}h^i\bigg),
\end{aligned}
\tag{4.38}
$$

where the constants a_j, $C_{i,j}$ are the same as those in (3.22). Now we deal with the second term (4.37). Applying the chain rule and regrouping the terms, we eventually get

$$
\begin{aligned}
(4.37) &= \frac{p_c+1}{2}W^{p_c-1}\bigg(\bigg|1+\frac{h+w^a}{W}\bigg|^{p_c-1}-1\bigg)\nabla h \qquad (4.39)\\
&\quad-\frac{p_c+1}{2}W^{p_c-2}\nabla W\bigg(\bigg|1+\frac{h+w^a}{W}\bigg|^{p_c-1}-1\bigg)h \qquad (4.40)\\
&\quad+\frac{p_c+1}{2}W^{p_c-1}\bigg(\bigg|1+\frac{h+w^a}{W}\bigg|^{p_c-1}-\bigg|1+\frac{w^a}{W}\bigg|^{p_c-1}\bigg)\nabla w^a \qquad (4.41)\\
&\quad-\frac{p_c+1}{2}W^{p_c-2}\nabla W\bigg(\bigg|1+\frac{h+w^a}{W}\bigg|^{p_c-1}-\bigg|1+\frac{w^a}{W}\bigg|^{p_c-1}\bigg)w^a \qquad (4.42)\\
&\quad+\frac{p_c-1}{2}W^{p_c-1}\bigg[\bigg(\bigg|1+\frac{h+w^a}{W}\bigg|^{p_c-3}\bigg(1+\frac{h+w^a}{W}\bigg)^2-1\bigg)\nabla\bar h \qquad (4.43)
\end{aligned}
$$

$$+\left(\left|1+\frac{h+w^a}{W}\right|^{p_c-3}\left(1+\frac{h+w^a}{W}\right)^2-\left|1+\frac{w^a}{W}\right|^{p_c-3}\left(1+\frac{w^a}{W}\right)^2\right)\nabla\bar{w^a}\Bigg] \tag{4.44}$$

$$-\frac{p_c-1}{2}W^{p_c-2}\nabla W\Bigg[\left(\left|1+\frac{h+w^a}{W}\right|^{p_c-3}\left(1+\frac{h+w^a}{W}\right)^2-1\right)\bar{h} \tag{4.45}$$

$$+\left(\left|1+\frac{h+w^a}{W}\right|^{p_c-3}\left(1+\frac{h+w^a}{W}\right)^2-\left|1+\frac{w^a}{W}\right|^{p_c-3}\left(1+\frac{w^a}{W}\right)^2\right)\bar{w^a}\Bigg]. \tag{4.46}$$

For (4.39) and (4.40) we use the fact

$$|1+z|^{p_c-1}-1\leqslant |z|^{p_c-1}$$

to bound them as:

$$\left|(4.39)\right|+\left|(4.40)\right|\lesssim \left|h+w^a\right|^{p_c-1}|\nabla h|+W^{-1}|\nabla W|\cdot\left|h+w^a\right|^{p_c-1}\cdot|h|.$$

For (4.41) we use the expansion

$$|1+z|^{p_c-1}=1+\frac{p_c-1}{2}z+\frac{p_c-1}{2}\bar{z}+\sum_{j_1+j_2\geqslant 2}b_{j_1,j_2}z^{j_1}\bar{z}^{j_2}$$

to write

$$\begin{aligned}(4.41)=&\frac{p_c^2-1}{4}W^{p_c-2}\left(h\nabla w^a+\bar{h}\nabla w^a\right)\\&+\frac{p_c+1}{2}W^{p_c-1}\nabla w^a\sum_{j_1+j_2\geqslant 2}b_{j_1,j_2}\\&\times\left[\left(\frac{h+w^a}{W}\right)^{j_1}\left(\frac{\bar{h}+\bar{w^a}}{W}\right)^{j_2}-\left(\frac{w^a}{W}\right)^{j_1}\left(\frac{\bar{w^a}}{W}\right)^{j_2}\right]\\=&O\left(W^{p_c-2}h\nabla w^a\right)+O\left(\sum_{j\geqslant 2,\,1\leqslant i\leqslant j}b_jC_{i,j}W^{p_c-1-j}\nabla w^a\mathcal{O}\left(\left(w^a\right)^{j-i}h^i\right)\right),\end{aligned}$$

