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Asymptotic enumeration of dense 0–1 matrices with specified line sums

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Abstract

Let $\boldsymbol{s} = (s_1, s_2, \ldots, s_m)$ and $\boldsymbol{t} = (t_1, t_2, \ldots, t_n)$ be vectors of non-negative integers with $\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} t_j$. Let $B(\boldsymbol{s}, \boldsymbol{t})$ be the number of $m \times n$ matrices over $\{0, 1\}$ with jth row sum equal to s_j for $1 \leqslant j \leqslant m$ and kth column sum equal to t_k for $1 \leqslant k \leqslant n$. Equivalently, $B(\boldsymbol{s}, \boldsymbol{t})$ is the number of bipartite graphs with m vertices in one part with degrees given by $\boldsymbol{s}$, and n vertices in the other part with degrees given by $\boldsymbol{t}$. Most research on the asymptotics of $B(\boldsymbol{s}, \boldsymbol{t})$ has focused on the sparse case, where the best result is that of Greenhill, McKay and Wang (2006). In the case of dense matrices, the only precise result is for the case of equal row sums and equal column sums (Canfield and McKay, 2005). This paper extends the analytic methods used by the latter paper to the case where the row and column sums can vary within certain limits. Interestingly, the result can be expressed by the same formula which holds in the sparse case.

Keywords: Asymptotic enumeration; Binary matrix; Bipartite graph; 0–1 matrix

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1. Introduction

Let $\boldsymbol{s} = (s_1, s_2, \ldots, s_m)$ and $\boldsymbol{t} = (t_1, t_2, \ldots, t_n)$ be vectors of positive integers with $\sum_{i=1}^m s_i = \sum_{j=1}^n t_j$. Let $B(\boldsymbol{s}, \boldsymbol{t})$ be the number of $m \times n$ matrices over $\{0, 1\}$ with jth row sum equal to s_j for $1 \leqslant j \leqslant m$ and kth column sum equal to t_k for $1 \leqslant k \leqslant n$. Equivalently, $B(\boldsymbol{s}, \boldsymbol{t})$ is the number of labeled bipartite graphs with m vertices in one part of the bipartition with degrees given by $\boldsymbol{s}$, and n vertices in the other part of the bipartition with degrees given by $\boldsymbol{t}$. Let s be the average value of $s_1, s_2, \ldots, s_m$ and let t be the average value of $t_1, t_2, \ldots, t_n$. Define the density $\lambda = s/n = t/m$, which is the fraction of entries in the matrix which equal 1.

The asymptotic value of $B(\boldsymbol{s}, \boldsymbol{t})$ has been much studied, especially since the celebrated Gale–Ryser Theorem [10] that characterizes $(\boldsymbol{s}, \boldsymbol{t})$ such that $B(\boldsymbol{s}, \boldsymbol{t}) > 0$. Various authors have considered the *semiregular* case, where $s_j = s$ for $1 \leqslant j \leqslant m$ and $t_k = t$ for $1 \leqslant k \leqslant n$. Write $B(m, s; n, t)$ for $B(\boldsymbol{s}, \boldsymbol{t})$ in this case. For the sparse (low-λ) semiregular case, the best result is by McKay and Wang [7] who gave an asymptotic expression for $B(m, s; n, t)$ which holds when $st = o((mn)^{1/2})$. Canfield and McKay [1] used analytic methods to obtain an asymptotic expression for $B(m, s; n, t)$ in two ranges: in the first, the matrix is relatively square and the density is not too close to 0 or 1, while in the second, the matrix is much wider than high (or vice versa) but the density is arbitrary. For the sparse irregular case, the best result is that of Greenhill, McKay and Wang [2], who gave an asymptotic expression for $B(\boldsymbol{s}, \boldsymbol{t})$ which holds when $\max\{s_j\}\max\{t_k\} = o((\lambda mn)^{2/3})$.

See [1,2] and [7] for a more extensive historical survey.

The contribution of this paper is to adapt the approach of [1] to the dense irregular case when the matrix is relatively square and the density is not too close to 0 or 1. See McKay and Wormald [8] for the corresponding calculation for symmetric matrices.

In keeping with these earlier papers, the asymptotic value of $B(\boldsymbol{s}, \boldsymbol{t})$ can be expressed by a very nice formula involving binomial coefficients. We now state our theorem.

Theorem 1. *Let $\boldsymbol{s} = \boldsymbol{s}(m, n) = (s_1, s_2, \ldots, s_m)$ and $\boldsymbol{t} = \boldsymbol{t}(m, n) = (t_1, t_2, \ldots, t_n)$ be vectors of positive integers such that $\sum_{j=1}^m s_j = \sum_{k=1}^n t_k$ for all m, n. Define $s = m^{-1}\sum_{j=1}^m s_j$, $t = n^{-1}\sum_{k=1}^n t_k$, $\lambda = s/n = t/m$ and $A = \frac{1}{2}\lambda(1-\lambda)$. For some $\varepsilon > 0$, suppose that $|s_j - s| = O(n^{1/2+\varepsilon})$ uniformly for $1 \leqslant j \leqslant m$, and $|t_k - t| = O(m^{1/2+\varepsilon})$ uniformly for $1 \leqslant k \leqslant n$. Define $R = \sum_{j=1}^m (s_j - s)^2$ and $C = \sum_{k=1}^n (t_k - t)^2$. Let $a, b > 0$ be constants such that $a + b < \frac{1}{2}$. Suppose that $m, n \to \infty$ with $n = o(A^2 m^{1+\varepsilon})$, $m = o(A^2 n^{1+\varepsilon})$ and*

$$\frac{(1-2\lambda)^2}{8A}\left(1 + \frac{5m}{6n} + \frac{5n}{6m}\right) \leqslant a \log n.$$

Then, provided $\varepsilon > 0$ is small enough, we have

$$B(\boldsymbol{s}, \boldsymbol{t}) = \binom{mn}{\lambda mn}^{-1} \prod_{j=1}^m \binom{n}{s_j} \prod_{k=1}^n \binom{m}{t_k} \times \exp\left(-\frac{1}{2}\left(1 - \frac{R}{2Amn}\right)\left(1 - \frac{C}{2Amn}\right) + O\left(n^{-b}\right)\right).$$

Proof. The proof of this theorem is the topic of the paper; here we will summarize the main phases and draw their conclusions together. The basic idea is to identify $B(\boldsymbol{s}, \boldsymbol{t})$ as a coefficient in a multivariable generating function and to extract that coefficient using the saddle-point method.

In Section 2, Eq. (1), we write $B(\boldsymbol{s},\boldsymbol{t}) = P(\boldsymbol{s},\boldsymbol{t})I(\boldsymbol{s},\boldsymbol{t})$, where $P(\boldsymbol{s},\boldsymbol{t})$ is a rational expression and $I(\boldsymbol{s},\boldsymbol{t})$ is an integral in $m+n$ complex dimensions. Both depend on the location of the saddle point, which is the solution of some non-linear equations. Those equations are solved in Section 3, and this leads to the value of $P(\boldsymbol{s},\boldsymbol{t})$ in (20). In Section 4, the integral $I(\boldsymbol{s},\boldsymbol{t})$ is estimated in a small region $\mathcal{R}'$ defined in (33). The result is given by Theorem 2 together with (24). Finally, in Section 5, it is shown that the integral $I(\boldsymbol{s},\boldsymbol{t})$ restricted to the exterior of $\mathcal{R}'$ is negligible. The present theorem thus follows from (1), (20), Theorems 2–3 and (24). □

Note that the error term in the above slightly improves the error term for the semiregular case proved in [1].

Theorem 1 has an instructive interpretation. Write it as $B(\boldsymbol{s},\boldsymbol{t}) = NP_1P_2E$, where

$$N = \binom{mn}{\lambda mn}, \qquad P_1 = N^{-1}\prod_{j=1}^{m}\binom{n}{s_j}, \qquad P_2 = N^{-1}\prod_{k=1}^{n}\binom{m}{t_k},$$

$$E = \exp\left(-\frac{1}{2}\left(1-\frac{R}{2Amn}\right)\left(1-\frac{C}{2Amn}\right)+O\left(n^{-b}\right)\right).$$

Clearly, N is the number of $m\times n$ binary matrices with λmn ones. P_1 is the probability that a matrix randomly chosen from this class has row sums $\boldsymbol{s}$, while P_2 is the probability of the similar event of having column sums $\boldsymbol{t}$. If these two events were independent, we would have $E=1$, so E can be taken as a measure of their non-independence. For the case when $\boldsymbol{s}$ and $\boldsymbol{t}$ are vectors of constants, that is, $R=C=0$, Ordentlich and Roth [9] proved that $E\leqslant 1$.

It is proved in [2] that the same formula for $B(\boldsymbol{s},\boldsymbol{t})$ modulo the error term also holds in the sparse case. Specifically, it holds with a different vanishing error term whenever $\max\{s_j\}\max\{t_k\} = o((\lambda mn)^{2/3})$, $R+C = O((\lambda mn)^{4/3})$ and $RC = O((\lambda mn)^{7/3})$. In [1], evidence is presented that the formula is universal in the semiregular case ($R=C=0$) and it is tempting to conjecture that the same is true in the irregular case for a wide range of R, C values.

We will use a shorthand notation for summation over doubly subscripted variables. If x_{jk} is a variable for $1\leqslant j\leqslant m$ and $1\leqslant k\leqslant n$, then

$$x_{j\bullet} = \sum_{k=1}^{n} x_{jk}, \qquad x_{\bullet k} = \sum_{j=1}^{m} x_{jk}, \qquad x_{\bullet\bullet} = \sum_{j=1}^{m}\sum_{k=1}^{n} x_{jk},$$

$$x_{j*} = \sum_{k=1}^{n-1} x_{jk}, \qquad x_{*k} = \sum_{j=1}^{m-1} x_{jk}, \qquad x_{**} = \sum_{j=1}^{m-1}\sum_{k=1}^{n-1} x_{jk},$$

for $1\leqslant j\leqslant m$ and $1\leqslant k\leqslant n$.

Throughout the paper, the asymptotic notation $O(f(m,n))$ refers to the passage of m and n to ∞. We also use a modified notation $\widetilde{O}(f(m,n))$, which is to be taken as a shorthand for $O(f(m,n)n^{O(1)\varepsilon})$. In this case it is important that the $O(1)$ factor is uniform over ε provided ε is small enough; for example we cannot write $f(m,n)n^{(\varepsilon^{-1})\varepsilon}$ as $\widetilde{O}(f(m,n))$ even though $\varepsilon^{-1} = O(1)$ (ε being defined as a constant). Under the assumptions of Theorem 1, we have $m = \widetilde{O}(n)$ and $n = \widetilde{O}(m)$. We also have that $8\leqslant A^{-1}\leqslant O(\log n)$, so $A^{-1} = \widetilde{O}(1)$. More generally, $A^{c_1}m^{c_2+c_3\varepsilon}n^{c_4+c_5\varepsilon} = \widetilde{O}(n^{c_2+c_4})$ if c_1, c_2, c_3, c_4, c_5 are constants.

2. Expressing the desired quantity as an integral

In this section we express $B(\boldsymbol{s},\boldsymbol{t})$ as a contour integral in $(m+n)$-dimensional complex space, then begin to estimate its value using the saddle-point method.

Firstly, notice that $B(\boldsymbol{s},\boldsymbol{t})$ is the coefficient of $x_1^{s_1}\cdots x_m^{s_m}y_1^{t_1}\cdots y_n^{t_n}$ in the function

$$\prod_{j=1}^{m}\prod_{k=1}^{n}(1+x_jy_k).$$

By Cauchy's coefficient theorem this equals

$$B(\boldsymbol{s},\boldsymbol{t})=\frac{1}{(2\pi i)^{m+n}}\oint\cdots\oint\frac{\prod_{j=1}^{m}\prod_{k=1}^{n}(1+x_jy_k)}{x_1^{s_1+1}\cdots x_m^{s_m+1}y_1^{t_1+1}\cdots y_n^{t_n+1}}\,dx_1\cdots dx_m\,dy_1\cdots dy_n,$$

where each integral is along a simple closed contour enclosing the origin anticlockwise. It will suffice to take each contour to be a circle; specifically, we will write

$$x_j=q_je^{i\theta_j}\quad\text{and}\quad y_k=r_ke^{i\phi_k}$$

for $1\leqslant j\leqslant m$ and $1\leqslant k\leqslant n$. Also define

$$\lambda_{jk}=\frac{q_jr_k}{1+q_jr_k}$$

for $1\leqslant j\leqslant m$ and $1\leqslant k\leqslant n$. Then $1+x_jy_k=(1+q_jr_k)(1+\lambda_{jk}(e^{i(\theta_j+\phi_k)}-1))$, so

$$B(\boldsymbol{s},\boldsymbol{t})=\frac{\prod_{j=1}^{m}\prod_{k=1}^{n}(1+q_jr_k)}{(2\pi)^{m+n}\prod_{j=1}^{m}q_j^{s_j}\prod_{k=1}^{n}r_k^{t_k}}\times\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\frac{\prod_{j=1}^{m}\prod_{k=1}^{n}(1+\lambda_{jk}(e^{i(\theta_j+\phi_k)}-1))}{\exp(i\sum_{j=1}^{m}s_j\theta_j+i\sum_{k=1}^{n}t_k\phi_k)}\,d\boldsymbol{\theta}\,d\boldsymbol{\phi},\tag{1}$$

where $\boldsymbol{\theta}=(\theta_1,\ldots,\theta_m)$ and $\boldsymbol{\phi}=(\phi_1,\ldots,\phi_n)$. Write $B(\boldsymbol{s},\boldsymbol{t})=P(\boldsymbol{s},\boldsymbol{t})I(\boldsymbol{s},\boldsymbol{t})$ where $P(\boldsymbol{s},\boldsymbol{t})$ denotes the factor in front of the integral in (1) and $I(\boldsymbol{s},\boldsymbol{t})$ denotes the integral. We will choose the radii q_j, r_k so that there is no linear term in the logarithm of the integrand of $I(\boldsymbol{s},\boldsymbol{t})$ when expanded for small $\boldsymbol{\theta},\boldsymbol{\phi}$. This gives the equation

$$\sum_{j=1}^{m}\sum_{k=1}^{n}\lambda_{jk}(\theta_j+\phi_k)-\sum_{j=1}^{m}s_j\theta_j-\sum_{k=1}^{n}t_k\phi_k=0.$$

For this to hold for all $\boldsymbol{\theta},\boldsymbol{\phi}$, we require

$$\begin{aligned}&\lambda_{j\bullet}=s_j\quad(1\leqslant j\leqslant m),\\&\lambda_{\bullet k}=t_k\quad(1\leqslant k\leqslant n).\end{aligned}\tag{2}$$

In Section 3 we show that (2) has a solution, and determine to sufficient accuracy the various functions of the radii, such as $P(\boldsymbol{s},\boldsymbol{t})$, that we require. In Section 4 we evaluate the integral $I(\boldsymbol{s},\boldsymbol{t})$ within a certain region $\mathcal{R}$ defined in (22). Section 5 contains the proof that the integral is concentrated within the region $\mathcal{R}$.

3. Locating the saddle-point

In this section we solve (2) and derive some of the consequences of the solution. As with the whole paper, we work under the assumptions of Theorem 1.

Change variables to $\{a_j\}_{j=1}^m$, $\{b_k\}_{k=1}^n$ as follows:

$$q_j = r\frac{1+a_j}{1-r^2a_j}, \qquad r_k = r\frac{1+b_k}{1-r^2b_k}, \tag{3}$$

where

$$r = \sqrt{\frac{\lambda}{1-\lambda}}.$$

Equation (2) is slightly underdetermined, which we will exploit to impose an additional condition. If $\{q_j\}, \{r_k\}$ satisfy (2) and $c > 0$ is a constant, then $\{cq_j\}, \{r_k/c\}$ also satisfy (2). From this we can see that, if there is a solution to (2) at all, there is one for which $\sum_{j=1}^m a_j < 0$ and $\sum_{k=1}^n b_k > 0$, and also a solution for which $\sum_{j=1}^m a_j > 0$ and $\sum_{k=1}^n b_k < 0$. It follows from the Intermediate Value Theorem that there is a solution for which

$$n\sum_{j=1}^m a_j = m\sum_{k=1}^n b_k, \tag{4}$$

so we will seek a common solution to (2) and (4).

From (3) we find that

$$\lambda_{jk}/\lambda = 1 + a_j + b_k + Z_{jk}, \tag{5}$$

where

$$Z_{jk} = \frac{a_jb_k(1-r^2-r^2a_j-r^2b_k)}{1+r^2a_jb_k}, \tag{6}$$

and that Eq. (2) can be rewritten as

$$\begin{aligned} a_j &= \frac{s_j - s}{\lambda n} - \frac{1}{n}\sum_{k=1}^n b_k - \frac{Z_{j\bullet}}{n} \quad (1 \leqslant j \leqslant m), \\ b_k &= \frac{t_k - t}{\lambda m} - \frac{1}{m}\sum_{j=1}^m a_j - \frac{Z_{\bullet k}}{m} \quad (1 \leqslant k \leqslant n). \end{aligned} \tag{7}$$

Summing (7) over j, k, we find in both cases that

$$n\sum_{j=1}^m a_j + m\sum_{k=1}^n b_k = -Z_{\bullet\bullet}. \tag{8}$$

Equations (4) and (8) together imply that

$$n\sum_{j=1}^m a_j = m\sum_{k=1}^n b_k = -\frac{1}{2}Z_{\bullet\bullet}.$$

Substituting back into (7), we obtain

$$a_j = \mathbb{A}_j(a_1,\dots,a_m,b_1,\dots,b_n),$$
$$b_k = \mathbb{B}_k(a_1,\dots,a_m,b_1,\dots,b_n), \tag{9}$$

for $1 \leqslant j \leqslant m$, $1 \leqslant k \leqslant n$, where

$$\mathbb{A}_j(a_1,\dots,a_m,b_1,\dots,b_n) = \frac{s_j - s}{\lambda n} - \frac{Z_{j\bullet}}{n} + \frac{Z_{\bullet\bullet}}{2mn},$$
$$\mathbb{B}_k(a_1,\dots,a_m,b_1,\dots,b_n) = \frac{t_k - t}{\lambda m} - \frac{Z_{\bullet k}}{m} + \frac{Z_{\bullet\bullet}}{2mn}.$$

Equation (9) suggests an iteration. Start with $a_j^{(0)} = b_k^{(0)} = 0$ for all j,k, and, for each $\ell \geqslant 0$, define

$$a_j^{(\ell+1)} = \mathbb{A}_j\big(a_1^{(\ell)},\dots,a_m^{(\ell)},b_1^{(\ell)},\dots,b_n^{(\ell)}\big),$$
$$b_k^{(\ell+1)} = \mathbb{B}_k\big(a_1^{(\ell)},\dots,a_m^{(\ell)},b_1^{(\ell)},\dots,b_n^{(\ell)}\big), \tag{10}$$

where $Z_{j\bullet} = Z_{j\bullet}^{(\ell)} = Z_{j\bullet}(a_1^{(\ell)},\dots,a_m^{(\ell)},b_1^{(\ell)},\dots,b_n^{(\ell)})$ and similarly for $Z_{\bullet k} = Z_{\bullet k}^{(\ell)}$ and $Z_{\bullet\bullet} = Z_{\bullet\bullet}^{(\ell)}$. We will show that this iteration converges to a solution of (9) using a standard contraction-mapping argument. Recall that $A^{-1} = O(\log n)$ under the assumptions of Theorem 1 (which we are adopting throughout). This implies that $r^2 = O(\log n)$. Therefore, within the region $\mathcal{A}$ defined by $|a_j|, |b_k| \leqslant n^{-1/3}$ for all j,k, we have that

$$\frac{\partial Z_{j,k}}{\partial a_j} = o\big(m^{-1/4}\big) \quad \text{and} \quad \frac{\partial Z_{j,k}}{\partial b_k} = o\big(n^{-1/4}\big),$$

which imply that, in the same region, we have

$$\frac{\partial \mathbb{A}_j}{\partial a_{j'}} = \begin{cases} o(m^{-1/4}) & (j' = j), \\ o(m^{-5/4}) & (j' \neq j), \end{cases} \qquad \frac{\partial \mathbb{A}_j}{\partial b_k} = o\big(n^{-5/4}\big),$$
$$\frac{\partial \mathbb{B}_k}{\partial a_j} = o\big(m^{-5/4}\big), \qquad \frac{\partial \mathbb{B}_k}{\partial b_{k'}} = \begin{cases} o(n^{-1/4}) & (k' = k), \\ o(n^{-5/4}) & (k' \neq k). \end{cases}$$

Therefore, by the mean value theorem, we have for $\ell \geqslant 1$ that

$$\max_j \big|a_j^{(\ell+1)} - a_j^{(\ell)}\big| + \max_k \big|b_k^{(\ell+1)} - b_k^{(\ell)}\big| = o\big(m^{-1/4}\big) \max_j \big|a_j^{(\ell)} - a_j^{(\ell-1)}\big| + o\big(n^{-1/4}\big) \max_k \big|b_k^{(\ell)} - b_k^{(\ell-1)}\big|,$$

provided $\{a_j^{(\ell-1)}\} \cup \{b_k^{(\ell-1)}\} \cup \{a_j^{(\ell)}\} \cup \{b_k^{(\ell)}\} \subseteq \mathcal{A}$.