where in the last equality we use the same conventions as in (3.22). In particular the constants $|b_j|\lesssim 1$ and $C_{i,j}\lesssim 2^j$. We therefore have the bound

$$\left|(4.41)\right|\lesssim W^{p_c-2}\left|\nabla w^a h\right|+\sum_{j\geqslant 2,\,1\leqslant i\leqslant j}2^j\left|W^{p_c-1-j}\nabla w^a\left(w^a\right)^{j-i}h^i\right|.$$

Similarly for (4.42) we have

$$\left|(4.42)\right|\lesssim W^{p_c-3}\nabla W\left|\nabla w^a h\right|+\sum_{j\geqslant 2,\,1\leqslant i\leqslant j}2^j\left|W^{p_c-2-j}\nabla W\left(w^a\right)^{j+1-i}h^i\right|.$$

Collecting all the estimates and noticing that (4.43) through (4.46) are just complex conjugates of (4.39) through (4.42), we therefore can bound (4.37) as follows:

$$
\begin{aligned}
(4.37) \lesssim {} & \left|h + w^a\right|^{p_c-1}|\nabla h| && (4.47)\\
& + W^{-1}|\nabla W| \cdot \left|h + w^a\right|^{p_c-1} \cdot |h| && (4.48)\\
& + W^{p_c-2}\left|\nabla w^a h\right| && (4.49)\\
& + W^{p_c-3}|\nabla W|\left|w^a h\right| && (4.50)\\
& + \sum_{j \geqslant 2,\, 1 \leqslant i \leqslant j} 2^j \left|W^{p_c-1-j}\nabla w^a \left(w^a\right)^{j-i} h^i\right| && (4.51)\\
& + \sum_{j \geqslant 2,\, 1 \leqslant i \leqslant j} 2^j \left|W^{p_c-2-j}\nabla W \left(w^a\right)^{j+1-i} h^i\right|. && (4.52)
\end{aligned}
$$

Now our task is reduced to bounding the $\dot{N}^0$ norm of (4.38) and (4.47) through (4.52). We start from (4.47), using Lemma 2.3 and Corollary 3.8 we have

$$
\begin{aligned}
\left\|(4.47)\right\|_{\dot{N}^0([t,\infty);|h| \leqslant \frac{1}{4}W)} &\lesssim \|h\|^{p_c}_{\dot{S}^1([t,\infty))} + \left\|w^a\right\|^{p_c-1}_{\dot{S}^1([t,\infty))}\|h\|_{\dot{S}^1([t,\infty))}\\
&\lesssim e^{-\frac{e_0}{2}(p_c-1)t}\|h\|_{\dot{S}^1([t,\infty))}. \qquad (4.53)
\end{aligned}
$$

Similarly we have

$$
\begin{aligned}
\left\|(4.48)\right\|_{\dot{N}^0([t,\infty);|h| \leqslant \frac{1}{4}W)} &\lesssim \|h\|^{\frac{d+\frac{3}{2}}{d-2}}_{\dot{S}^1([t,\infty))} + \left\|w^a\right\|^{\frac{7}{2(d-2)}}_{\dot{S}^1([t,\infty))}\|h\|_{\dot{S}^1([t,\infty))}\\
&\lesssim e^{-\frac{7}{4(d-2)}e_0 t}\|h\|_{\dot{S}^1([t,\infty))}. \qquad (4.54)
\end{aligned}
$$