Applying the iteration once, we have

$$a_j^{(1)} = (s_j - s)/(\lambda n) \quad \text{and} \quad b_k^{(1)} = (t_k - t)/(\lambda m).$$

Since $\{a_j^{(0)}\}$, $\{b_k^{(0)}\}$ and $\{a_j^{(1)}\}$, $\{b_k^{(1)}\}$ lie inside $\frac{1}{2}\mathcal{A}$, we find by induction that $\{a_j^{(\ell)}\}$, $\{b_k^{(\ell)}\}$ lie in $\frac{\ell}{\ell+1}\mathcal{A}$ for all ℓ. Moreover, the iteration is Cauchy-convergent in the maximum norm, and the error in stopping at $\{a_j^{(\ell)}\}$, $\{b_k^{(\ell)}\}$ is at most $\max_j |a_j^{(\ell)} - a_j^{(\ell-1)}| + \max_k |b_k^{(\ell)} - b_k^{(\ell-1)}|$.

When we carry out this iteration, we find that all the encountered $a_j^{(l)}$ and $b_k^{(l)}$ values are $\widetilde{O}(n^{-1/2})$. It helps to know that the following approximation holds in that case:

$$Z_{jk} = (1-r^2)a_j b_k - r^2 a_j b_k^2 - r^2 a_j^2 b_k - r^2(1-r^2)a_j^2 b_k^2 + \widetilde{O}(n^{-5/2}).$$

Using the fact that $\sum_{j=1}^m a_j^{(1)} = 0$ and $\sum_{k=1}^n b_k^{(1)} = 0$, we find that

$$Z_{j\bullet}^{(1)} = -r^2 a_j^{(1)} \sum_{k=1}^n \bigl(b_k^{(1)}\bigr)^2 + \widetilde{O}(n^{-1}),$$

$$Z_{\bullet k}^{(1)} = -r^2 b_k^{(1)} \sum_{j=1}^m \bigl(a_j^{(1)}\bigr)^2 + \widetilde{O}(n^{-1}),$$

$$Z_{\bullet\bullet}^{(1)} = \widetilde{O}(1).$$

Therefore,

$$a_j^{(2)} = \frac{s_j - s}{\lambda n} + \frac{(s_j - s)C}{\lambda^2(1-\lambda)m^2n^2} + \widetilde{O}(n^{-2}) \quad (1 \leqslant j \leqslant m),$$

$$b_k^{(2)} = \frac{t_k - t}{\lambda m} + \frac{(t_k - t)R}{\lambda^2(1-\lambda)m^2n^2} + \widetilde{O}(n^{-2}) \quad (1 \leqslant k \leqslant n).$$

Similarly,

$$Z_{j\bullet}^{(2)} = -r^2 a_j^{(2)} \sum_{k=1}^n \bigl(b_k^{(2)}\bigr)^2 - r^2(1-r^2)\bigl(a_j^{(2)}\bigr)^2 \sum_{k=1}^n \bigl(b_k^{(2)}\bigr)^2 + \widetilde{O}(n^{-3/2}),$$

$$Z_{\bullet k}^{(2)} = -r^2 b_k^{(2)} \sum_{j=1}^m \bigl(a_j^{(2)}\bigr)^2 - r^2(1-r^2)\bigl(b_k^{(2)}\bigr)^2 \sum_{j=1}^m \bigl(a_j^{(2)}\bigr)^2 + \widetilde{O}(n^{-3/2}),$$

$$Z_{\bullet\bullet}^{(2)} = -r^2(1-r^2) \sum_{j=1}^m \bigl(a_j^{(2)}\bigr)^2 \sum_{k=1}^n \bigl(b_k^{(2)}\bigr)^2 + \widetilde{O}(n^{-1/2}),$$

which gives

$$\begin{aligned} a_j^{(3)} &= \frac{s_j - s}{\lambda n} + \frac{(s_j - s)C}{\lambda^2(1-\lambda)m^2n^2} + \frac{(1-2\lambda)(s_j - s)^2 C}{\lambda^3(1-\lambda)^2m^2n^3} \\ &\quad - \frac{(1-2\lambda)RC}{2\lambda^3(1-\lambda)^2m^3n^3} + \widetilde{O}(n^{-5/2}) \quad (1 \leqslant j \leqslant m), \\ b_k^{(3)} &= \frac{t_k - t}{\lambda m} + \frac{(t_k - t)R}{\lambda^2(1-\lambda)m^2n^2} + \frac{(1-2\lambda)(t_k - t)^2 R}{\lambda^3(1-\lambda)^2m^3n^2} \\ &\quad - \frac{(1-2\lambda)RC}{2\lambda^3(1-\lambda)^2m^3n^3} + \widetilde{O}(n^{-5/2}) \quad (1 \leqslant k \leqslant n). \end{aligned} \tag{11}$$

Further iterations make no change to this accuracy, so we have that $a_j = a_j^{(3)} + \widetilde{O}(n^{-5/2})$ and $b_k = b_k^{(3)} + \widetilde{O}(n^{-5/2})$. We also have that

$$\begin{aligned} Z_{jk} &= \frac{(1-2\lambda)(s_j - s)(t_k - t)}{\lambda^2(1-\lambda)mn} - \frac{(s_j - s)(t_k - t)^2}{\lambda^2(1-\lambda)m^2n} - \frac{(s_j - s)^2(t_k - t)}{\lambda^2(1-\lambda)mn^2} \\ &\quad - \frac{(1-2\lambda)(s_j - s)^2(t_k - t)^2}{\lambda^3(1-\lambda)^2m^2n^2} + \frac{(1-2\lambda)(s_j - s)(t_k - t)R}{\lambda^3(1-\lambda)^2m^2n^3} \\ &\quad + \frac{(1-2\lambda)(s_j - s)(t_k - t)C}{\lambda^3(1-\lambda)^2m^3n^2} + \widetilde{O}(n^{-5/2}). \end{aligned} \tag{12}$$

A sufficient approximation of λ_{jk} is given by substituting (11) and (12) into (5). In evaluating the integral $I(\boldsymbol{s},\boldsymbol{t})$, the following approximations will be required:

$$\lambda_{jk}(1-\lambda_{jk}) = \lambda(1-\lambda) + \frac{(1-2\lambda)(s_j-s)}{n} + \frac{(1-2\lambda)(t_k-t)}{m} - \frac{(s_j-s)^2}{n^2} - \frac{(t_k-t)^2}{m^2} + \frac{(1-6\lambda+6\lambda^2)(s_j-s)(t_k-t)}{\lambda(1-\lambda)mn} + \widetilde{O}\big(n^{-3/2}\big), \tag{13}$$

$$\lambda_{jk}(1-\lambda_{jk})(1-2\lambda_{jk}) = \lambda(1-\lambda)(1-2\lambda) + \frac{(1-6\lambda+6\lambda^2)(s_j-s)}{n} + \frac{(1-6\lambda+6\lambda^2)(t_k-t)}{m} + \widetilde{O}\big(n^{-1}\big), \tag{14}$$

$$\lambda_{jk}(1-\lambda_{jk})\big(1-6\lambda_{jk}+6\lambda_{jk}^2\big) = \lambda(1-\lambda)\big(1-6\lambda+6\lambda^2\big) + \widetilde{O}\big(n^{-1/2}\big). \tag{15}$$

3.1. Estimating the factor $P(\boldsymbol{s},\boldsymbol{t})$

Let

$$\Lambda = \prod_{j=1}^{m}\prod_{k=1}^{n} \lambda_{jk}^{\lambda_{jk}}(1-\lambda_{jk})^{1-\lambda_{jk}}.$$

Then

$$\begin{aligned}\Lambda^{-1} &= \prod_{j=1}^{m}\prod_{k=1}^{n}\left(\frac{1+q_jr_k}{q_jr_k}\right)^{\lambda_{jk}}(1+q_jr_k)^{1-\lambda_{jk}}\\ &= \prod_{j=1}^{m}\prod_{k=1}^{n}(1+q_jr_k)\left(\prod_{j=1}^{m}q_j^{\lambda_{j\bullet}}\prod_{k=1}^{n}r_k^{\lambda_{\bullet k}}\right)^{-1}\\ &= \prod_{j=1}^{m}\prod_{k=1}^{n}(1+q_jr_k)\prod_{j=1}^{m}q_j^{-s_j}\prod_{k=1}^{n}r_k^{-t_k}\end{aligned}$$

using (2). Therefore the factor $P(\boldsymbol{s},\boldsymbol{t})$ in front of the integral in (1) is given by

$$P(\boldsymbol{s},\boldsymbol{t}) = (2\pi)^{-(m+n)}\Lambda^{-1}.$$

We proceed to estimate Λ. Writing $\lambda_{jk} = \lambda(1+x_{jk})$, we have

$$\log\left(\frac{\lambda_{jk}^{\lambda_{jk}}(1-\lambda_{jk})^{1-\lambda_{jk}}}{\lambda^{\lambda}(1-\lambda)^{1-\lambda}}\right) = \lambda x_{jk}\log\left(\frac{\lambda}{1-\lambda}\right) + \frac{\lambda}{2(1-\lambda)}x_{jk}^2 - \frac{\lambda(1-2\lambda)}{6(1-\lambda)^2}x_{jk}^3 + \frac{\lambda(1-3\lambda+3\lambda^2)}{12(1-\lambda)^3}x_{jk}^4 + O\left(\frac{x_{jk}^5}{(1-\lambda)^4}\right). \tag{16}$$

We know from (2) that $\lambda_{\bullet\bullet} = mn\lambda$, which implies that $x_{\bullet\bullet} = 0$, hence the first term on the right side of (16) does not contribute to Λ. Now using (5) we can write $x_{jk} = a_j + b_k + Z_{jk}$ and apply the estimates in (11) and (12) to obtain

$$\Lambda = \left(\lambda^{\lambda}(1-\lambda)^{1-\lambda}\right)^{mn} \exp\Bigg(\frac{R}{4An} + \frac{C}{4Am} + \frac{RC}{8A^2m^2n^2} - \frac{(1-2\lambda)R_3}{24A^2n^2} - \frac{(1-2\lambda)C_3}{24A^2m^2} + \frac{(1-3\lambda+3\lambda^2)R_4}{96A^3n^3} + \frac{(1-3\lambda+3\lambda^2)C_4}{96A^3m^3} + \widetilde{O}\left(n^{-1/2}\right)\Bigg), \tag{17}$$

where $R_\ell = \sum_{j=1}^{m}(s_j - s)^\ell$ and $C_\ell = \sum_{k=1}^{n}(t - t_k)^\ell$ for any ℓ. Note that $R_2 = R$ and $C_2 = C$.

To match the formula from the sparse case solved in [2], we will write (17) in terms of binomial coefficients. First, by Stirling's expansion of the logarithm of the gamma function, we have that

$$\binom{N}{(x+d)N} = \frac{(x^{x+d}(1-x)^{1-x-d})^{-N}}{2\sqrt{\pi XN}} \times \exp\Bigg(-\frac{1-2X}{24XN} - \frac{d^2N}{4X} - \frac{(1-2x)d}{4X} + \frac{(1-4X)d^2}{16X^2} + \frac{(1-2x)d^3N}{24X^2} - \frac{(1-6X)d^4N}{96X^3} + O\left(\frac{d^5N}{X^4} + \frac{d}{X^2N} + \frac{1}{X^3N^3}\right)\Bigg) \tag{18}$$

as $N \to \infty$, provided $x = x(N)$, $X = X(N) = \frac{1}{2}x(1-x)$ and $d = d(N)$ are such that $0 < x < 1$, $0 < x + d < 1$ and provided that the error term in the above is $o(1)$. From this we infer that

$$\binom{mn}{\lambda mn}^{-1} \prod_{j=1}^{m}\binom{n}{s_j}\prod_{k=1}^{n}\binom{m}{t_k} = \frac{(\lambda^{\lambda}(1-\lambda)^{1-\lambda})^{-mn}}{(4\pi A)^{(m+n-1)/2}m^{(n-1)/2}n^{(m-1)/2}} \times \exp\Bigg(-\frac{R}{4An} - \frac{C}{4Am} - \frac{1-2A}{24A}\left(\frac{m}{n} + \frac{n}{m}\right) + \frac{1-4A}{16A^2}\left(\frac{R}{n^2} + \frac{C}{m^2}\right) + \frac{1-2\lambda}{24A^2}\left(\frac{R_3}{n^2} + \frac{C_3}{m^2}\right) - \frac{1-6A}{96A^3}\left(\frac{R_4}{n^3} + \frac{C_4}{m^3}\right) + \widetilde{O}\left(n^{-1/2}\right)\Bigg). \tag{19}$$

Putting (17) and (19) together, we find that

$$\begin{aligned} P(\boldsymbol{s},\boldsymbol{t}) &= \Lambda^{-1}(2\pi)^{-(m+n)} \\ &= \frac{A^{(m+n-1)/2}m^{(n-1)/2}n^{(m-1)/2}}{2\pi^{(m+n+1)/2}}\binom{mn}{\lambda mn}^{-1}\prod_{j=1}^{m}\binom{n}{s_j}\prod_{k=1}^{n}\binom{m}{t_k} \\ &\quad\times \exp\Bigg(\frac{1-2A}{24A}\left(\frac{m}{n} + \frac{n}{m}\right) - \frac{RC}{8A^2m^2n^2} - \frac{1-4A}{16A^2}\left(\frac{R}{n^2} + \frac{C}{m^2}\right) + \widetilde{O}\left(n^{-1/2}\right)\Bigg). \end{aligned} \tag{20}$$

4. Evaluating the integral

Our next task is to evaluate the integral $I(\boldsymbol{s},\boldsymbol{t})$ given by

$$I(\boldsymbol{s},\boldsymbol{t}) = \int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} \frac{\prod_{j=1}^{m}\prod_{k=1}^{n}(1+\lambda_{j,k}(e^{i(\theta_j+\phi_k)}-1))}{\exp(i\sum_{j=1}^{m}s_j\theta_j + i\sum_{k=1}^{n}t_k\phi_k)}\,d\boldsymbol{\theta}\,d\boldsymbol{\phi}. \tag{21}$$

It is convenient to think of θ_j, ϕ_k as points on the unit circle. We wish to define "averages" of the angles θ_j, ϕ_k. To do this cleanly we make the following definitions, as in [1]. Let C be the ring of real numbers modulo 2π, which we can interpret as points on a circle in the usual way. Let z be the canonical mapping from C to the real interval $(-\pi, \pi]$. An *open half-circle* is $C_t = (t - \pi/2, t + \pi/2) \subseteq C$ for some t. Now define

$$\hat{C}^N = \bigcup_t \hat{C}_t^N = \left\{ \boldsymbol{x} = (x_1, \ldots, x_N) \in C^N \;\middle|\; x_1, \ldots, x_N \in C_t \text{ for some } t \in \mathbb{R} \right\}.$$

If $\boldsymbol{x} = (x_1, \ldots, x_N) \in C_0^N$ then define

$$\bar{\boldsymbol{x}} = z^{-1}\left(\frac{1}{N} \sum_{j=1}^{N} z(x_j) \right).$$

More generally, if $\boldsymbol{x} \in C_t^N$ then define $\bar{\boldsymbol{x}} = t + \overline{(x_1 - t, \ldots, x_N - t)}$. The function $\boldsymbol{x} \to \bar{\boldsymbol{x}}$ is well defined and continuous for $\boldsymbol{x} \in \hat{C}^N$.

Let $\mathcal{R}$ denote the set of vector pairs $(\boldsymbol{\theta}, \boldsymbol{\phi}) \in \hat{C}^m \times \hat{C}^n$ such that

$$\begin{aligned}
&|\bar{\boldsymbol{\theta}} + \bar{\boldsymbol{\phi}}| \leqslant (mn)^{-1/2+2\varepsilon}, \\
&|\hat{\theta}_j| \leqslant n^{-1/2+\varepsilon} \quad (1 \leqslant j \leqslant m), \\
&|\hat{\phi}_k| \leqslant m^{-1/2+\varepsilon} \quad (1 \leqslant k \leqslant n),
\end{aligned} \tag{22}$$

where $\hat{\theta}_j = \theta_j - \bar{\boldsymbol{\theta}}$ and $\hat{\phi}_k = \phi_k - \bar{\boldsymbol{\phi}}$. In this definition, values are considered in C. The constant ε is the sufficiently-small value required by Theorem 1.

Let $I_{\mathcal{R}''}(\boldsymbol{s}, \boldsymbol{t})$ denote the integral $I(\boldsymbol{s}, \boldsymbol{t})$ restricted to any region $\mathcal{R}''$. In this section, we estimate $I_{\mathcal{R}'}(\boldsymbol{s}, \boldsymbol{t})$ in a certain region $\mathcal{R}' \supseteq \mathcal{R}$. In Section 5 we will show that the remaining parts of $I(\boldsymbol{s}, \boldsymbol{t})$ are negligible. We begin by analyzing the integrand in $\mathcal{R}$, but for future use when we expand the region to $\mathcal{R}'$ (to be defined in (33)), note that all the approximations we establish for the integrand in $\mathcal{R}$ also hold in the superset of $\mathcal{R}'$ defined by

$$\begin{aligned}
&|\bar{\boldsymbol{\theta}} + \bar{\boldsymbol{\phi}}| \leqslant 3(mn)^{-1/2+2\varepsilon}, \\
&|\hat{\theta}_j| \leqslant 3n^{-1/2+\varepsilon} \quad (1 \leqslant j \leqslant m-1), \\
&|\hat{\theta}_m| \leqslant 2n^{-1/2+3\varepsilon}, \\
&|\hat{\phi}_k| \leqslant 3m^{-1/2+\varepsilon} \quad (1 \leqslant k \leqslant n-1), \\
&|\hat{\phi}_n| \leqslant 2m^{-1/2+3\varepsilon}.
\end{aligned} \tag{23}$$

Define $\hat{\boldsymbol{\theta}} = (\hat{\theta}_1, \ldots, \hat{\theta}_{m-1})$ and $\hat{\boldsymbol{\phi}} = (\hat{\phi}_1, \ldots, \hat{\phi}_{n-1})$. Let T_1 be the transformation $T_1(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}, \nu, \delta) = (\boldsymbol{\theta}, \boldsymbol{\phi})$ defined by

$$\nu = \bar{\boldsymbol{\theta}} + \bar{\boldsymbol{\phi}}, \qquad \delta = \bar{\boldsymbol{\theta}} - \bar{\boldsymbol{\phi}},$$

together with $\hat{\theta}_j = \theta_j - \bar{\boldsymbol{\theta}}$ $(1 \leqslant j \leqslant m-1)$ and $\hat{\phi}_k = \phi_k - \bar{\boldsymbol{\phi}}$ $(1 \leqslant k \leqslant n-1)$. We also define the 1-many transformation T_1^* by

$$T_1^*(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}, \nu) = \bigcup_{\delta} T_1(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}, \nu, \delta).$$

After applying the transformation T_1 to $I_{\mathcal{R}}(\boldsymbol{s}, \boldsymbol{t})$, the new integrand is easily seen to be independent of δ, so we can multiply by the range of δ and remove it as an independent variable. Therefore, we can continue with an $(m + n - 1)$-dimensional integral over $\mathcal{S}$ such that

$\mathcal{R} = T_1^*(\mathcal{S})$. More generally, if $\mathcal{S}'' \subseteq (-\frac{1}{2}\pi, \frac{1}{2}\pi)^{m+n-2} \times (-2\pi, 2\pi]$ and $\mathcal{R}'' = T_1^*(\mathcal{S}'')$, we have

$$I_{\mathcal{R}''}(\boldsymbol{s}, \boldsymbol{t}) = 2\pi mn \int_{\mathcal{S}''} G(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}, \nu)\, d\hat{\boldsymbol{\theta}}\, d\hat{\boldsymbol{\phi}}\, d\nu, \tag{24}$$

where $G(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}, \nu) = F(T_1(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}, \nu, 0))$ with $F(\boldsymbol{\theta}, \boldsymbol{\phi})$ being the integrand of (21). The factor $2\pi mn$ combines the range of δ, which is 4π, and the Jacobian of T_1, which is $mn/2$.

Note that $\mathcal{S}$ is defined by the same inequalities (22) as define $\mathcal{R}$. The first inequality is now $|\nu| \leqslant (mn)^{-1/2+2\varepsilon}$ and the bounds on

$$\hat{\theta}_m = -\sum_{j=1}^{m-1} \hat{\theta}_j \quad \text{and} \quad \hat{\phi}_m = -\sum_{k=1}^{n-1} \hat{\phi}_k$$

still apply even though these are no longer variables of integration.

Our main result in this section is the following.

Theorem 2. *Under the conditions of Theorem* 1*, there is a region* $\mathcal{S}' \supseteq \mathcal{S}$ *such that*

$$\begin{aligned}\int_{\mathcal{S}'} G(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}, \nu)\, d\hat{\boldsymbol{\theta}}\, d\hat{\boldsymbol{\phi}}\, d\nu = (mn)^{-1/2} &\left(\frac{\pi}{Amn}\right)^{1/2} \left(\frac{\pi}{An}\right)^{(m-1)/2} \left(\frac{\pi}{Am}\right)^{(n-1)/2} \\ &\times \exp\left(-\frac{1}{2} - \frac{1-2A}{24A}\left(\frac{m}{n} + \frac{n}{m}\right) + \frac{1}{4A}\left(\frac{1}{m} + \frac{1}{n}\right)\left(\frac{R}{n} + \frac{C}{m}\right)\right. \\ &\left. + \frac{1-8A}{16A^2}\left(\frac{R}{n^2} + \frac{C}{m^2}\right) + O\big(n^{-b}\big)\right).\end{aligned}$$

In the region $\mathcal{S}$, the integrand of (24) can be expanded as

$$\begin{aligned}G(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}, \nu) = \exp\Bigg(&-\sum_{j=1}^{m}\sum_{k=1}^{n}(A + \alpha_{jk})(\nu + \hat{\theta}_j + \hat{\phi}_k)^2 \\ &- i\sum_{j=1}^{m}\sum_{k=1}^{n}(A_3 + \beta_{jk})(\nu + \hat{\theta}_j + \hat{\phi}_k)^3 \\ &+ \sum_{j=1}^{m}\sum_{k=1}^{n}(A_4 + \gamma_{jk})(\nu + \hat{\theta}_j + \hat{\phi}_k)^4 + O\left(A\sum_{j=1}^{m}\sum_{k=1}^{n}|\nu + \hat{\theta}_j + \hat{\phi}_k|^5\right)\Bigg).\end{aligned}$$

Here α_{jk}, β_{jk}, and γ_{jk} are defined by

$$\begin{aligned}&\frac{1}{2}\lambda_{jk}(1-\lambda_{jk}) = A + \alpha_{jk}, \\ &\frac{1}{6}\lambda_{jk}(1-\lambda_{jk})(1-2\lambda_{jk}) = A_3 + \beta_{jk}, \\ &\frac{1}{24}\lambda_{jk}(1-\lambda_{jk})\big(1 - 6\lambda_{jk} + 6\lambda_{jk}^2\big) = A_4 + \gamma_{jk},\end{aligned} \tag{25}$$

where

$$A = \frac{1}{2}\lambda(1-\lambda), \qquad A_3 = \frac{1}{6}\lambda(1-\lambda)(1-2\lambda), \quad \text{and} \quad A_4 = \frac{1}{24}\lambda(1-\lambda)\bigl(1-6\lambda+6\lambda^2\bigr).$$

Approximations for α_{jk}, β_{jk}, γ_{jk} were given in (13)–(15).

4.1. Another change of variables

We now make a second change of variables $(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}, \nu) = T_2(\boldsymbol{\zeta}, \boldsymbol{\xi}, \nu)$, where $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_{m-1})$ and $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_{n-1})$, whose purpose is to almost diagonalize the quadratic part of G. The diagonalization will be completed in the next subsection. The transformation T_2 is defined as follows. For $1 \leqslant j \leqslant m-1$ and $1 \leqslant k \leqslant n-1$ let

$$\hat{\theta}_j = \zeta_j + c\pi_1, \qquad \hat{\phi}_k = \xi_k + d\rho_1,$$

where

$$c = -\frac{1}{m+m^{1/2}} \quad \text{and} \quad d = -\frac{1}{n+n^{1/2}}$$

and, for $1 \leqslant h \leqslant 4$,

$$\pi_h = \sum_{j=1}^{m-1} \zeta_j^h, \qquad \rho_h = \sum_{k=1}^{n-1} \xi_k^h.$$

The Jacobian of the transformation is $(mn)^{-1/2}$. In [1], this transformation was seen to exactly diagonalize the quadratic part of the integrand in the semiregular case. In the present irregular case, the diagonalization is no longer exact but still provides useful progress.

By summing the equations $\hat{\theta}_j = \zeta_j + c\pi_1$ and $\hat{\phi}_k = \xi_k + d\rho_1$, we find that

$$\begin{aligned}
&\pi_1 = m^{1/2} \sum_{j=1}^{m-1} \hat{\theta}_j, \quad |\pi_1| \leqslant m^{1/2} n^{-1/2+\varepsilon}, \\
&\rho_1 = n^{1/2} \sum_{k=1}^{n-1} \hat{\phi}_k, \quad |\rho_1| \leqslant n^{1/2} m^{-1/2+\varepsilon},
\end{aligned} \tag{26}$$

where the right sides come from the bounds on $\hat{\theta}_m$ and $\hat{\phi}_n$. This implies that

$$\begin{aligned}
&\zeta_j = \hat{\theta}_j + \widetilde{O}\bigl(n^{-1}\bigr) \quad (1 \leqslant j \leqslant m-1), \\
&\xi_k = \hat{\phi}_k + \widetilde{O}(n^{-1}) \quad (1 \leqslant k \leqslant n-1).
\end{aligned} \tag{27}$$

The transformed region of integration is $T_2^{-1}(\mathcal{S})$, but for convenience we will expand it a little to be the region defined by the inequalities

$$\begin{aligned}
&|\zeta_j| \leqslant \tfrac{3}{2} n^{-1/2+\varepsilon} \quad (1 \leqslant j \leqslant m-1), \\
&|\xi_k| \leqslant \tfrac{3}{2} m^{-1/2+\varepsilon} \quad (1 \leqslant k \leqslant n-1), \\
&|\pi_1| \leqslant m^{1/2} n^{-1/2+\varepsilon}, \\
&|\rho_1| \leqslant n^{1/2} m^{-1/2+\varepsilon}, \\
&|\nu| \leqslant (mn)^{-1/2+2\varepsilon}.
\end{aligned} \tag{28}$$

We now consider the new integrand $E_1 = \exp(L_1) = G \circ T_2$. As in [1], the semiregular parts of the integrand (those not involving α_{jk}, β_{jk} or γ_{jk}) transform to

$$\begin{aligned}&-Amn\nu^2 - An\pi_2 - Am\rho_2 - 3iA_3n\nu\pi_2 - 3iA_3m\nu\rho_2 + 6A_4\pi_2\rho_2\\&\quad - iA_3n\pi_3 - iA_3n\rho_3 - 3iA_3cn\pi_1\pi_2 - 3iA_3dm\rho_1\rho_2 + A_4n\pi_4 + A_4m\rho_4 + \widetilde{O}\big(n^{-1/2}\big).\end{aligned} \tag{29}$$

To see the effect of the transformation on the irregular parts of the integrand, write $\zeta_m = \hat{\theta}_m - c\pi_1$ and $\xi_n = \hat{\theta}_n - d\rho_1$. From (26) we can see that $\zeta_m = \widetilde{O}(n^{-1/2})$ and $\xi_n = \widetilde{O}(n^{-1/2})$. Thus we have, for all $1 \leqslant j \leqslant m$ and $1 \leqslant k \leqslant n$, $\zeta_j + \xi_k = \widetilde{O}(n^{-1/2})$ and $c\pi_1 + d\rho_1 + \nu = \widetilde{O}(n^{-1})$. Recalling also that $\alpha_{jk}, \beta_{jk}, \gamma_{jk} = \widetilde{O}(n^{-1/2})$, we have

$$\begin{aligned}&\sum_{j=1}^{m}\sum_{k=1}^{n}\alpha_{jk}(\nu + \hat{\theta}_j + \hat{\phi}_k)^2\\&\quad = \sum_{j=1}^{m}\sum_{k=1}^{n}\alpha_{jk}\big((\zeta_j + \xi_k)^2 + 2(\zeta_j + \xi_k)(\nu + c\pi_1 + d\rho_1)\big) + \widetilde{O}\big(n^{-1/2}\big),\\&\sum_{j=1}^{m}\sum_{k=1}^{n}\beta_{jk}(\nu + \hat{\theta}_j + \hat{\phi}_k)^3 = \sum_{j=1}^{m}\sum_{k=1}^{n}\beta_{jk}(\zeta_j + \xi_k)^3 + \widetilde{O}\big(n^{-1/2}\big),\\&\sum_{j=1}^{m}\sum_{k=1}^{n}\gamma_{jk}(\nu + \hat{\theta}_j + \hat{\phi}_k)^4 = \widetilde{O}\big(n^{-1/2}\big).\end{aligned}$$

Moreover, the terms on the right sides of the above that involve ζ_m or ξ_n contribute only $\widetilde{O}\big(n^{-1/2}\big)$ in total, so we can drop them. Combining this with (29), we have

$$\begin{aligned}L_1 = {}&-Amn\nu^2 - An\pi_2 - Am\rho_2 - 3iA_3n\nu\pi_2 - 3iA_3m\nu\rho_2 + 6A_4\pi_2\rho_2\\&- iA_3n\pi_3 - iA_3n\rho_3 - 3iA_3cn\pi_1\pi_2 - 3iA_3dm\rho_1\rho_2 + A_4n\pi_4 + A_4m\rho_4\\&- \sum_{j=1}^{m-1}\sum_{k=1}^{n-1}\alpha_{jk}\big((\zeta_j + \xi_k)^2 + 2(\zeta_j + \xi_k)(\nu + c\pi_1 + d\rho_1)\big)\\&- i\sum_{j=1}^{m-1}\sum_{k=1}^{n-1}\beta_{jk}(\zeta_j + \xi_k)^3 + \widetilde{O}\big(n^{-1/2}\big).\end{aligned} \tag{30}$$

4.2. Completing the diagonalization

The quadratic form in E_1 is the following function of the $m + n - 1$ variables $\boldsymbol{\zeta}, \boldsymbol{\xi}, \nu$:

$$\begin{aligned}Q = {}&-Amn\nu^2 - An\pi_2 - Am\rho_2\\&- \sum_{j=1}^{m-1}\sum_{k=1}^{n-1}\alpha_{jk}\big((\zeta_j + \xi_k)^2 + 2(\zeta_j + \xi_k)(\nu + c\pi_1 + d\rho_1)\big).\end{aligned} \tag{31}$$

We will make a third change of variables, $(\boldsymbol{\zeta}, \boldsymbol{\xi}, \nu) = T_3(\boldsymbol{\sigma}, \boldsymbol{\tau}, \mu)$, that diagonalizes this quadratic form, where $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_{m-1})$ and $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_{n-1})$. This is achieved using a slight extension of [6, Lemma 3.2].

Lemma 1. *Let X and Y be square matrices of the same order, such that X^{-1} exists and all the eigenvalues of $X^{-1}Y$ are less than* 1 *in absolute value. Then*

$$\bigl(I + YX^{-1}\bigr)^{-1/2}(X+Y)\bigl(I + X^{-1}Y\bigr)^{-1/2} = X,$$

where the fractional powers are defined by the binomial expansion.

Note that $X^{-1}Y$ and YX^{-1} have the same eigenvalues, so the eigenvalue condition on $X^{-1}Y$ applies equally to YX^{-1}. If we also have that both X and Y are symmetric, then

$$\sum_{r\geqslant 0}\binom{-\frac{1}{2}}{r}\bigl(YX^{-1}\bigr)^T = \sum_{r\geqslant 0}\binom{-\frac{1}{2}}{r}\bigl(X^{-1}\bigr)^T Y^T = \sum_{r\geqslant 0}\binom{-\frac{1}{2}}{r}X^{-1}Y$$

so $(I + YX^{-1})^{-1/2}$ is the transpose of $(I + X^{-1}Y)^{-1/2}$. Let V be the symmetric matrix associated with the quadratic form Q. Write $V = V_{\mathrm{d}} + V_{\mathrm{nd}}$ where V_{d} has all off-diagonal entries equal to zero and matches V on the diagonal entries, and V_{nd} has all diagonal entries zero and matches V on the off-diagonal entries. We will apply Lemma 1 with $X = V_{\mathrm{d}}$ and $Y = V_{\mathrm{nd}}$. Note that V_{d} is invertible and that both V_{d} and V_{nd} are symmetric. Let T_3 be the transformation given by $T_3(\boldsymbol{\sigma}, \boldsymbol{\tau}, \mu)^T = (\boldsymbol{\zeta}, \boldsymbol{\xi}, \nu)^T = (I + V_{\mathrm{d}}^{-1}V_{\mathrm{nd}})^{-1/2}(\boldsymbol{\sigma}, \boldsymbol{\tau}, \mu)^T$. If the eigenvalue condition of Lemma 1 is satisfied then this transformation diagonalizes the quadratic form Q, keeping the diagonal entries unchanged.

From the formula for Q we extract the following coefficients, which tell us the diagonal and off-diagonal entries of V:

$$\begin{aligned}
&\bigl[\zeta_j^2\bigr]Q = -An - (1+2c)\alpha_{j*},\\
&\bigl[\xi_k^2\bigr]Q = -Am - (1+2d)\alpha_{*k},\\
&\bigl[\nu^2\bigr]Q = -Amn,\\
&[\zeta_{j_1}\zeta_{j_2}]Q = -2c(\alpha_{j_1*} + \alpha_{j_2*}) \quad (j_1 \neq j_2),\\
&[\zeta_j\xi_k]Q = -2\alpha_{jk} - 2d\alpha_{j*} - 2c\alpha_{*k},\\
&[\xi_{k_1}\xi_{k_2}]Q = -2d(\alpha_{*k_1} + \alpha_{*k_2}) \quad (k_1 \neq k_2),\\
&[\zeta_j\nu]Q = -2\alpha_{j*},\\
&[\xi_k\nu]Q = -2\alpha_{*k}.
\end{aligned}$$

Using these equations we find that all off-diagonal entries of $V_{\mathrm{d}}^{-1}V_{\mathrm{nd}}$ are $\widetilde{O}(n^{-3/2})$, except for the column corresponding to ν which has off-diagonal entries of size $\widetilde{O}(n^{-1/2})$. Similarly, the off-diagonal entries of $V_{\mathrm{nd}}V_{\mathrm{d}}^{-1}$ are all $\widetilde{O}(n^{-3/2})$, except for the row corresponding to ν, which has off-diagonal entries of size $\widetilde{O}(n^{-1/2})$. To see that these conditions imply that the eigenvalues of $V_{\mathrm{d}}^{-1}V_{\mathrm{nd}}$ are less than one, recall that the value of any matrix norm is greater than or equal to the greatest absolute value of an eigenvalue. The ∞-norm (maximum row sum of absolute values) of $V_{\mathrm{d}}^{-1}V_{\mathrm{nd}}$ is $\widetilde{O}(n^{-1/2})$, so the eigenvalues are all $\widetilde{O}(n^{-1/2})$.

We also need to know the Jacobian of the transformation T_3.