For (4.49), (4.50), we use Hölder's inequality and Corollary 3.8 to get

$$
\begin{aligned}
\left\|(4.49)\right\|_{\dot{N}^0([t,\infty);|h| \leqslant \frac{1}{4}W)} &\leqslant \left\|W^{p_c-2}\nabla w^a h\right\|_{L_s^2 L_x^{\frac{2d}{d+2}}([t,\infty))}\\
&\leqslant \left\|W^{p_c-2}\nabla w^a\right\|_{L_s^\infty L_x^{\frac{d}{3}}([t,\infty))}\|h\|_{L_s^2 L_x^{\frac{2d}{d-4}}([t,\infty))}\\
&\leqslant e^{-\frac{e_0}{2}t}\|h\|_{\dot{S}^1([t,\infty))}. \qquad (4.55)
\end{aligned}
$$

$$
\left\|(4.50)\right\|_{\dot{N}^0([t,\infty);|h| \leqslant \frac{1}{4}W)} \leqslant \|h\|_{L_s^2 L_x^{\frac{2d}{d-4}}([t,\infty))}\left\|w^a\right\|_{L_x^{\frac{d}{3}}} \leqslant e^{-\frac{e_0}{2}t}\|h\|_{\dot{S}^1([t,\infty))}. \qquad (4.56)
$$

Now we are left with the estimates of the summation terms (4.38), (4.51) and (4.52). We first treat (4.38). We have

$$
\begin{aligned}
&\left\|(4.38)\right\|_{\dot{N}^0([t,\infty);|h| \leqslant \frac{1}{4}W)}\\
&\quad \leqslant \sum_{j \geqslant 2,\, 1 \leqslant i \leqslant j} 2^j \left\|W^{\frac{d+3}{d-2}-j}\left(w^a\right)^{j-i}h^{i-1}h\right\|_{L_s^2 L_x^{\frac{2d}{d+2}}([t,\infty);|h| \leqslant \frac{1}{4}W)} \qquad (4.57)
\end{aligned}
$$

$$\leqslant \sum_{j\geqslant 2} 2^j \left\| W^{\frac{d+3}{d-2}-j} h^j \right\|_{L_s^2 L_x^{\frac{2d}{d+2}}([t,\infty);|h|\leqslant \frac14 W)} \tag{4.58}$$

$$+ \sum_{j\geqslant 2,\,1\leqslant i\leqslant j-1} 2^j \left\| W^{\frac{d+3}{d-2}-j} \big(w^a\big)^{j-i} h^{i-1} h \right\|_{L_s^2 L_x^{\frac{2d}{d+2}}([t,\infty);|h|\leqslant \frac14 W)}. \tag{4.59}$$

For (4.58) we have by Lemma 2.3,

$$\begin{aligned}
\big|(4.58)\big| &\lesssim \sum_{j\geqslant 2} 2^j \left\| W^{\frac{d+3}{d-2}-2} h^2 \right\|_{L_s^2 L_x^{\frac{d}{d+2}}([t,\infty);|h|\leqslant \frac14 W)} \left\| W^{2-j} h^{j-2} \right\|_{L^\infty_{s,x}([t,\infty);|h|\leqslant \frac14 W)} \\
&\lesssim \sum_{j\geqslant 2} 2^j \|h\|_{\dot S^1([t,\infty)}^{\frac{d+\frac32}{d-2}} \cdot \left(\frac14\right)^{j-2} \\
&\lesssim \|h\|_{\dot S^1([t,\infty)}^{\frac{d+\frac32}{d-2}} \\
&\lesssim e^{-\frac{7m_1}{2(d-2)}t} \|h\|_{\dot S^1([t,\infty)}.
\end{aligned}$$

For (4.59) we estimate

$$\begin{aligned}
\big|(4.59)\big| \lesssim & \sum_{j\geqslant 2,\,1\leqslant i\leqslant j-1} 2^j \|h\|_{L_s^2 L_x^{\frac{2d}{d+4}}([t,\infty))} \left\| W^{-1} h \right\|^{i-1}_{L^\infty_{s,x}([t,\infty))} \\
& \times \left\| W^{\frac{d+3}{d-2}-1+i-j} \big(w^a\big)^{j-i} \right\|_{L_s^\infty L_x^{\frac d3}([t,\infty))} \\
\lesssim & \sum_{j\geqslant 2,\,1\leqslant i\leqslant j-1} 2^j \|h\|_{\dot S^1([t,\infty))} \left(\frac14\right)^{i-1} e^{-\frac12 (j-i)e_0 t} \\
\lesssim & \|h\|_{\dot S^1([t,\infty))} e^{-\frac{e_0}{2}t} \sum_{j\geqslant 2,\,1\leqslant i\leqslant j-1} 2^j \left(\frac14\right)^{i-1} e^{-\frac{e_0}{2}(j-i-1)t} \\
\lesssim & \|h\|_{\dot S^1([t,\infty))} e^{-\frac{e_0}{2}t} \sum_{j\geqslant 2,\,1\leqslant i\leqslant j-1} 2^{-j} 4^{j-i-1} e^{-\frac{e_0}{2}(j-i-1)t} \\
\lesssim & e^{-\frac{e_0}{2}t} \|h\|_{\dot S^1([t,\infty))}.
\end{aligned}$$