Lemma 2. *Let M be a matrix of order $O(m+n)$ with all eigenvalues uniformly $\widetilde{O}(n^{-1/2})$. Then*

$$\det(I+M) = \exp\left(\operatorname{tr} M - \frac{1}{2}\operatorname{tr} M^2 + \widetilde{O}\bigl(n^{-1/2}\bigr)\right).$$

Proof. The eigenvalue condition ensures that the Taylor series for $\log(I+M)$ converges and that

$$\det(I+M) = \exp\bigl(\operatorname{tr}\log(I+M)\bigr).$$

Expanding the logarithm and noting that $|\operatorname{tr} M^r| = \widetilde{O}(n^{-(r-2)/2})$ for $r \geqslant 3$ gives the result. □

Let $M = V_{\mathrm{d}}^{-1} V_{\mathrm{nd}}$. As noted before, the eigenvalues of M are all $\widetilde{O}(n^{-1/2})$ so Lemma 2 applies. Noting that $\operatorname{tr}(M) = 0$ and calculating that $\operatorname{tr}(M^2) = \widetilde{O}(n^{-1})$, we conclude that the Jacobian of T_3 is

$$\det\bigl((I+M)^{-1/2}\bigr) = \bigl(\det(I+M)\bigr)^{-1/2} = 1 + \widetilde{O}\bigl(n^{-1/2}\bigr).$$

To derive T_3 explicitly, we can expand $(I + V_{\mathrm{d}}^{-1} V_{\mathrm{nd}})^{-1/2}$ while noting that $\alpha_{j*} = O(n^{1/2+\varepsilon})$ for all j, $\alpha_{*k} = O(m^{1/2+\varepsilon})$ for all k, $\alpha_{**} = O(mn^{2\varepsilon} + nm^{2\varepsilon})$, $R \leqslant mn^{1+2\varepsilon}$ and $C \leqslant nm^{1+2\varepsilon}$.

This gives

$$\begin{aligned}
\sigma_j &= \zeta_j + \sum_{j'=1}^{m-1}\left(\frac{c(\alpha_{j*}+\alpha_{j'*})}{2An} + \widetilde{O}\bigl(n^{-2}\bigr)\right)\zeta_{j'} + \sum_{k=1}^{n-1}\left(\frac{\alpha_{jk}+d\alpha_{j*}+c\alpha_{*k}}{2An} + \widetilde{O}\bigl(n^{-2}\bigr)\right)\xi_k \\
&\quad + \left(\frac{\alpha_{j*}}{2An} + \widetilde{O}\bigl(n^{-1}\bigr)\right)\nu + \widetilde{O}\bigl(n^{-2}\bigr), \\
\tau_k &= \xi_k + \sum_{j=1}^{m-1}\left(\frac{\alpha_{jk}+d\alpha_{j*}+c\alpha_{*k}}{2Am} + \widetilde{O}\bigl(n^{-2}\bigr)\right)\zeta_j + \sum_{k'=1}^{n-1}\left(\frac{d(\alpha_{*k}+\alpha_{*k'})}{2Am} + \widetilde{O}\bigl(n^{-2}\bigr)\right)\xi_{k'} \\
&\quad + \left(\frac{\alpha_{*k}}{2Am} + \widetilde{O}\bigl(n^{-1}\bigr)\right)\nu + \widetilde{O}\bigl(n^{-2}\bigr), \\
\mu &= \nu + \sum_{j=1}^{m-1}\left(\frac{\alpha_{j*}}{2Amn} + \widetilde{O}\bigl(n^{-2}\bigr)\right)\zeta_j + \sum_{k=1}^{n-1}\left(\frac{\alpha_{*k}}{2Amn} + \widetilde{O}\bigl(n^{-2}\bigr)\right)\xi_k + \widetilde{O}\bigl(n^{-1}\bigr)\nu,
\end{aligned}$$

for $1 \leqslant j \leqslant m-1$, $1 \leqslant k \leqslant n-1$.

The transformation T_3^{-1} perturbs the region of integration in an irregular fashion that we must bound. From the explicit form of T_3 above, we have

$$\begin{aligned}
\sigma_j &= \zeta_j + \sum_{j'=1}^{m-1}\widetilde{O}\bigl(n^{-3/2}\bigr)\zeta_{j'} + \sum_{k=1}^{n-1}\widetilde{O}\bigl(n^{-3/2}\bigr)\xi_k + \widetilde{O}\bigl(n^{-1/2}\bigr)\nu + \widetilde{O}\bigl(n^{-2}\bigr) = \zeta_j + \widetilde{O}\bigl(n^{-1}\bigr), \\
\tau_k &= \xi_k + \sum_{j=1}^{m-1}\widetilde{O}\bigl(n^{-3/2}\bigr)\zeta_j + \sum_{k'=1}^{n-1}\widetilde{O}\bigl(n^{-3/2}\bigr)\xi_{k'} + \widetilde{O}\bigl(n^{-1/2}\bigr)\nu + \widetilde{O}\bigl(n^{-2}\bigr) = \xi_k + \widetilde{O}\bigl(n^{-1}\bigr)
\end{aligned}$$

for $1 \leqslant j \leqslant m-1$, $1 \leqslant k \leqslant n-1$, so $\boldsymbol{\sigma}, \boldsymbol{\tau}$ are only slightly different from $\boldsymbol{\zeta}, \boldsymbol{\xi}$.

For μ versus ν we have

$$\mu = \nu + O\bigl(n^{-1+2\varepsilon}/A\bigr) + O\bigl(m^{-1+2\varepsilon}/A\bigr) = \nu + o\bigl((mn)^{-1/2+2\varepsilon}\bigr),$$

where the second step requires our assumptions $m = o(A^2 n^{1+\varepsilon})$ and $n = o(A^2 m^{1+\varepsilon})$. This shows that the bound $|\nu| \leqslant (mn)^{-1/2+2\varepsilon}$ is adequately covered by $|\mu| \leqslant 2(mn)^{-1/2+2\varepsilon}$.

For $1 \leqslant h \leqslant 4$, define

$$\mu_h = \sum_{j=1}^{m-1} {\sigma_j}^h, \qquad \nu_h = \sum_{k=1}^{n-1} {\tau_k}^h.$$

From (28), we see that $|\pi_1| \leqslant m^{1/2}n^{-1/2+\varepsilon}$ and $|\rho_1| \leqslant m^{-1/2+\varepsilon}n^{1/2}$ are the remaining constraints that define the region of integration. We next apply these constraints to bound μ_1 and ν_1. From the explicit form of T_3, we have

$$\begin{aligned}
\mu_1 &= \pi_1 + \sum_{j=1}^{m-1}\sum_{j'=1}^{m-1}\left(\frac{c(\alpha_{j*}+\alpha_{j'*})}{2An} + \widetilde{O}\bigl(n^{-2}\bigr)\right)\zeta_{j'} \\
&\quad + \sum_{j=1}^{m-1}\sum_{k=1}^{n-1}\left(\frac{\alpha_{jk}+d\alpha_{j*}+c\alpha_{*k}}{2An} + \widetilde{O}\bigl(n^{-2}\bigr)\right)\xi_k + \sum_{j=1}^{m-1}\left(\frac{\alpha_{j*}}{2An} + \widetilde{O}\bigl(n^{-1}\bigr)\right)\nu + \widetilde{O}\bigl(n^{-1}\bigr) \\
&= \pi_1 + \frac{c\alpha_{**}}{2An}m^{1/2}n^{-1/2+\varepsilon} + \frac{d\alpha_{**}}{2An}m^{-1/2+\varepsilon}n^{1/2} + \frac{\alpha_{**}}{2An}\nu \\
&\quad + \bigl(1+c(m-1)\bigr)\sum_{k=1}^{n-1}\frac{\alpha_{*k}}{2An}\xi_k + \frac{c(m-1)}{2An}\sum_{j'=1}^{m-1}\alpha_{j'*}\zeta_{j'} + \widetilde{O}\bigl(n^{-1/2}\bigr) \\
&= \pi_1 + \frac{c(m-1)}{2An}\sum_{j'=1}^{m-1}\alpha_{j'*}\zeta_{j'} + \widetilde{O}\bigl(n^{-1/2}\bigr) \\
&= \pi_1 + O\bigl(A^{-1}mn^{-1+2\varepsilon}\bigr) \\
&= \pi_1 + o\bigl(m^{1/2}n^{-1/2+5\varepsilon/2}\bigr).
\end{aligned} \tag{32}$$

To derive the above we have used $1+c(m-1) = m^{1/2}$ and the bounds we have established on the various variables. For the last step, we need the assumption $m = o(A^2n^{1+\varepsilon})$, which implies that $A^{-1}mn^{-1+2\varepsilon} = o(m^{1/2}n^{-1/2+5\varepsilon/2})$. Since our region of integration has $|\pi_1| \leqslant m^{1/2}n^{-1/2+\varepsilon}$, we see that this implies the bound $|\mu_1| \leqslant m^{1/2}n^{-1/2+3\varepsilon}$. By a parallel argument, we have

$$\nu_1 = \rho_1 + o\bigl(m^{-1/2+5\varepsilon/2}n^{1/2}\bigr),$$

which implies $|\nu_1| \leqslant n^{1/2}m^{-1/2+3\varepsilon}$. Putting together all the bounds we have derived, we see that

$$T_3^{-1}\bigl(T_2^{-1}(\mathcal{S})\bigr) \subseteq \mathcal{Q} \cap \mathcal{M},$$

where

$$\begin{aligned}
\mathcal{Q} &= \bigl\{|\sigma_j| \leqslant 2n^{-1/2+\varepsilon},\, j = 1,\ldots,m-1\bigr\} \cap \bigl\{|\tau_k| \leqslant 2m^{-1/2+\varepsilon},\, k = 1,\ldots,n-1\bigr\} \\
&\quad \cap \bigl\{|\mu| \leqslant 2(mn)^{-1/2+2\varepsilon}\bigr\}, \\
\mathcal{M} &= \bigl\{|\mu_1| \leqslant m^{1/2}n^{-1/2+3\varepsilon}\bigr\} \cap \bigl\{|\nu_1| \leqslant n^{1/2}m^{-1/2+3\varepsilon}\bigr\}.
\end{aligned}$$

Now define

$$\begin{aligned}
\mathcal{S}' &= T_2\bigl(T_3(\mathcal{Q}\cap\mathcal{M})\bigr), \\
\mathcal{R}' &= T_1^*(\mathcal{S}').
\end{aligned} \tag{33}$$

We have proved that $\mathcal{S}' \supseteq \mathcal{S}$, so it is valid to take $\mathcal{S}'$ to be the region required by Theorem 2. Also notice that $\mathcal{R}'$ is contained in the region defined by the inequalities (23). As we forecast at that time, our estimates of the integrand have been valid inside this expanded region. It remains to apply the transformation T_3^{-1} to the integrand (30) so that we have it in terms of $(\boldsymbol{\sigma}, \boldsymbol{\tau}, \mu)$. The explicit form of T_3^{-1} is similar to the explicit form for T_3, namely:

$$\begin{aligned}
\zeta_j &= \sigma_j - \sum_{j'=1}^{m-1}\left(\frac{c(\alpha_{j*}+\alpha_{j'*})}{2An} + \widetilde{O}\big(n^{-2}\big)\right)\sigma_{j'} - \sum_{k=1}^{n-1}\left(\frac{\alpha_{jk}+d\alpha_{j*}+c\alpha_{*k}}{2An} + \widetilde{O}\big(n^{-2}\big)\right)\tau_k \\
&\quad - \left(\frac{\alpha_{j*}}{2An} + \widetilde{O}\big(n^{-1}\big)\right)\mu + \widetilde{O}\big(n^{-2}\big), \\
\xi_k &= \tau_k - \sum_{j=1}^{m-1}\left(\frac{\alpha_{jk}-d\alpha_{j*}+c\alpha_{*k}}{2Am} + \widetilde{O}\big(n^{-2}\big)\right)\sigma_j - \sum_{k'=1}^{n-1}\left(\frac{d(\alpha_{*k}+\alpha_{*k'})}{2Am} + \widetilde{O}\big(n^{-2}\big)\right)\tau_{k'} \\
&\quad - \left(\frac{\alpha_{*k}}{2Am} + \widetilde{O}\big(n^{-1}\big)\right)\mu + \widetilde{O}\big(n^{-2}\big), \\
\nu &= \mu - \sum_{j=1}^{m-1}\left(\frac{\alpha_{j*}}{2Amn} + \widetilde{O}\big(n^{-2}\big)\right)\sigma_j - \sum_{k=1}^{n-1}\left(\frac{\alpha_{*k}}{2Amn} + \widetilde{O}\big(n^{-2}\big)\right)\tau_k + \widetilde{O}\big(n^{-1}\big)\mu,
\end{aligned}$$

for $1 \leqslant j \leqslant m-1$, $1 \leqslant k \leqslant n-1$. In addition to the relationships between the old and new variables that we proved before, we can note that $\pi_2 = \mu_2 + \widetilde{O}(n^{-1/2})$, $\rho_2 = \nu_2 + \widetilde{O}(n^{-1/2})$, $\pi_3 = \mu_3 + \widetilde{O}(n^{-1})$, $\rho_3 = \nu_3 + \widetilde{O}(n^{-1})$, $\pi_4 = \mu_4 + \widetilde{O}(n^{-3/2})$, and $\rho_4 = \nu_4 + \widetilde{O}(n^{-3/2})$.

The quadratic part of L_1, which we called Q in (31), loses its off-diagonal parts according to our design of T_3. Thus, what remains is

$$\begin{aligned}
&-Amn\mu^2 - \sum_{j=1}^{m-1}\big(An + (1+2c)\alpha_{j*}\big)\sigma_j^2 - \sum_{k=1}^{n-1}\big(Am + (1+2d)\alpha_{*k}\big)\tau_k^2 \\
&\quad = -Amn\mu^2 - An\mu_2 - Am\nu_2 - \sum_{j=1}^{m-1}\alpha_{j*}\sigma_j^2 - \sum_{k=1}^{n-1}\alpha_{*k}\tau_k^2 + \widetilde{O}\big(n^{-1/2}\big).
\end{aligned}$$

Next consider the cubic terms of L_1. These are

$$\begin{aligned}
&-3iA_3 n\nu\pi_2 - 3iA_3 m\nu\rho_2 - iA_3 n\pi_3 - iA_3 n\rho_3 \\
&\quad - 3iA_3 cn\pi_1\pi_2 - 3iA_3 dn\rho_1\rho_2 - i\sum_{j=1}^{m-1}\sum_{k=1}^{n-1}\beta_{jk}(\zeta_j+\xi_k)^3.
\end{aligned}$$

We calculate the following in $\mathcal{Q}\cap\mathcal{M}$:

$$-3iA_3 n\nu\pi_2 = -3iA_3 n\mu\mu_2 + \frac{3iA_3\mu_2}{2Am}\left(\sum_{j=1}^{m-1}\alpha_{j*}\sigma_j + \sum_{k=1}^{n-1}\alpha_{*k}\tau_k\right) + \widetilde{O}\big(n^{-1/2}\big),$$

$$-iA_3n\pi_3 = -iA_3n\mu_3 + \frac{3iA_3}{2A}\Bigg(\sum_{j,j'=1}^{m-1} c(\alpha_{j*}+\alpha_{j'*})\sigma_j^2\sigma_{j'} + \sum_{j=1}^{m-1}\sum_{k=1}^{n-1}(\alpha_{jk}+d\alpha_{j*}+c\alpha_{*k})\sigma_j^2\tau_k\Bigg) + \widetilde{O}\big(n^{-1/2}\big),$$

$$-3iA_3cn\pi_1\pi_2 = -3iA_3cn\mu_1\mu_2 + \frac{3iA_3c^2m\mu_2}{2A}\sum_{j=1}^{m-1}\alpha_{j*}\sigma_j + \widetilde{O}\big(n^{-1/2}\big), \tag{34}$$

$$-i\sum_{j=1}^{m-1}\sum_{k=1}^{n-1}\beta_{jk}(\zeta_j+\xi_k)^3 = -i\sum_{j=1}^{m-1}\sum_{k=1}^{n-1}\beta_{jk}(\sigma_j+\tau_k)^3 + \widetilde{O}\big(n^{-1/2}\big), \tag{35}$$

and the remaining cubic terms are each parallel to one of those. The proof of (34) is similar to the proof of (32).

Finally we come to the quartic part of E_1, which is

$$6A_4\pi_2\rho_2 + A_4n\pi_4 + A_4m\rho_4 = 6A_4\mu_2\nu_2 + A_4n\mu_4 + A_4m\nu_4 + \widetilde{O}\big(n^{-1/2}\big).$$

In summary, the value of the integrand for $(\boldsymbol{\sigma},\boldsymbol{\tau},\mu)\in\mathcal{Q}\cap\mathcal{M}$ is $\exp(L_2+\widetilde{O}(n^{-1/2}))$, where

$$\begin{aligned} L_2 = {} & -Amn\mu^2 - An\mu_2 - Am\nu_2 - \sum_{j=1}^{m-1}\alpha_{j*}\sigma_j^2 - \sum_{k=1}^{n-1}\alpha_{*k}\tau_k^2 + 6A_4\mu_2\nu_2 \\ & + A_4n\mu_4 + A_4m\nu_4 - iA_3n\mu_3 - iA_3m\nu_3 - 3iA_3cn\mu_1\mu_2 - 3iA_3dm\nu_1\nu_2 \\ & - 3iA_3n\mu\mu_2 - 3iA_3m\mu\nu_2 - i\sum_{j=1}^{m-1}\beta_{j*}\sigma_j^3 - i\sum_{k=1}^{n-1}\beta_{*k}\tau_k^3 \\ & + i\sum_{j,j'=1}^{m-1} g_{jj'}\sigma_j\sigma_{j'}^2 + i\sum_{k,k'=1}^{n-1} h_{kk'}\tau_k\tau_{k'}^2 + i\sum_{j=1}^{m-1}\sum_{k=1}^{n-1}\big(u_{jk}\sigma_j\tau_k^2 + v_{jk}\sigma_j^2\tau_k\big), \end{aligned} \tag{36}$$

with

$$\begin{aligned} g_{jj'} &= \frac{3A_3}{2Am}\big((1+cm+c^2m^2)\alpha_{j*} + cm\alpha_{j'*}\big) = O\big(n^{-1/2+\varepsilon}\big), \\ h_{kk'} &= \frac{3A_3}{2An}\big((1+dn+d^2n^2)\alpha_{*k} + dn\alpha_{*k'}\big) = O\big(m^{-1/2+\varepsilon}\big), \\ u_{jk} &= \frac{3A_3}{2An}\big(n\alpha_{jk} + (1+dn)\alpha_{j*} + cn\alpha_{*k}\big) - 3\beta_{jk} = O\big(m^{-1/2+2\varepsilon} + n^{-1/2+2\varepsilon}\big), \\ v_{jk} &= \frac{3A_3}{2Am}\big(m\alpha_{jk} + (1+cm)\alpha_{*k} + dm\alpha_{j*}\big) - 3\beta_{jk} = O\big(m^{-1/2+2\varepsilon} + n^{-1/2+2\varepsilon}\big). \end{aligned}$$

Note that the $O(\,)$ estimates in the last four lines are uniform over j, j', k, k'.

4.3. Estimating the main part of the integral

Define $E_2 = \exp(L_2)$. We have shown that the value of the integrand in $\mathcal{Q}\cap\mathcal{M}$ is $E_1 = E_2(1+\widetilde{O}(n^{-1/2}))$. Denote the complement of the region $\mathcal{M}$ by $\mathcal{M}^c$. We can approximate our integral as follows:

$$\begin{aligned}\int_{\mathcal{Q}\cap\mathcal{M}} E_1 &= \int_{\mathcal{Q}\cap\mathcal{M}} E_2 + \widetilde{O}\big(n^{-1/2}\big) \int_{\mathcal{Q}\cap\mathcal{M}} |E_2| \\ &= \int_{\mathcal{Q}\cap\mathcal{M}} E_2 + \widetilde{O}\big(n^{-1/2}\big) \int_{\mathcal{Q}} |E_2| \\ &= \int_{\mathcal{Q}} E_2 + O(1) \int_{\mathcal{Q}\cap\mathcal{M}^c} |E_2| + \widetilde{O}\big(n^{-1/2}\big) \int_{\mathcal{Q}} |E_2|. \end{aligned} \tag{37}$$

It suffices to estimate the value of each integral in (37).