This ends the estimate of (4.38). Using the fact that $|\nabla w^a| \leqslant |\nabla W|$ and $|w^a| \leqslant W$, (4.50) and (4.52) can be bounded by (4.38), thus has the same estimate

$$(4.50) + (4.52) \lesssim e^{-\frac{e_0}{2}t} \|h\|_{\dot S^1([t,\infty))}. \tag{4.60}$$

Collecting the estimates (4.35), (4.53) through (4.60), we have

$$\left\| R\big(h + w^a\big) - R\big(w^a\big) \right\|_{\dot N^1([t,\infty))} \lesssim e^{-\frac{7}{4(d-2)}e_0 t} \|h\|_{\dot S^1([t,\infty))} \leqslant e^{(-m_1 - \frac{e_0}{d-2})t}.$$

(4.25) is proved and we conclude the proof of the proposition. □

As the last step of the argument, we show that any solution h of Eq. (4.23) which has enough exponential decay must be identically 0. This would imply $u = W^a$ and we can conclude the proof of Theorem 4.1. To this end, we have

Proposition 4.6. *Let h be the solution of Eq.* (4.23) *satisfying the following*: $\forall m > 0$, *there exists* $t_m > 0$ *such that*

$$\|h\|_{\dot{S}^1([t,\infty))} \leqslant e^{-mt}, \quad \forall t > t_m. \tag{4.61}$$

Then $h \equiv 0$.

Proof. Note first that in an equivalent form, h satisfies

$$i\partial_t h + \Delta h = -\Gamma(h) + i\bigl(-R(v + w^a) + R\bigl(w^a\bigr)\bigr), \tag{4.62}$$

hence the following Duhamel's formula holds

$$h(t) = i\int_t^\infty e^{i(t-s)\Delta}\bigl(-\Gamma(h) - iR\bigl(h + w^a\bigr) + iR\bigl(w^a\bigr)\bigr)(s)\,ds,$$

since $\|h(t)\|_{\dot{H}^1} \to 0$ as $t \to \infty$. Using Strichartz estimate we then have

$$\|h\|_{\dot{S}^1([t,\infty))} \leqslant \bigl\|\Gamma(h)\bigr\|_{\dot{N}^1([t,\infty))} + \bigl\|R\bigl(h + w^a\bigr) - R\bigl(w^a\bigr)\bigr\|_{\dot{N}^1([t,\infty))}.$$

Denote $\|h\|_{\Sigma_t} := \sup_{s\geqslant t} e^{ms}\|h\|_{\dot{S}^1([s,\infty))}$, and we have for $\eta > 0$ small enough

$$\begin{aligned}
\bigl\|\Gamma(h)\bigr\|_{\dot{N}^1([t,\infty))} &\leqslant \sum_{j\geqslant 0}\bigl\|\Gamma(h)\bigr\|_{\dot{N}^1([t+\eta j, t+\eta(j+1)])} \\
&\leqslant \sum_{j\geqslant 0}\eta\|h\|_{\dot{S}^1([t+\eta j, t+(j+1)\eta])} \\
&\leqslant \sum_{j\geqslant 0}\eta e^{-m(t+\eta j)}\|h\|_{\Sigma_{t_m}} \\
&\leqslant e^{-mt}\|h\|_{\Sigma_{t_m}}\frac{\eta}{1 - e^{-\eta m}} \\
&\leqslant \frac{2}{m}e^{-mt}\|h\|_{\Sigma_{t_m}}.
\end{aligned}$$