We first compute the integral of E_2 over $\mathcal{Q}$. We proceed in three stages, starting with integration with respect to μ. For the latter, we can use the formula

$$\int_{-(mn)^{-1/2+2\varepsilon}}^{(mn)^{-1/2+2\varepsilon}} \exp\big(-Amn\mu^2 - i\beta\mu\big)\,d\mu = \left(\frac{\pi}{Amn}\right)^{1/2} \exp\left(-\frac{\beta^2}{4Amn} + O\big(n^{-1}\big)\right),$$

provided $\beta = o(A(mn)^{1/2+2\varepsilon})$. In our case, $\beta = 3A_3(n\mu_2 + m\nu_2)$, which is small enough because of the assumptions $m = o(A^2 n^{1+\varepsilon})$ and $n = o(A^2 m^{1+\varepsilon})$. Therefore, integration over μ contributes

$$\left(\frac{\pi}{Amn}\right)^{1/2} \exp\left(\frac{-9A_3^2(n\mu_2 + m\nu_2)^2}{4Amn} + O\big(n^{-1}\big)\right). \tag{38}$$

The second step is to integrate with respect to $\boldsymbol{\sigma}$ the integrand

$$\begin{aligned}&\exp\Bigg(-An\mu_2 - \sum_{j=1}^{m-1} \alpha_{j*}\sigma_j^2 - \frac{9A_3^2 n}{4Am}\mu_2^2 - iA_3 n\mu_3 - 3iA_3 cn\mu_1\mu_2 \\ &\quad - i\sum_{j=1}^{m-1} \beta_{j*}\sigma_j^3 + i\sum_{j,j'=1}^{m-1} g_{jj'}\sigma_j\sigma_{j'}^2 + i\sum_{j=1}^{m-1}\sum_{k=1}^{n-1}\big(u_{jk}\sigma_j\tau_k^2 + v_{jk}\sigma_j^2\tau_k\big) \\ &\quad + \left(6A_4 - \frac{9A_3^2}{2A}\right)\mu_2\nu_2 + A_4 n\mu_4 + O\big(n^{-1}\big)\Bigg). \end{aligned} \tag{39}$$

This is accomplished by an appeal to Theorem 4, presented in Appendix A. In the terminology of that theorem, we have $N = m-1$, $\delta(N) = O(n^{-1})$, $\varepsilon' = \frac{3}{2}\varepsilon$, $\varepsilon'' = \frac{5}{3}\varepsilon$, $\varepsilon''' = 3\varepsilon$, $\bar{\varepsilon} = 6\varepsilon$, and $\hat{\varepsilon}(N) = \varepsilon + o(1)$ is defined by $2n^{-1/2+\varepsilon} = N^{-1/2+\hat{\varepsilon}}$. Furthermore,

$$\begin{aligned}&\hat{A} = \frac{An}{m-1}, \qquad \hat{a}_j = -\alpha_{j*} + \left(6A_4 - \frac{9A_3^2}{2A}\right)\nu_2 + i\sum_{k=1}^{n-1} v_{jk}\tau_k, \\ &\hat{B}_j = -\frac{iA_3 n}{m-1} - \frac{i}{m-1}\beta_{j*}, \qquad \hat{C}_{jj'} = -3iA_3 cn + ig_{jj'}, \\ &\hat{E}_j = \frac{A_4 n}{m-1}, \qquad \hat{F}_{jj'} = -\frac{9A_3^2 n}{4Am}, \\ &\hat{J}_j = i\sum_{k=1}^{n-1} u_{jk}\tau_k^2. \end{aligned}$$

We can take $\Delta = \frac{3}{4}$, and calculate that

$$\frac{3}{4\hat{A}^2 N}\sum_{j=1}^{N}\hat{E}_j + \frac{1}{4\hat{A}^2N^2}\sum_{j,j'=1}^{N}\hat{F}_{jj'} = \frac{m}{n}\left(\frac{3A_4}{4A^2} - \frac{9A_3^2}{16A^3}\right) + \widetilde{O}\big(n^{-1}\big),$$

$$\frac{15}{16\hat{A}^3 N}\sum_{j=1}^{N}\hat{B}_j^2 + \frac{3}{8\hat{A}^3N^2}\sum_{j,j'=1}^{N}\hat{B}_j\hat{C}_{jj'} + \frac{1}{16\hat{A}^3N^3}\sum_{j,j',j''=1}^{N}\hat{C}_{jj'}\hat{C}_{jj''}$$
$$= -\frac{3A_3^2 m}{8A^3 n} + \widetilde{O}\big(n^{-1}\big),$$

$$\frac{1}{2\hat{A}N}\sum_{j=1}^{N}\hat{a}_j + \frac{1}{4\hat{A}^2N^2}\sum_{j=1}^{N}\hat{a}_j^2 = -\frac{1}{2An}\alpha_{**} + \frac{1}{4A^2n^2}\sum_{j=1}^{m-1}(\alpha_{j*})^2$$
$$+ \frac{m}{n}\left(\frac{3A_4}{A} - \frac{9A_3^2}{4A^2}\right)\nu_2 + \frac{i}{2An}\sum_{k=1}^{n-1}\upsilon_{*k}\tau_k + \widetilde{O}\big(n^{-1/2}\big),$$

$$\hat{Z} = Z_1 = \exp\left(\frac{3A_3^2 m}{8A^3 n} + \widetilde{O}(n^{-1})\right) = O(1)\exp\left(\frac{(1-2\lambda)^2 m}{24An}\right). \tag{40}$$

Applying Theorem 4, we see that $\Theta_2 = \widetilde{O}(n^{-1/2})$, and so integration with respect to $\boldsymbol{\sigma}$ contributes a $\boldsymbol{\tau}$-free factor

$$\left(\frac{\pi}{An}\right)^{(m-1)/2}\exp\left(\frac{m}{n}\left(\frac{3A_4}{4A^2} - \frac{15A_3^2}{16A^3}\right) - \frac{1}{2An}\alpha_{**}\right.$$
$$\left. + \frac{1}{4A^2n^2}\sum_{j=1}^{m-1}(\alpha_{j*})^2 + \widetilde{O}\big(n^{-1/2}\big) + O\big(n^{-3/4}Z_1\big)\right). \tag{41}$$

By the conditions of Theorem 1, $Z_1 \leqslant n^{1/5}$, so $\widetilde{O}(n^{-1/2}) + O(n^{-3/4}Z_1) = \widetilde{O}(n^{-1/2}) = o(1)$ as required by Theorem 4.

Finally, we need to integrate over $\boldsymbol{\tau}$. Collecting the remaining terms from (36), and the terms involving $\boldsymbol{\tau}$ from (38) and (40), we have an integrand equal to

$$\exp\left(-Am\nu_2 + \left(\frac{3A_4 m}{An} - \frac{9A_3^2 m}{4A^2 n}\right)\nu_2 - \frac{9A_3^2 m}{4An}\nu_2^2 + A_4 m\nu_4 - iA_3 m\nu_3 - 3iA_3 dm\nu_2\nu_1\right.$$
$$\left. - \sum_{k=1}^{n-1}\alpha_{*k}\tau_k^2 - i\sum_{k=1}^{n-1}\beta_{*k}\tau_k^3 + \frac{i}{2An}\sum_{k=1}^{n-1}\upsilon_{*k}\tau_k + i\sum_{k,k'=1}^{n-1}h_{kk'}\tau_k\tau_{k'}^2 + \widetilde{O}\big(n^{-1/2}\big)\right).$$

In the terminology of Theorem 4, $N = n-1$, $\delta(N) = \widetilde{O}(n^{-1/2})$, $\varepsilon' = \frac{3}{2}\varepsilon$, $\varepsilon'' = \frac{5}{3}\varepsilon$, $\varepsilon''' = 3\varepsilon$, $\bar{\varepsilon} = 4\varepsilon$, and $\hat{\varepsilon}(N) = \varepsilon + o(1)$ is defined by $2m^{-1/2+\varepsilon} = N^{-1/2+\hat{\varepsilon}}$. Furthermore,

$$\hat{A} = \frac{Am}{n-1}, \qquad \hat{a}_k = \frac{3A_4 m}{An} - \frac{9A_3^2 m}{4A^2 n} - \alpha_{*k},$$
$$\hat{B}_k = -\frac{iA_3 m}{n-1} - \frac{i}{n-1}\beta_{*k}, \qquad \hat{C}_{kk'} = -3iA_3 dm + ih_{kk'},$$

$$\hat{E}_k = \frac{A_4 m}{n-1}, \qquad \hat{F}_{kk'} = -\frac{9A_3^2 m}{4An},$$
$$\hat{J}_k = \frac{i}{2An} v_{*k}.$$

We can take $\Delta = \frac{3}{4}$ again and calculate that

$$\frac{3}{4\hat{A}^2 N}\sum_{k=1}^{N}\hat{E}_k + \frac{1}{4\hat{A}^2N^2}\sum_{k,k'=1}^{N}\hat{F}_{kk'} = \frac{n}{m}\left(\frac{3A_4}{4A^2} - \frac{9A_3^2}{16A^3}\right) + \widetilde{O}\big(n^{-1}\big),$$
$$\frac{15}{16\hat{A}^3 N}\sum_{k=1}^{N}\hat{B}_k^2 + \frac{3}{8\hat{A}^3N^2}\sum_{k,k'=1}^{N}\hat{B}_j\hat{C}_{kk'} + \frac{1}{16\hat{A}^3N^3}\sum_{k,k',k''=1}^{N}\hat{C}_{kk'}\hat{C}_{kk''}$$
$$= -\frac{3A_3^2 n}{8A^3 m} + \widetilde{O}\big(n^{-1}\big),$$
$$\frac{1}{2\hat{A}N}\sum_{k=1}^{N}\hat{a}_k + \frac{1}{4\hat{A}^2N^2}\sum_{k=1}^{N}\hat{a}_k^2 = -\frac{1}{2Am}\alpha_{**} + \frac{1}{4A^2m^2}\sum_{k=1}^{n-1}(\alpha_{*k})^2$$
$$- \frac{9A_3^2}{8A^3} + \frac{3A_4}{2A^2} + \widetilde{O}\big(n^{-1/2}\big),$$
$$\hat{Z} = Z_2 = \exp\left(\frac{3A_3^2 n}{8A^3 m} + \widetilde{O}\big(n^{-1}\big)\right) = O(1)\exp\left(\frac{(1-2\lambda)^2 n}{24Am}\right).$$

We again find that $\Theta_2 = \widetilde{O}(n^{-1/2})$. Including the contributions from (38) and (41), we obtain

$$\int_{\mathcal{Q}} E_2 = \left(\frac{\pi}{Amn}\right)^{1/2}\left(\frac{\pi}{An}\right)^{(m-1)/2}\left(\frac{\pi}{Am}\right)^{(n-1)/2}$$
$$\times \exp\left(-\frac{9A_3^2}{8A^3} + \frac{3A_4}{2A^2} + \left(\frac{m}{n} + \frac{n}{m}\right)\left(\frac{3A_4}{4A^2} - \frac{15A_3^2}{16A^3}\right)\right.$$
$$- \left(\frac{1}{2Am} + \frac{1}{2An}\right)\alpha_{**} + \frac{1}{4A^2m^2}\sum_{k=1}^{n-1}(\alpha_{*k})^2$$
$$\left. + \frac{1}{4A^2n^2}\sum_{j=1}^{m-1}(\alpha_{j*})^2 + \widetilde{O}\big(n^{-1/2}\big)Z_2\right). \tag{42}$$

Using (13) and the conditions of Theorem 1, we calculate that

$$\alpha_{**} = -\frac{1}{2}\left(\frac{R}{n} + \frac{C}{m}\right) + \widetilde{O}\big(n^{1/2}\big),$$
$$\sum_{j=1}^{m-1}(\alpha_{j*})^2 = \frac{1}{4}(1-2\lambda)^2 R + \widetilde{O}\big(n^{3/2}\big),$$
$$\sum_{k=1}^{n-1}(\alpha_{*k})^2 = \frac{1}{4}(1-2\lambda)^2 C + \widetilde{O}\big(n^{3/2}\big),$$
$$\widetilde{O}\big(n^{-1/2}\big)Z_2 = \widetilde{O}\big(n^{-1/2}\big)n^{2a/5} = O\big(n^{-b}\big).$$

Substituting these values into (42) together with the actual values of A, A_3, A_4, we conclude that

$$\begin{aligned}\int_{\mathcal{Q}} E_2 &= \left(\frac{\pi}{Amn}\right)^{1/2}\left(\frac{\pi}{An}\right)^{(m-1)/2}\left(\frac{\pi}{Am}\right)^{(n-1)/2}\\ &\quad\times\exp\left(-\frac{1}{2}-\frac{1-2A}{24A}\left(\frac{m}{n}+\frac{n}{m}\right)+\frac{1}{4A}\left(\frac{1}{m}+\frac{1}{n}\right)\left(\frac{R}{n}+\frac{C}{m}\right)\right.\\ &\qquad\left.+\frac{1-8A}{16A^2}\left(\frac{R}{n^2}+\frac{C}{m^2}\right)+O\bigl(n^{-b}\bigr)\right).\end{aligned}\tag{43}$$

We next infer a estimate of $\int_{\mathcal{Q}}|E_2|$. The calculation that leads to (42) remains valid if we set all the values A_3, β_{jk}, $g_{jj'}$, $h_{kk'}$, u_{jk} and v_{jk} to zero, which is the same as replacing L_2 by its real part. Since $|E_2| = \exp(\mathrm{Re}(L_2))$, this gives

$$\begin{aligned}\int_{\mathcal{Q}}|E_2| &= \exp\left(\frac{9A_3^2}{8A^3}+\frac{15A_3^2}{16A^3}\left(\frac{m}{n}+\frac{n}{m}\right)+o(1)\right)\int_{\mathcal{Q}} E_2\\ &= \exp\left(\frac{(1-2\lambda)^2}{8A}\left(1+\frac{5n}{6m}+\frac{5m}{6n}\right)+o(1)\right)\int_{\mathcal{Q}} E_2\\ &= O\bigl(n^a\bigr)\int_{\mathcal{Q}} E_2\end{aligned}\tag{44}$$

under the assumptions of Theorem 1. The third term of (37) can now be identified:

$$\widetilde{O}\bigl(n^{-1/2}\bigr)\int_{\mathcal{Q}}|E_2| = \widetilde{O}\bigl(n^{-1/2}\bigr)n^a\int_{\mathcal{Q}} E_2 = O\bigl(n^{-b}\bigr)\int_{\mathcal{Q}} E_2,\tag{45}$$

where, as always, we suppose that ε is sufficiently small.

Finally, we consider the second term of (37), namely

$$\int_{\mathcal{Q}\cap\mathcal{M}^c}|E_2|,$$

which we will bound as a fraction of $\int_{\mathcal{Q}}|E_2|$ using a statistical technique. The following is a well-known result of Hoeffding [3].

Lemma 3. *Let $X_1, X_2, \ldots, X_N$ be independent random variables such that $\mathbb{E}X_i = 0$ and $|X_i| \leqslant M$ for all i. Then, for any $t \geqslant 0$,*

$$\mathrm{Prob}\left(\sum_{i=1}^{N} X_i \geqslant t\right) \leqslant \exp\left(-\frac{t^2}{2NM^2}\right).$$

Now consider $|E_2| = \exp(\mathrm{Re}(L_2))$. Write $\mathcal{M} = \mathcal{M}_1 \cap \mathcal{M}_2$, where

$$\mathcal{M}_1 = \bigl\{|\mu_1| \leqslant m^{1/2}n^{-1/2+3\varepsilon}\bigr\} \quad\text{and}\quad \mathcal{M}_2 = \bigl\{|\nu_1| \leqslant n^{1/2}m^{-1/2+3\varepsilon}\bigr\}.$$

For fixed values of μ and $\boldsymbol{\sigma}$, $\mathrm{Re}(L_2)$ separates over $\tau_1, \tau_2, \ldots, \tau_{n-1}$ and therefore, apart from normalization, it is the joint density of independent random variables $X_1, X_2, \ldots, X_{n-1}$ which

satisfy $\mathbb{E}\,X_k = 0$ (by symmetry) and $|X_k| \leqslant 2m^{-1/2+\varepsilon}$ (by the definition of $\mathcal{Q}$). By Lemma 3, the fraction of the integral over $\boldsymbol{\tau}$ (for fixed $\mu, \boldsymbol{\sigma}$) that has $\nu_1 \geqslant n^{1/2}m^{-1/2+3\varepsilon}$ is at most $\exp(-m^{4\varepsilon}/2)$. By symmetry, the same bound holds for $\nu_1 \leqslant -n^{1/2}m^{-1/2+3\varepsilon}$. Since these bounds are independent of μ and $\boldsymbol{\sigma}$, we have

$$\int\limits_{\mathcal{Q}\cap\mathcal{M}_2^c} |E_2| \leqslant 2\exp\bigl(-m^{4\varepsilon}/2\bigr) \int\limits_{\mathcal{Q}} |E_2|.$$

By the same argument,

$$\int\limits_{\mathcal{Q}\cap\mathcal{M}_1^c} |E_2| \leqslant 2\exp\bigl(-n^{4\varepsilon}/2\bigr) \int\limits_{\mathcal{Q}} |E_2|.$$

Therefore we have in total that

$$\int\limits_{\mathcal{Q}\cap\mathcal{M}^c} |E_2| \leqslant 2\bigl(\exp\bigl(-m^{4\varepsilon}/2\bigr) + \exp\bigl(-n^{4\varepsilon}/2\bigr)\bigr) \int\limits_{\mathcal{Q}} |E_2| \leqslant O\bigl(n^{-b}\bigr) \int\limits_{\mathcal{Q}} E_2, \tag{46}$$

as for (45). Applying (37) with (43), (45) and (46), we find that $\int_{\mathcal{Q}\cap\mathcal{M}} E_1$ is given by (43). Multiplying by the Jacobians of the transformations T_2 and T_3, we find that Theorem 2 is proved for $\mathcal{S}'$ given by (33).

5. Bounding the remainder of the integral

In the previous section, we estimated the value of the integral $I_{\mathcal{R}'}(\boldsymbol{s}, \boldsymbol{t})$, which is the same as $I(\boldsymbol{s}, \boldsymbol{t})$ except that it is restricted to a certain region $\mathcal{R}' \supseteq \mathcal{R}$ (see (21)–(23)). In this section, we extend this to an estimate of $I(\boldsymbol{s}, \boldsymbol{t})$ by showing that the remainder of the region of integration contributes negligibly.

Precisely, we show the following.

Theorem 3. *Let $F(\boldsymbol{\theta}, \boldsymbol{\phi})$ be the integrand of $I(\boldsymbol{s}, \boldsymbol{t})$ as defined in* (21)*. Then, under the conditions of Theorem* 1*,*

$$\int\limits_{\mathcal{R}^c} \bigl|F(\boldsymbol{\theta}, \boldsymbol{\phi})\bigr|\, d\boldsymbol{\theta}\, d\boldsymbol{\phi} = O\bigl(n^{-1}\bigr) \int\limits_{\mathcal{R}'} F(\boldsymbol{\theta}, \boldsymbol{\phi})\, d\boldsymbol{\theta}\, d\boldsymbol{\phi}.$$

For $1 \leqslant j \leqslant m$, $1 \leqslant k \leqslant n$, let $A_{jk} = A + \alpha_{jk} = \frac{1}{2}\lambda_{jk}(1 - \lambda_{jk})$ (recall (25)), and define $A_{\min} = \min_{jk} A_{jk} = A + \widetilde{O}(n^{-1/2})$. We begin with two technical lemmas whose proofs are omitted.

Lemma 4.

$$\bigl|F(\boldsymbol{\theta}, \boldsymbol{\phi})\bigr| = \prod_{j=1}^{m} \prod_{k=1}^{n} f_{jk}(\theta_j + \phi_k),$$

where

$$f_{jk}(z) = \sqrt{1 - 4A_{jk}(1 - \cos z)}.$$

Moreover, for all real z,

$$0 \leqslant f_{jk}(z) \leqslant \exp\left(-A_{jk}z^2 + \frac{1}{12}A_{jk}z^4\right).$$

Lemma 5. *For all $c > 0$,*

$$\int_{-8\pi/75}^{8\pi/75} \exp\left(c\left(-x^2 + \frac{7}{3}x^4\right)\right) dx \leqslant \sqrt{\pi/c}\exp(3/c).$$

Proof of Theorem 3. Our approach will be to bound $\int |F(\boldsymbol{\theta},\boldsymbol{\phi})|$ over a variety of regions whose union covers $\mathcal{R}^c$. To make the comparison of these bounds with $\int_{\mathcal{R}'} F(\boldsymbol{\theta},\boldsymbol{\phi})$ easier, we note that

$$\int_{\mathcal{R}'} F(\boldsymbol{\theta},\boldsymbol{\phi})\, d\boldsymbol{\theta}\, d\boldsymbol{\phi} = \exp\bigl(O\bigl(m^{\varepsilon}+n^{\varepsilon}\bigr)\bigr)I_0 = \exp\bigl(O\bigl(m^{3\varepsilon}+n^{3\varepsilon}\bigr)\bigr)I_1, \tag{47}$$

where

$$I_0 = \left(\frac{\pi}{A_{\bullet\bullet}}\right)^{1/2} \prod_{j=1}^{m}\left(\frac{\pi}{A_{j\bullet}}\right)^{1/2} \prod_{k=1}^{n}\left(\frac{\pi}{A_{\bullet k}}\right)^{1/2},$$

$$I_1 = \left(\frac{\pi}{An}\right)^{m/2}\left(\frac{\pi}{Am}\right)^{n/2}.$$

To see this, expand

$$A_{j\bullet} = An + \alpha_{j\bullet} = An\exp\left(\frac{\alpha_{j\bullet}}{An} - \frac{\alpha_{j\bullet}^2}{2A^2n^2} + \cdots\right),$$

and similarly for $A_{\bullet k}$, and compare the result to Theorem 2 using the assumptions of Theorem 1. It may help to recall the calculation following (42).

Take $\kappa = \pi/300$ and define $x_0, x_1, \ldots, x_{299}$ by $x_\ell = 2\ell\kappa$. For any ℓ, let $\mathcal{S}_1(\ell)$ be the set of $(\boldsymbol{\theta},\boldsymbol{\phi})$ such that $\theta_j \in [x_\ell - \kappa, x_\ell + \kappa]$ for at least $\kappa m/\pi$ values of j and $\phi_k \notin [-x_\ell - 2\kappa, -x_\ell + 2\kappa]$ for at least n^{ε} values of k. For $(\boldsymbol{\theta},\boldsymbol{\phi}) \in \mathcal{S}_1(\ell)$, $\theta_j + \phi_k \notin [-\kappa,\kappa]$ for at least $\kappa m n^{\varepsilon}/\pi$ pairs (j,k) so, by Lemma 4, $|F(\boldsymbol{\theta},\boldsymbol{\phi})| \leqslant \exp(-c_1 A_{\min} m n^{\varepsilon})$ for some $c_1 > 0$ which is independent of ℓ.

Next define $\mathcal{S}_2(\ell)$ to be the set of $(\boldsymbol{\theta},\boldsymbol{\phi})$ such that $\theta_j \in [x_\ell - \kappa, x_\ell + \kappa]$ for at least $\kappa m/\pi$ values of j, $\phi_k \in [-x_\ell - 2\kappa, -x_\ell + 2\kappa]$ for at least $n - n^{\varepsilon}$ values of k and $\theta_j \notin [x_\ell - 3\kappa, x_\ell + 3\kappa]$ for at least m^{ε} values of j. By the same argument with the roles of $\boldsymbol{\theta}$ and $\boldsymbol{\phi}$ reversed, $|F(\boldsymbol{\theta},\boldsymbol{\phi})| \leqslant \exp(-c_2 A_{\min} m^{\varepsilon} n)$ for some $c_2 > 0$ independent of ℓ when $(\boldsymbol{\theta},\boldsymbol{\phi}) \in \mathcal{S}_2(\ell)$.

Now define $\mathcal{R}_1(\ell)$ to be the set of pairs $(\boldsymbol{\theta},\boldsymbol{\phi})$ such that $\theta_j \in [x_\ell - 3\kappa, x_\ell + 3\kappa]$ for at least $m - m^{\varepsilon}$ values of j, and $\phi_k \in [-x_\ell - 3\kappa, -x_\ell + 3\kappa]$ for at least $n - n^{\varepsilon}$ values of k. By the pigeonhole principle, for any $\boldsymbol{\theta}$ there is some ℓ such that $[x_\ell - \kappa, x_\ell + \kappa]$ contains at least $\kappa m/\pi$ values of θ_j. Therefore,

$$\left(\bigcup_{\ell=0}^{299} \mathcal{R}_1(\ell)\right)^c \subseteq \bigcup_{\ell=0}^{299} \bigl(\mathcal{S}_1(\ell) \cup \mathcal{S}_2(\ell)\bigr).$$

Since the total volume of $(\bigcup_\ell \mathcal{R}_1(\ell))^c$ is at most $(2m)^{m+n}$, we find that for some $c_3 > 0$,

$$\int\limits_{(\bigcup_\ell \mathcal{R}_1(\ell))^c} \bigl|F(\boldsymbol{\theta},\boldsymbol{\phi})\bigr|\,d\boldsymbol{\theta}\,d\boldsymbol{\phi} \leqslant (2\pi)^{m+n}\bigl(\exp(-c_3 A_{\min} m n^{\varepsilon}) + \exp\bigl(-c_3 A_{\min} m^{\varepsilon} n\bigr)\bigr)$$
$$\leqslant e^{-n} I_1. \tag{48}$$

We are left with $(\boldsymbol{\theta},\boldsymbol{\phi}) \in \bigcup_\ell \mathcal{R}_1(\ell)$. If we subtract x_ℓ from each θ_j and add x_ℓ to each ϕ_k the integrand $F(\boldsymbol{\theta},\boldsymbol{\phi})$ is unchanged, so we can assume for convenience that $\ell = 0$ and that $(\boldsymbol{\theta},\boldsymbol{\phi}) \in \mathcal{R}_1 = \mathcal{R}_1(0)$. The bounds we obtain on parts of the integral we seek to reject will be at least $1/300$ of the total and thus be of the right order of magnitude. We will not mention this point again.

For a given $\boldsymbol{\theta}$, partition $\{1,2,\ldots,m\}$ into sets $J_0 = J_0(\boldsymbol{\theta})$, $J_1 = J_1(\boldsymbol{\theta})$ and $J_2 = J_2(\boldsymbol{\theta})$, containing the indices j such that $|\theta_j| \leqslant 3\kappa$, $3\kappa < |\theta_j| \leqslant 15\kappa$ and $|\theta_j| > 15\kappa$, respectively. Similarly partition $\{1,2,\ldots,n\}$ into $K_0 = K_0(\boldsymbol{\phi})$, $K_1 = K_1(\boldsymbol{\phi})$ and $K_2 = K_2(\boldsymbol{\phi})$. The value of $|F(\boldsymbol{\theta},\boldsymbol{\phi})|$ can now be bounded using

$$f_{jk}(\theta_j+\phi_k) \leqslant \begin{cases} \exp(-A_{\min}(\theta_j+\phi_k)^2 + \frac{1}{12}A_{\min}(\theta_j+\phi_k)^4) & \text{if } (j,k) \in (J_0\cup J_1)\times(K_0\cup K_1),\\ \sqrt{1-4A_{\min}(1-\cos(12\kappa))} \leqslant e^{-A_{\min}/64} & \text{if } (j,k) \in (J_0\times K_2)\cup(J_2\times K_0),\\ 1 & \text{otherwise.}\end{cases}$$

Let $I_2(m_2,n_2)$ be the contribution to $\int_{\mathcal{R}_1}|F(\boldsymbol{\theta},\boldsymbol{\phi})|$ of those $(\boldsymbol{\theta},\boldsymbol{\phi})$ with $|J_2| = m_2$ and $|K_2| = n_2$. Recall that $|J_0| > m - m^{\varepsilon}$ and $|K_0| > n - n^{\varepsilon}$. We have

$$I_2(m_2,n_2) \leqslant \binom{m}{m_2}\binom{n}{n_2}(2\pi)^{m_2+n_2} \times \exp\biggl(-\frac{1}{64}A_{\min}\bigl(n-n^{\varepsilon}\bigr)m_2 - \frac{1}{64}A_{\min}\bigl(m-m^{\varepsilon}\bigr)n_2\biggr) I_2'(m_2,n_2), \tag{49}$$

where

$$I_2'(m_2,n_2) = \int_{-15\kappa}^{15\kappa}\cdots\int_{-15\kappa}^{15\kappa} \exp\biggl(-A_{\min}\sum_{jk}{}'(\theta_j+\phi_k)^2 + \frac{1}{12}A_{\min}\sum_{jk}{}'(\theta_j+\phi_k)^4\biggr)\,d\boldsymbol{\theta}'\,d\boldsymbol{\phi}',$$

and the primes denote restriction to $j \in J_0\cup J_1$ and $k \in K_0\cup K_1$. Write $m' = m - m_2$ and $n' = n - n_2$ and define $\bar{\boldsymbol{\theta}}' = (m')^{-1}\sum_j{}'\theta_j$, $\breve{\theta}_j = \theta_j - \bar{\boldsymbol{\theta}}'$ for $j \in J_0\cup J_1$, $\bar{\boldsymbol{\phi}}' = (n')^{-1}\sum_k{}'\phi_k$, $\breve{\phi}_k = \phi_k - \bar{\boldsymbol{\phi}}'$ for $k \in K_0\cup K_1$, $\nu' = \bar{\boldsymbol{\phi}}' + \bar{\boldsymbol{\theta}}'$ and $\delta' = \bar{\boldsymbol{\theta}}' - \bar{\boldsymbol{\phi}}'$. Change variables from $(\boldsymbol{\theta}',\boldsymbol{\phi}')$ to $\{\breve{\theta}_j \mid j \in J_3\}\cup\{\breve{\phi}_k \mid k \in K_3\}\cup\{\nu',\delta'\}$, where J_3 is some subset of $m'-1$ elements of $J_0\cup J_1$ and K_3 is some subset of $n'-1$ elements of $K_0\cup K_1$. From Section 4 we know that the Jacobian of this transformation is $m'n'/2$. The integrand of I_2' can now be bounded using

$$\sum_{jk}{}'(\theta_j+\phi_k)^2 = n'\sum_j{}'\breve{\theta}_j^2 + m'\sum_k{}'\breve{\phi}_k^2 + m'n'\nu'^2$$

and

$$\sum_{jk}{}'(\theta_j+\phi_k)^4 \leqslant 27n'\sum_j{}'\breve{\theta}_j^4 + 27m'\sum_k{}'\breve{\phi}_k^4 + 27m'n'\nu'^4.$$

The latter follows from the inequality $(x+y+z)^4 \leqslant 27(x^4+y^4+z^4)$ valid for all x,y,z. Therefore,

$$I_2'(m_2,n_2) \leqslant \frac{O(1)}{m'n'} \int_{-30\kappa}^{30\kappa}\int_{-30\kappa}^{30\kappa}\cdots\int_{-30\kappa}^{30\kappa} \exp\Bigg(A_{\min}n'\sum_j{}' g(\breve{\theta}_j) + A_{\min}m'\sum_k{}' g(\breve{\phi}_k) + A_{\min}m'n'g(\nu')\Bigg) d\breve{\theta}_{j\in J_3}\, d\breve{\phi}_{k\in K_3}\, d\nu',$$

where $g(z) = -z^2 + \frac{9}{4}z^4$. Since $g(z) \leqslant 0$ for $|z| \leqslant 30\kappa$, and we only need an upper bound, we can restrict the summations in the integrand to $j \in J_3$ and $k \in K_3$. The integral now separates into $m' + n' - 1$ one-dimensional integrals and Lemma 5 (by monotonicity) gives that

$$I_2'(m_2,n_2) = O(1)\frac{\pi^{(m'+n')/2}}{A_{\min}^{(m'+n'-1)/2}(m')^{n'/2-1}(n')^{m'/2-1}} \times \exp\big(O\big(m'/(A_{\min}n') + n'/(A_{\min}m')\big)\big).$$

Applying (47) and (49), we find that

$$\sum_{\substack{m_2=0\\ m_2+n_2\geqslant 1}}^{m^\varepsilon}\sum_{n_2=0}^{n^\varepsilon} I_2(m_2,n_2) = O\big(e^{-c_4Am} + e^{-c_4An}\big)I_1 \tag{50}$$

for some $c_4 > 0$.

We have now bounded contributions to the integral of $|F(\boldsymbol{\theta},\boldsymbol{\phi})|$ from everywhere outside the region

$$\mathcal{X} = \big\{(\boldsymbol{\theta},\boldsymbol{\phi}) \;\big|\; |\theta_j|, |\phi_k| \leqslant 15\kappa \text{ for } 1 \leqslant j \leqslant m,\ 1 \leqslant k \leqslant n\big\}.$$

By Lemma 4, we have for $(\boldsymbol{\theta},\boldsymbol{\phi}) \in \hat{C}^{m+n}$ (which includes $\mathcal{X}$) that

$$\big|F(\boldsymbol{\theta},\boldsymbol{\phi})\big| \leqslant \exp\Bigg(-\sum_{j=1}^m\sum_{k=1}^n A_{jk}(\hat{\theta}_j + \hat{\phi}_k + \nu)^2 + \frac{1}{12}\sum_{j=1}^m\sum_{k=1}^n A_{jk}(\hat{\theta}_j + \hat{\phi}_k + \nu)^4\Bigg),$$

where $\hat{\theta}_j = \theta_j - \bar{\boldsymbol{\theta}}$, $\hat{\phi}_k = \phi_k - \bar{\boldsymbol{\phi}}$ and $\nu = \bar{\boldsymbol{\theta}} + \bar{\boldsymbol{\phi}}$. As before, the integrand is independent of $\delta = \bar{\boldsymbol{\theta}} - \bar{\boldsymbol{\phi}}$ and our notation will tend to ignore δ for that reason; for our bounds it will suffice to remember that δ has a bounded range.

We proceed by exactly diagonalizing the $(m+n+1)$-dimensional quadratic form. Since $\sum_{j=1}^m \hat{\theta}_j = \sum_{k=1}^n \hat{\phi}_k = 0$, we have

$$\sum_{j=1}^m\sum_{k=1}^n A_{jk}(\hat{\theta}_j + \hat{\phi}_k + \nu)^2 = \sum_{j=1}^m A_{j\bullet}\hat{\theta}_j^2 + \sum_{k=1}^n A_{\bullet k}\hat{\phi}_k^2 + A_{\bullet\bullet}\nu^2 + 2\sum_{j=1}^m\sum_{k=1}^n \alpha_{jk}\hat{\theta}_j\hat{\phi}_k + 2\nu\sum_{j=1}^m \alpha_{j\bullet}\hat{\theta}_j + 2\nu\sum_{k=1}^n \alpha_{\bullet k}\hat{\phi}_k.$$

This is almost diagonal, because $\alpha_{jk} = \widetilde{O}(n^{-1/2})$, and we can correct it with the slight additional transformation $(I + X^{-1}Y)^{-1/2}$ described by Lemma 1, where X is a diagonal matrix with diagonal entries $A_{j\bullet}$, $A_{\bullet k}$ and $A_{\bullet\bullet}$. The matrix Y has zero diagonal and other entries of magnitude $\widetilde{O}(n^{-1/2})$ apart from the row and column indexed by ν, which have entries of magnitude $\widetilde{O}(n^{1/2})$. By the same argument as used in Section 4.2, all eigenvalues of $X^{-1}Y$ have

magnitude $\widetilde{O}(n^{-1/2})$, so the transformation is well defined. The new variables $\{\hat{\vartheta}_j\}$, $\{\hat{\varphi}_k\}$ and $\dot{\nu}$ are related to the old by

$$(\hat{\theta}_1,\dots,\hat{\theta}_m,\hat{\phi}_1,\dots,\hat{\phi}_n,\nu)^T=\big(I+X^{-1}Y\big)^{-1/2}(\hat{\vartheta}_1,\dots,\hat{\vartheta}_m,\hat{\varphi}_1,\dots,\hat{\varphi}_n,\dot{\nu})^T.$$

We will keep the variable δ as a variable of integration but, as noted before, our notation will generally ignore it.

More explicitly, for some $d_1,\dots,d_m,d'_1,\dots,d'_n=\widetilde{O}(n^{-3/2})$, we have uniformly over $j=1,\dots,m$, $k=1,\dots,n$ that

$$\begin{aligned}
\hat{\theta}_j&=\hat{\vartheta}_j+\sum_{q=1}^{m}\widetilde{O}\big(n^{-2}\big)\hat{\vartheta}_q+\sum_{k=1}^{n}\widetilde{O}\big(n^{-3/2}\big)\hat{\varphi}_k+\widetilde{O}\big(n^{-1/2}\big)\dot{\nu},\\
\hat{\phi}_k&=\hat{\varphi}_k+\sum_{j=1}^{m}\widetilde{O}\big(n^{-3/2}\big)\hat{\vartheta}_j+\sum_{q=1}^{n}\widetilde{O}\big(n^{-2}\big)\hat{\varphi}_q+\widetilde{O}\big(n^{-1/2}\big)\dot{\nu},\\
\nu&=\dot{\nu}+\sum_{j=1}^{m}d_j\hat{\vartheta}_j+\sum_{k=1}^{n}d'_k\hat{\varphi}_k+\widetilde{O}\big(n^{-1}\big)\dot{\nu}.
\end{aligned}\tag{51}$$

Note that the expressions $O()$ in (51) represent values that depend on $m,n,\boldsymbol{s},\boldsymbol{t}$ but not on $\{\hat{\vartheta}_j\}$, $\{\hat{\varphi}_k\}$, $\dot{\nu}$.