From the estimate (4.25), we get

$$\bigl\|R\bigl(w^a + h\bigr) - R\bigl(w^a\bigr)\bigr\|_{\dot{N}^1([t,\infty))} \leqslant \frac{1}{10}e^{-mt}\|h\|_{\Sigma_{t_m}}.$$

Combining these two estimates, we get for m large enough that

$$\|h\|_{\Sigma_{t_m}} \leqslant \frac{1}{2}\|h\|_{\Sigma_{t_m}},$$

which implies that $h = 0$ on $[t_m, \infty)$. Recall that $h = u - W^a$ we obtain $u = W^a$ on $[t_m, \infty)$. Therefore $u \equiv W^a$ by uniqueness of solutions to (1.1). The proposition is proved and we have Theorem 4.1. □

Proof of Corollary 4.2. The proof is almost the same as Corollary 6.6 in [6]. Let $a \neq 0$ and T_a be such that $|a|e^{-e_0 T_a} = 1$. By (3.11) we have

$$\left\| W^a(t + T_a) - W \mp e^{-e_0 t}\mathcal{Y}_+ \right\|_{H^{m,m}} \lesssim e^{-\frac{3}{2}e_0 t}. \tag{4.63}$$

Moreover $W^a(\cdot + T_a)$ satisfies the assumption in Theorem 4.1, thus there exists a' such that $W^a(\cdot + T_a) = W^{a'}$. By (4.63), $a' = 1$ if $a > 0$ and $a' = -1$ if $a < 0$. Corollary 4.2 is proved. □

Finally, we give the proof of the main Theorem 1.4.

Proof of Theorem 1.4. We first note that (2) is just the variational characterization of W. More precisely we have

Theorem 4.7. *(See [1,20].) Let $c(d)$ denote the sharp constant in Sobolev-embedding*

$$\|f\|_{\frac{2d}{d-2}} \leqslant c(d)\|\nabla f\|_2.$$

Then the equality holds iff f is W up to symmetries. More precisely, there exists $(\theta_0, \lambda_0, x_0) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^d$ such that

$$f(x) = e^{i\theta_0}\lambda_0^{-\frac{d-2}{2}} W\left(\frac{x - x_0}{\lambda_0}\right).$$

In particular, if u_0 satisfies

$$E(u_0) = E(W), \qquad \|\nabla u_0\|_2 = \|\nabla W\|_2,$$

then u_0 coincides with W up to symmetries, hence the corresponding solution u coincides with W up to symmetries.

It remains for us to show (1), (3). We first prove (1). Let u be the maximal-lifespan solution of (1.1) on I satisfying $E(u) = E(W)$, $\|\nabla u_0\|_2 < \|\nabla W\|_2$. Then by the Proposition 1.5, we have $I = \mathbb{R}$. Assume that u blows up forward in time. Applying Proposition 1.5 again, we conclude that there exist $\theta_0, \mu_0, \gamma_0$ such that

$$\left\| u(t) - W_{[\theta_0, \mu_0]} \right\|_{\dot{H}^1} \leqslant e^{-\gamma_0 t}.$$

This implies

$$\left\| u_{[-\theta_0, \mu_0^{-1}]}(t) - W \right\|_{\dot{H}^1} \leqslant e^{-\gamma_0 \mu_0^2 t}$$

where

$$u_{[-\theta_0, \mu_0^{-1}]}(t, x) = e^{-i\theta_0}\mu_0^{\frac{d-2}{2}} u\left(\mu_0^2 t, \mu_0 x\right)$$

is also a solution of Eq. (1.1). By Theorem 4.1 with γ_0 now replaced by $\gamma_0\mu_0^2$, we conclude there exists $a < 0$ such that $u_{[-\theta_0, \mu_0^{-1}]} = W^a$.

Using Corollary 4.2, we get

$$u(t,x) = e^{i\theta_0}\mu_0^{-\frac{d-2}{2}} W^-\big(\mu_0^{-2}t + T_a, \mu_0^{-1}x\big).$$

This shows that $u = W^-$ up to symmetries. The proof of (3) is similar so we omit it. This ends the proof of Theorem 1.4. □

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