The region of integration $\mathcal{X}$ is $(m+n)$-dimensional. In place of the variables $(\boldsymbol{\theta},\boldsymbol{\phi})$ we can use $(\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\phi}},\nu,\delta)$ by applying the identities $\hat{\theta}_m=-\sum_{j=1}^{m-1}\hat{\theta}_j$ and $\hat{\phi}_n=-\sum_{k=1}^{n-1}\hat{\phi}_k$. (Recall that $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$ do not include $\hat{\theta}_m$ and $\hat{\phi}_n$.) The additional transformation (51) maps the two just-mentioned identities into identities that define $\hat{\vartheta}_m$ and $\hat{\varphi}_n$ in terms of $(\hat{\boldsymbol{\vartheta}},\hat{\boldsymbol{\varphi}},\dot{\nu})$, where $\hat{\boldsymbol{\vartheta}}=(\hat{\vartheta}_1,\dots,\hat{\vartheta}_{m-1})$ and $\hat{\boldsymbol{\varphi}}=(\hat{\varphi}_1,\dots,\hat{\varphi}_{n-1})$. These have the form

$$\begin{aligned}
\hat{\vartheta}_m&=-\sum_{j=1}^{m-1}\big(1+\widetilde{O}\big(n^{-1}\big)\big)\hat{\vartheta}_j+\sum_{k=1}^{n-1}\widetilde{O}\big(n^{-1/2}\big)\hat{\varphi}_k+\widetilde{O}\big(n^{1/2}\big)\dot{\nu},\\
\hat{\varphi}_n&=\sum_{j=1}^{m-1}\widetilde{O}\big(n^{-1/2}\big)\hat{\vartheta}_j-\sum_{k=1}^{n-1}\big(1+\widetilde{O}\big(n^{-1}\big)\big)\hat{\varphi}_k+\widetilde{O}\big(n^{1/2}\big)\dot{\nu}.
\end{aligned}\tag{52}$$

Therefore, we can now integrate over $(\hat{\boldsymbol{\vartheta}},\hat{\boldsymbol{\varphi}},\dot{\nu},\delta)$. The Jacobian of the transformation from $(\boldsymbol{\theta},\boldsymbol{\phi})$ to $(\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\phi}},\nu,\delta)$ is $mn/2$, as in Section 4. The Jacobian of the transformation $T_4(\hat{\boldsymbol{\vartheta}},\hat{\boldsymbol{\varphi}},\dot{\nu})=(\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\phi}},\nu)$ defined by (51) can be seen to be $1+\widetilde{O}(n^{-1/2})$ by Lemma 2, using the fact that the ∞-norm of the matrix of partial derivatives is $\widetilde{O}(n^{-1/2})$. This matrix has order $m+n-1$ and can be obtained by substituting (52) into (51).

The transformation T_4 changes the region of integration only by a factor $1+\widetilde{O}(n^{-1/2})$ in each direction, since the inverse of (51) has exactly the same form except that the constants $\{d_j\}$, $\{d'_k\}$, while still of magnitude $\widetilde{O}(n^{-3/2})$, may be different. Therefore, the image of region $\mathcal{X}$ lies inside the region

$$\mathcal{Y}=\big\{(\hat{\boldsymbol{\vartheta}},\hat{\boldsymbol{\varphi}},\dot{\nu})\;\big|\;|\hat{\vartheta}_j|,|\hat{\varphi}_k|\leqslant 31\kappa\ (1\leqslant j\leqslant m,\ 1\leqslant k\leqslant n),\ |\dot{\nu}|\leqslant 31\kappa\big\}.$$

We next bound the value of the integrand in $\mathcal{Y}$. By repeated application of the inequality $xy\leqslant\frac{1}{2}x^2+\frac{1}{2}y^2$, we find that

$$\frac{1}{12}\sum_{j=1}^{m}\sum_{k=1}^{n} A_{jk}(\hat{\theta}_j+\hat{\phi}_k+\nu)^4 \leqslant \frac{7}{3}\left(\sum_{j=1}^{m} A_{j\bullet}\hat{\vartheta}_j^4+\sum_{k=1}^{n} A_{\bullet k}\hat{\varphi}_k^4+A_{\bullet\bullet}\dot{\nu}^4\right),$$

where we have chosen $\frac{7}{3}$ as a convenient value greater than $\frac{9}{4}$. Now define $h(z)=-z^2+\frac{7}{3}z^4$. Then, for $(\hat{\boldsymbol{\vartheta}},\hat{\boldsymbol{\varphi}},\dot{\nu})\in\mathcal{Y}$,

$$\begin{aligned}\bigl|F(\boldsymbol{\theta},\boldsymbol{\phi})\bigr| &\leqslant \exp\left(\sum_{j=1}^{m} A_{j\bullet}h(\hat{\vartheta}_j)+\sum_{k=1}^{n} A_{\bullet k}h(\hat{\varphi}_k)+A_{\bullet\bullet}h(\dot{\nu})\right)\\ &\leqslant \exp\left(\sum_{j=1}^{m-1} A_{j\bullet}h(\hat{\vartheta}_j)+\sum_{k=1}^{n-1} A_{\bullet k}h(\hat{\varphi}_k)+A_{\bullet\bullet}h(\dot{\nu})\right) \qquad (53)\\ &= \exp\bigl(A_{\bullet\bullet}h(\dot{\nu})\bigr)\prod_{j=1}^{m-1}\exp\bigl(A_{j\bullet}h(\hat{\vartheta}_j)\bigr)\prod_{k=1}^{n-1}\exp\bigl(A_{\bullet k}h(\hat{\varphi}_k)\bigr), \qquad (54)\end{aligned}$$

where the second line holds because $h(z)\leqslant 0$ for $|z|\leqslant 31\kappa$.

Define

$$\begin{aligned}\mathcal{W}_0 &= \left\{(\hat{\boldsymbol{\vartheta}},\hat{\boldsymbol{\varphi}},\dot{\nu})\in\mathcal{Y}\;\middle|\; |\hat{\vartheta}_j|\leqslant\frac{1}{2}n^{-1/2+\varepsilon}\ (1\leqslant j\leqslant m-1),\right.\\ &\qquad\left. |\hat{\varphi}_k|\leqslant\frac{1}{2}m^{-1/2+\varepsilon}\ (1\leqslant k\leqslant n-1),\ |\dot{\nu}|\leqslant\frac{1}{2}(mn)^{-1/2+2\varepsilon}\right\},\\ \mathcal{W}_1 &= \mathcal{Y}-\mathcal{W}_0,\\ \mathcal{W}_2 &= \left\{(\hat{\boldsymbol{\vartheta}},\hat{\boldsymbol{\varphi}},\dot{\nu})\in\mathcal{Y}\;\middle|\;\left|\sum_{j=1}^{m-1} d_j\hat{\vartheta}_j+\sum_{k=1}^{n-1} d'_k\hat{\varphi}_k\right|\leqslant n^{-5/4}\right\}.\end{aligned}$$

Also define similar regions $\mathcal{W}'_0,\mathcal{W}'_1,\mathcal{W}'_2$ by omitting the variables $\hat{\vartheta}_1,\hat{\varphi}_1$ instead of $\hat{\vartheta}_m,\hat{\varphi}_n$ starting at (53). Using (51), we see that T_4, and the corresponding transformation that omits $\hat{\vartheta}_1$ and $\hat{\varphi}_1$, map $\mathcal{R}$ to a superset of $\mathcal{W}_0\cap\mathcal{W}_2\cap\mathcal{W}'_1\cap\mathcal{W}'_2$. Therefore, $\mathcal{X}-\mathcal{R}$ is mapped to a subset of $\mathcal{W}_1\cup(\mathcal{W}_0-\mathcal{W}_2)\cup\mathcal{W}'_1\cup(\mathcal{W}'_0-\mathcal{W}'_2)$ and it will suffice to find a tight bound on the integral in each of the four latter regions. Denoting the right side of (54) by $F_0(\hat{\boldsymbol{\vartheta}},\hat{\boldsymbol{\varphi}},\dot{\nu})$, Lemma 5 gives

$$\int_{\mathcal{Y}} F_0(\hat{\boldsymbol{\vartheta}},\hat{\boldsymbol{\varphi}},\dot{\nu})\,d\hat{\boldsymbol{\vartheta}}\,d\hat{\boldsymbol{\varphi}}\,d\dot{\nu}=\exp\bigl(O\bigl(m^{\varepsilon}+n^{\varepsilon}\bigr)\bigr)I_0. \qquad (55)$$

Also note that

$$\int_{z_0}^{31\kappa}\exp\bigl(ch(z)\bigr)=O(1)\exp\bigl(ch(z_0)\bigr) \qquad (56)$$

for $c,z_0>0$ and $z_0=o(1)$, since $h(z)\leqslant h(z_0)$ for $z_0\leqslant z\leqslant 31\kappa$. By applying (56) to each of the factors of (54) in turn,

$$\int_{\mathcal{W}_1} F_0(\hat{\boldsymbol{\vartheta}},\hat{\boldsymbol{\varphi}},\dot{\nu})\,d\hat{\boldsymbol{\vartheta}}\,d\hat{\boldsymbol{\varphi}}\,d\dot{\nu}=O\bigl(e^{-c_6Am^{2\varepsilon}}+e^{-c_6An^{2\varepsilon}}\bigr)I_0 \qquad (57)$$

for some $c_6 > 0$ and so, by (55) and (57),

$$\int_{\mathcal{W}_0} F_0(\hat{\boldsymbol{\vartheta}}, \hat{\boldsymbol{\varphi}}, \dot{\nu})\, d\hat{\boldsymbol{\vartheta}}\, d\hat{\boldsymbol{\varphi}}\, d\dot{\nu} = \exp\bigl(O\bigl(m^{\varepsilon} + n^{\varepsilon}\bigr)\bigr) I_0.$$

Applying Lemma 3 twice, once to the variables $d_1\hat{\vartheta}_1, \ldots, d_{m-1}\hat{\vartheta}_{m-1}, d'_1\hat{\varphi}_1, \ldots, d'_{n-1}\hat{\varphi}_{n-1}$ and once to their negatives, using $M = \widetilde{O}(n^{-2})$, $N = m + n - 2$ and $t = n^{-5/4}$, we find that

$$\begin{aligned}\int_{\mathcal{W}_0 - \mathcal{W}_2} F_0(\hat{\boldsymbol{\vartheta}}, \hat{\boldsymbol{\varphi}}, \dot{\nu})\, d\hat{\boldsymbol{\vartheta}}\, d\hat{\boldsymbol{\varphi}}\, d\dot{\nu} &= O\bigl(e^{-n^{1/4}}\bigr) \int_{\mathcal{W}_0} F_0(\hat{\boldsymbol{\vartheta}}, \hat{\boldsymbol{\varphi}}, \dot{\nu})\, d\hat{\boldsymbol{\vartheta}}\, d\hat{\boldsymbol{\varphi}}\, d\dot{\nu} \\ &= O\bigl(e^{-n^{1/5}}\bigr) I_0. \end{aligned} \tag{58}$$

Finally, parallel computations give the same bounds on the integrals over $\mathcal{W}'_1$ and $\mathcal{W}'_0 - \mathcal{W}'_2$.

We have now bounded $\int |F(\boldsymbol{\theta}, \boldsymbol{\phi})|$ in regions that together cover the complement of $\mathcal{R}$. Collecting these bounds from (48), (50), (57), (58), and the above-mentioned analogues of (57) and (58), we conclude that

$$\int_{\mathcal{R}^c} \bigl|F(\boldsymbol{\theta}, \boldsymbol{\phi})\bigr|\, d\boldsymbol{\theta}\, d\boldsymbol{\phi} = O\bigl(e^{-c_7 A m^{2\varepsilon}} + e^{-c_7 A n^{2\varepsilon}}\bigr) I_0$$

for some $c_7 > 0$, which implies the theorem by (47). □

Appendix A. Estimating an integral

In this appendix we estimate the value of a certain multi-dimensional integral. A similar integral appeared in [7] and variations of it appeared in [4–6]. However, none of the previously published variations meet our present requirements entirely. We will meet them here, and also introduce a new method of proof that gives a better error term.

It is intended that this appendix be notationally independent of the rest of the paper. We have used new symbols where possible, but even in the few remaining exceptions, assumptions about the values of variables stated earlier do not apply here.

Theorem 4. *Let $\varepsilon', \varepsilon'', \varepsilon''', \bar{\varepsilon}, \Delta$ be constants such that $0 < \varepsilon' < \varepsilon'' < \varepsilon'''$, $\bar{\varepsilon} \geqslant 0$, and $0 < \Delta < 1$. The following is true if ε''' and $\bar{\varepsilon}$ are sufficiently small.*

Let $\hat{A} = \hat{A}(N)$ be a real-valued function such that $\hat{A}(N) = \Omega(N^{-\varepsilon'})$. Let $\hat{a}_j = \hat{a}_j(N)$, $\hat{B}_j = \hat{B}_j(N)$, $\hat{C}_{jk} = \hat{C}_{jk}(N)$, $\hat{E}_j = \hat{E}_j(N)$, $\hat{F}_{jk} = \hat{F}_{jk}(N)$ and $\hat{J}_j = \hat{J}_j(N)$ be complex-valued functions $(1 \leqslant j, k \leqslant N)$ such that $\hat{B}_j, \hat{C}_{jk}, \hat{E}_j, \hat{F}_{jk} = O(N^{\bar{\varepsilon}})$, $\hat{a}_j = O(N^{1/2+\bar{\varepsilon}})$, and $\hat{J}_j = O(N^{-1/2+\bar{\varepsilon}})$, uniformly over $1 \leqslant j, k \leqslant N$. Suppose that

$$\begin{aligned} f(\boldsymbol{z}) = \exp\Biggl(&-\hat{A}N\sum_{j=1}^{N} z_j^2 + \sum_{j=1}^{N} \hat{a}_j z_j^2 + N\sum_{j=1}^{N} \hat{B}_j z_j^3 + \sum_{j,k=1}^{N} \hat{C}_{jk} z_j z_k^2 \\ &+ N\sum_{j=1}^{N} \hat{E}_j z_j^4 + \sum_{j,k=1}^{N} \hat{F}_{jk} z_j^2 z_k^2 + \sum_{j=1}^{N} \hat{J}_j z_j + \delta(\boldsymbol{z})\Biggr) \end{aligned}$$

is integrable for $\boldsymbol{z} = (z_1, z_2, \ldots, z_N) \in U_N$ and $\delta(N) = \max_{\boldsymbol{z}\in U_N} |\delta(\boldsymbol{z})| = o(1)$, where

$$U_N = \left\{ z \subseteq \mathbb{R}^N \;\middle|\; |z_j| \leqslant N^{-1/2+\hat{\varepsilon}} \text{ for } 1 \leqslant j \leqslant N \right\},$$

where $\hat{\varepsilon} = \hat{\varepsilon}(N)$ satisfies $\varepsilon'' \leqslant 2\hat{\varepsilon} \leqslant \varepsilon'''$. Then, provided the $O()$ term in the following converges to zero,

$$\int_{U_N} f(z)\,dz = \left(\frac{\pi}{\hat{A}N}\right)^{N/2} \exp\bigl(\Theta_1 + \Theta_2 + O\bigl(\bigl(N^{-\Delta} + \delta(N)\bigr)\hat{Z}\bigr)\bigr),$$

where

$$\begin{aligned}
\Theta_1 &= \frac{1}{2\hat{A}N}\sum_{j=1}^{N}\hat{a}_j + \frac{1}{4\hat{A}^2N^2}\sum_{j=1}^{N}\hat{a}_j^2 + \frac{15}{16\hat{A}^3N}\sum_{j=1}^{N}\hat{B}_j^2 + \frac{3}{8\hat{A}^3N^2}\sum_{j,k=1}^{N}\hat{B}_j\hat{C}_{jk} \\
&\quad + \frac{1}{16\hat{A}^3N^3}\sum_{j,k,\ell=1}^{N}\hat{C}_{jk}\hat{C}_{j\ell} + \frac{3}{4\hat{A}^2N}\sum_{j=1}^{N}E_j + \frac{1}{4\hat{A}^2N^2}\sum_{j,k=1}^{N}F_{jk}, \\
\Theta_2 &= \frac{1}{6\hat{A}^3N^3}\sum_{j=1}^{N}\hat{a}_j^3 + \frac{3}{2\hat{A}^3N^2}\sum_{j=1}^{N}\hat{a}_j\hat{E}_j + \frac{45}{16\hat{A}^4N^2}\sum_{j=1}^{N}\hat{a}_j\hat{B}_j^2 \\
&\quad + \frac{1}{4\hat{A}^3N^3}\sum_{j,k=1}^{N}(\hat{a}_j + \hat{a}_k)\hat{F}_{jk} + \frac{3}{4\hat{A}^2N}\sum_{j=1}^{N}\hat{B}_j\hat{J}_j + \frac{1}{4\hat{A}^2N^2}\sum_{j,k=1}^{N}\hat{C}_{jk}\hat{J}_j \\
&\quad + \frac{1}{16\hat{A}^4N^4}\sum_{j,k,\ell=1}^{N}(\hat{a}_j + 2\hat{a}_k)\hat{C}_{jk}\hat{C}_{j\ell} + \frac{3}{8\hat{A}^4N^3}\sum_{j,k=1}^{N}(2\hat{a}_j + \hat{a}_k)\hat{B}_j\hat{C}_{jk}, \\
\hat{Z} &= \exp\Biggl(\frac{1}{4\hat{A}^2N^2}\sum_{j=1}^{N}\operatorname{Im}(\hat{a}_j)^2 + \frac{15}{16\hat{A}^3N}\sum_{j=1}^{N}\operatorname{Im}(\hat{B}_j)^2 \\
&\qquad + \frac{3}{8\hat{A}^3N^2}\sum_{j,k=1}^{N}\operatorname{Im}(\hat{B}_j)\operatorname{Im}(\hat{C}_{jk}) + \frac{1}{16\hat{A}^3N^3}\sum_{j,k,\ell=1}^{N}\operatorname{Im}(\hat{C}_{jk})\operatorname{Im}(\hat{C}_{j\ell})\Biggr).
\end{aligned}$$

Proof. Our method of proof will be integration over one variable at a time. This method is conceptually simple but technically challenging. Assistance from a computer-algebra system is recommended.

Let $H_{j_1,j_2,\ldots,j_k}$ be a functions of N for each $1 \leqslant j_1, j_2, \ldots, j_k \leqslant N$ and let $p_1, p_2, \ldots, p_k$ be non-negative integers. Let $1 \leqslant j \leqslant N+1$. Define the generalized moment

$$\eta_j(H_{j_1,\ldots,j_k} \mid p_1, \ldots, p_k)(z) = \sum_{j_1,\ldots,j_k} H_{j_1,\ldots,j_k} z_{j_1}^{p_1} \cdots z_{j_k}^{p_k},$$

where the summation is over

$$\left\{ (j_1, \ldots, j_k) \;\middle|\; \bigl|\{j_1, \ldots, j_k\}\bigr| = k,\ 1 \leqslant j_i \leqslant j-1 \text{ if } p_i = 0,\ j \leqslant j_i \leqslant N \text{ if } p_i > 0 \right\}.$$

We will customarily omit the argument z as it will be clear from the context. Note that the indices $j_1, \ldots, j_k$ are reserved to this notation and always index the position their name suggests; for example

$$\eta_j(\alpha_{j_2} \mid 3,0) = \sum_{\substack{j \leqslant j_1 \leqslant N \\ 1 \leqslant j_2 \leqslant j-1}} \alpha_{j_2} z_{j_1}^3.$$

We also need the defective moment $\eta'_j(H_{j_1,\dots,j_k} \mid p_1, p_2, \dots, p_k)$ which is the same as $\eta_j(H_{j_1,\dots,j_k} \mid p_1, p_2, \dots, p_k)$ except that the index value j is forbidden; that is, the condition $j_1, \dots, j_k \neq j$ is added to the domain of summation.

Some properties of these moments that we require are listed below. Assume that $z \in U_N$. Then

$$\left|\eta_j(H_{j_1,\dots,j_k} \mid p_1, \dots, p_k)\right| \leqslant \max|H_{j_1,\dots,j_k}|\, N^{k+(-1/2+\bar{\varepsilon})(p_1+\cdots+p_k)}, \tag{59}$$

$$\eta_1(H_{j_1,\dots,j_k} \mid p_1, \dots, p_k) = 0 \quad \text{if } p_i = 0 \text{ for any } i, \tag{60}$$

$$\eta_{N+1}(H_{j_1,\dots,j_k} \mid p_1, \dots, p_k) = 0 \quad \text{if } p_i > 0 \text{ for any } i, \tag{61}$$

$$\begin{aligned}
&\eta_j(H_{j_1,\dots,j_k} \mid p_1, \dots, p_k) \\
&\quad = \eta'_j(H_{j_1,\dots,j_k} \mid p_1, \dots, p_k) \\
&\qquad + \sum_{i \mid p_i > 0} z_j^{p_i} \eta'_j(H_{j_1,\dots,j_{i-1},j,j_i,\dots,j_{k-1}} \mid p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_k),
\end{aligned} \tag{62}$$

$$\begin{aligned}
&\eta_{j+1}(H_{j_1,\dots,j_k} \mid p_1, \dots, p_k) \\
&\quad = \eta'_j(H_{j_1,\dots,j_k} \mid p_1, \dots, p_k) \\
&\qquad - \sum_{i \mid p_i = 0} \eta'_j(H_{j_1,\dots,j_{i-1},j,j_i,\dots,j_{k-1}} \mid p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_k).
\end{aligned} \tag{63}$$

The last two equalities require $j \leqslant N$.

The product of generalized moments $\eta_j(P_{j_1,\dots,j_k} \mid p_1, \dots, p_k)$ and $\eta_j(Q_{j_1,\dots,j_\ell} \mid q_1, \dots, q_\ell)$ can be written as a sum of generalized moments. Define Φ to be the set of injections $\phi : \{1, 2, \dots, \ell\} \to \{1, 2, \dots, k+\ell\}$ such that (a) $\phi(\{1, 2, \dots, \ell\}) \cup \{1, 2, \dots, k\} = \{1, 2, \dots, |\phi|\}$ for some integer $|\phi|$ depending on ϕ, (b) for $1 \leqslant i < j \leqslant \ell$, if $\phi(i), \phi(j) > k$ then $\phi(i) < \phi(j)$, and (c) for $1 \leqslant i \leqslant \ell$, $q_i = 0 \Leftrightarrow (\phi(i) > k$ or $p_{\phi(i)} = 0)$. For $\phi \in \Phi$ and $1 \leqslant i \leqslant |\phi|$, define $r_i = p_i + q_{\phi^{-1}(i)}$, where the first term is omitted if $i > k$ and the second term is omitted if i is not in the range of ϕ. Then

$$\begin{aligned}
&\eta_j(P_{j_1,\dots,j_k} \mid p_1, \dots, p_k)\eta_j(Q_{j_1,\dots,j_\ell} \mid q_1, \dots, q_\ell) \\
&\quad = \sum_{\phi \in \Phi} \eta_j(P_{j_1,\dots,j_k} Q_{j_{\phi(1)},\dots,j_{\phi(\ell)}} \mid r_1, \dots, r_{|\phi|}).
\end{aligned} \tag{64}$$

For example,

$$\eta_j(\alpha_{j_1,j_2} \mid 0,2)\eta_j(\beta_{j_1} \mid 3) = \eta_j(\alpha_{j_1,j_2}\beta_{j_3} \mid 0,2,3) + \eta_j(\alpha_{j_1,j_2}\beta_{j_2} \mid 0,5),$$

where the two terms correspond to the injections $\phi(1) = 3$ and $\phi(1) = 2$. Exactly the same formula holds for defective moments.

For $1 \leqslant j \leqslant N+1$, define

$$\begin{aligned}
F_j(z) = {} & \eta_j(-\hat{A}N + \hat{a}_{j_1} \mid 2) + \eta_j(\hat{B}_{j_1}N + \hat{C}_{j_1 j_1} \mid 3) + \eta_j(\hat{E}_{j_1}N + \hat{F}_{j_1 j_1} \mid 4) + \eta_j(\hat{J}_{j_1} \mid 1) \\
& + \eta_j(\hat{C}_{j_2 j_1} \mid 2,1) + \eta_j(\hat{F}_{j_2 j_1} \mid 2,2) + \eta_j(\Gamma_0 \mid 0) + \eta_j(\Gamma_{0,0} \mid 0,0) \\
& + \eta_j(\Gamma_{1,0} \mid 1,0) + \eta_j(\Gamma_{2,0} \mid 2,0) + \eta_j(\Gamma_{0,0,0} \mid 0,0,0) + \eta_j(\Gamma_{1,0,0} \mid 1,0,0)
\end{aligned}$$

$$+\eta_j(\Gamma_{1,1,0}\mid 1,1,0)+\eta_j(\Gamma_{2,0,0}\mid 2,0,0)+\eta_j(\Gamma_{2,1,0}\mid 2,1,0)$$
$$+\eta_j(\Gamma_{2,2,0}\mid 2,2,0)+\eta_j(\Gamma_{1,0,0,0}\mid 1,0,0,0)+\eta_j(\Gamma_{1,1,1,0}\mid 1,1,1,0)$$
$$+\eta_j(\Gamma_{2,1,0,0}\mid 2,1,0,0)+\eta_j(\Gamma_{2,2,1,0}\mid 2,2,1,0),$$

where

$$\Gamma_0=\frac{\hat a_{j_1}}{2\hat A N}+\frac{\hat a_{j_1}^2}{4\hat A^2N^2}+\frac{\hat a_{j_1}^3}{6\hat A^3N^3}+\frac{3\hat J_{j_1}\hat B_{j_1}}{4\hat A^2N}+\frac{3\hat E_{j_1}}{4\hat A^2N}+\frac{15\hat B_{j_1}^2}{16\hat A^3N}$$
$$+\frac{45\hat a_{j_1}\hat B_{j_1}^2}{16\hat A^4N^2}+\frac{3\hat a_{j_1}\hat E_{j_1}}{2\hat A^3N^2},$$
$$\Gamma_{0,0}=\frac{\hat F_{j_2j_1}}{4\hat A^2N^2}+\frac{\hat C_{j_2j_1}\hat J_{j_2}}{4\hat A^2N^2}+\frac{3\hat C_{j_2j_1}\hat B_{j_2}}{8\hat A^3N^2}+\frac{(\hat a_{j_1}+\hat a_{j_2})\hat F_{j_2j_1}}{4\hat A^3N^3}+\frac{3(\hat a_{j_1}+2\hat a_{j_2})\hat C_{j_2j_1}\hat B_{j_2}}{8\hat A^4N^3},$$
$$\Gamma_{1,0}=\frac{\hat C_{j_1j_2}}{2\hat A N}+\frac{\hat a_{j_2}\hat C_{j_1j_2}}{2\hat A^2N^2}+\frac{45\hat C_{j_1j_2}\hat B_{j_2}^2}{16\hat A^4N^2}+\frac{\hat a_{j_2}^2\hat C_{j_1j_2}}{2\hat A^3N^3}+\frac{3\hat C_{j_1j_2}\hat E_{j_2}}{2\hat A^3N^2},$$
$$\Gamma_{2,0}=\frac{3\hat C_{j_2j_1}\hat B_{j_2}}{4\hat A^2N}+\frac{\hat F_{j_1j_2}+\hat F_{j_2j_1}}{2\hat A N}+\frac{\hat C_{j_2j_1}\hat J_{j_2}}{2\hat A N}+\frac{\hat a_{j_2}(\hat F_{j_1j_2}+\hat F_{j_2j_1})}{2\hat A^2N^2}+\frac{3\hat a_{j_2}\hat C_{j_2j_1}\hat B_{j_2}}{2\hat A^3N^2},$$
$$\Gamma_{0,0,0}=\frac{\hat C_{j_3j_1}\hat C_{j_3j_2}}{16\hat A^3N^3}+\frac{(2\hat a_{j_2}+\hat a_{j_3})\hat C_{j_3j_1}\hat C_{j_3j_2}}{16\hat A^4N^4},$$
$$\Gamma_{1,0,0}=\frac{(\hat F_{j_2j_3}+\hat F_{j_3j_2})\hat C_{j_1j_2}}{4\hat A^3N^3}+\frac{3\hat C_{j_1j_3}\hat C_{j_3j_2}\hat B_{j_3}}{4\hat A^4N^3}+\frac{3\hat C_{j_3j_2}\hat C_{j_1j_2}\hat B_{j_3}}{8\hat A^4N^3},$$
$$\Gamma_{1,1,0}=\frac{\hat C_{j_1j_3}\hat C_{j_2j_3}}{4\hat A^2N^2}+\frac{\hat a_{j_3}\hat C_{j_1j_3}\hat C_{j_2j_3}}{2\hat A^3N^3},$$
$$\Gamma_{2,0,0}=\frac{(\hat a_{j_2}+\hat a_{j_3})\hat C_{j_3j_2}\hat C_{j_3j_1}}{4\hat A^3N^3}+\frac{\hat C_{j_3j_1}\hat C_{j_3j_2}}{4\hat A^2N^2},$$
$$\Gamma_{2,1,0}=\frac{3\hat C_{j_2j_3}\hat C_{j_3j_1}\hat B_{j_3}}{2\hat A^3N^2}+\frac{(\hat F_{j_1j_3}+\hat F_{j_3j_1})\hat C_{j_2j_3}}{2\hat A^2N^2},$$
$$\Gamma_{2,2,0}=\frac{\hat C_{j_3j_1}\hat C_{j_3j_2}}{4\hat A N}+\frac{\hat a_{j_3}\hat C_{j_3j_1}\hat C_{j_3j_2}}{4\hat A^2N^2},$$
$$\Gamma_{1,0,0,0}=\frac{\hat C_{j_1j_4}\hat C_{j_4j_2}\hat C_{j_4j_3}}{16\hat A^4N^4}+\frac{\hat C_{j_4j_3}\hat C_{j_4j_2}\hat C_{j_1j_2}}{8\hat A^4N^4},$$
$$\Gamma_{1,1,1,0}=\frac{\hat C_{j_1j_4}\hat C_{j_2j_4}\hat C_{j_3j_4}}{6\hat A^3N^3},$$
$$\Gamma_{2,1,0,0}=\frac{\hat C_{j_2j_4}\hat C_{j_4j_1}\hat C_{j_4j_3}}{4\hat A^3N^3}+\frac{\hat C_{j_4j_1}\hat C_{j_4j_3}\hat C_{j_2j_3}}{4\hat A^3N^3},$$
$$\Gamma_{2,2,1,0}=\frac{\hat C_{j_3j_4}\hat C_{j_4j_1}\hat C_{j_4j_2}}{4\hat A^2N^2}.$$

Note that $F_j(z)$ is independent of z_i for $i<j$. The key properties of $F_j(z)$ for $z\in U_n$ are

$$f(z)=\exp\bigl(F_1(z)+\delta(z)\bigr),\tag{65}$$

$$\int_{-N^{-1/2+\hat{\varepsilon}}}^{N^{-1/2+\hat{\varepsilon}}} \exp\bigl(F_j(z)\bigr)\,dz_j = \sqrt{\frac{\pi}{\hat{A}N}}\exp\bigl(F_{j+1}(z) + O\bigl(N^{-1-\Delta}\bigr)\bigr) \qquad (j \leqslant N). \tag{66}$$

Equation (65) is easily seen after applying (60) to eliminate most of the terms. Proof of (66) requires a tedious calculation which we now outline.

First, apply (62) to make explicit the dependence of $F_j(z)$ on z_j (a polynomial of degree 4). Then expand

$$\exp\bigl(F_j(z)\bigr) = \exp\bigl(R_0(z)\bigr)\exp\bigl(-\hat{A}Nz_j^2\bigr)\bigl(1 + R_1(z)z_j + R_2(z)z_j^2 + \cdots + O\bigl(N^{-1-\Delta}\bigr)\bigr), \tag{67}$$

where each $R_i(z)$ is independent of z_j and contains defective moments only. As seen by applying (59), only a finite number of terms are required to achieve the requested error term. The factor $\exp\bigl(-\hat{A}Nz_j^2\bigr)$ comes from the first term $\eta_j(-\hat{A}N \mid 2)$ of $F_j(z)$. Products of moments that occur need to be rewritten as sums using (64). Next, integrate (67) over z_j using

$$\int_{-N^{-1/2+\hat{\varepsilon}}}^{N^{-1/2+\hat{\varepsilon}}} z^{2k}e^{-\hat{A}Nz^2}\,dz = \frac{(2k)!}{k!\,(4\hat{A}N)^k}\sqrt{\frac{\pi}{\hat{A}N}}\bigl(1 + O\bigl(\exp\bigl(-cN^{2\hat{\varepsilon}-\varepsilon'}\bigr)\bigr)\bigr),$$

for fixed $k \geqslant 0$, for some $c > 0$. Here we have used the assumptions that $\hat{A} = \Omega(N^{-\varepsilon'})$ and $\varepsilon' < \varepsilon'' < 2\hat{\varepsilon}$. The result of the integration has the form

$$\sqrt{\frac{\pi}{\hat{A}N}}\exp\bigl(R_0(z)\bigr)\bigl(1 + S(z)\bigr) = \sqrt{\frac{\pi}{\hat{A}N}}\exp\bigl(R_0(z) + \log\bigl(1 + S(z)\bigr)\bigr).$$

Since $S(z) = o(1)$ (in fact $S(z) = O(N^{-1/2+k(\varepsilon''+\bar{\varepsilon})})$ for some k), we can expand the logarithm using (59) again to limit the expansion to finitely many terms. Finally, apply (63) to rewrite the defective moments in terms of ordinary generalized moments. The result is the right side of (66).

If all the coefficients $\hat{a}_j, \hat{B}_j, \hat{C}_{jk}, \hat{E}_j, \hat{F}_{jk}, \hat{J}_j$ were real, we could apply (65) and (66) immediately to find that

$$\int_{U_N} f(z)\,dz = \left(\frac{\pi}{\hat{A}N}\right)^{N/2}\exp\bigl(F_{N+1} + O\bigl(\delta(N) + N^{-\Delta}\bigr)\bigr), \tag{68}$$

noting that $F_{N+1}(z)$ is independent of z.

When the coefficients are complex, we must take more care. Equation (65) only allows us to write

$$\int_{U_N} f(z)\,dz = \int_{U_N}\exp\bigl(F_1(z)\bigr)\,dz + O\bigl(\delta(N)\bigr)\int_{U_N}\bigl|\exp\bigl(F_1(z)\bigr)\bigr|\,dz. \tag{69}$$

Let $F_j^*(z)$ be the same as $F_j(z)$ except that the coefficients $\hat{a}_j, \hat{B}_j, \hat{C}_{jk}, \hat{E}_j, \hat{F}_{jk}, \hat{J}_j$ are all replaced by their real parts. Clearly

$$\bigl|\exp\bigl(F_1(z)\bigr)\bigr| = \exp\bigl(F_1^*(z)\bigr),$$

and so, as in (68),

$$\int_{U_N} \bigl|\exp\bigl(F_1(z)\bigr)\bigr|\,dz = O(1)\biggl(\frac{\pi}{\hat{A}N}\biggr)^{N/2}\exp\bigl(F^*_{N+1}\bigr). \tag{70}$$

From (66) we have for $1 \leqslant j \leqslant N$ that

$$\begin{aligned}\bigl|\exp\bigl(F_{j+1}(z)\bigr)\bigr| &= \biggl(\frac{\pi}{\hat{A}N}\biggr)^{-1/2}\bigl(1+O\bigl(N^{-1-\Delta}\bigr)\bigr)\biggl|\int \exp\bigl(F_j(z)\bigr)\,dz_j\biggr| \\ &\leqslant \biggl(\frac{\pi}{\hat{A}N}\biggr)^{-1/2}\bigl(1+O\bigl(N^{-1-\Delta}\bigr)\bigr)\int \bigl|\exp\bigl(F_j(z)\bigr)\bigr|\,dz_j,\end{aligned}$$

so we have by induction starting with (70) that

$$\int \bigl|\exp\bigl(F_{j+1}(z)\bigr)\bigr|\,dz_{j+1}\cdots dz_N \leqslant O(1)\biggl(\frac{\pi}{\hat{A}N}\biggr)^{(N-j)/2}\exp\bigl(F^*_{N+1}\bigr).$$

Returning to (66), we find that, for $1 \leqslant j \leqslant N$,

$$\begin{aligned}\int \exp\bigl(F_j(z)\bigr)\,dz_j\cdots dz_N &= \sqrt{\frac{\pi}{\hat{A}N}}\int \exp\bigl(F_{j+1}(z)+O\bigl(N^{-1-\Delta}\bigr)\bigr)\,dz_{j+1}\cdots dz_N \\ &= \sqrt{\frac{\pi}{\hat{A}N}}\int \exp\bigl(F_{j+1}(z)\bigr)\,dz_{j+1}\cdots dz_N \\ &\quad + O\bigl(N^{-1-\Delta}\bigr)\sqrt{\frac{\pi}{\hat{A}N}}\int \bigl|\exp\bigl(F_{j+1}(z)\bigr)\bigr|\,dz_{j+1}\cdots dz_N \\ &= \sqrt{\frac{\pi}{\hat{A}N}}\int \exp\bigl(F_{j+1}(z)\bigr)\,dz_{j+1}\cdots dz_N \\ &\quad + O\bigl(N^{-1-\Delta}\bigr)\biggl(\frac{\pi}{\hat{A}N}\biggr)^{(N-j+1)/2}\exp\bigl(F^*_{N+1}\bigr).\end{aligned}$$

By induction on j, this gives

$$\int_{U_N} \exp\bigl(F_1(z)\bigr)\,dz = \biggl(\frac{\pi}{\hat{A}N}\biggr)^{N/2}\bigl(\exp(F_{N+1})+O\bigl(N^{-\Delta}\bigr)\exp\bigl(F^*_{N+1}\bigr)\bigr),$$

which, together with (69) and (70) gives

$$\begin{aligned}\int_{U_N} f(z)\,dz &= \biggl(\frac{\pi}{\hat{A}N}\biggr)^{N/2}\bigl(\exp(F_{N+1})+O\bigl(N^{-\Delta}+\delta(N)\bigr)\exp\bigl(F^*_{N+1}\bigr)\bigr) \\ &= \biggl(\frac{\pi}{\hat{A}N}\biggr)^{N/2}\exp\bigl(F_{N+1}+O\bigl(N^{-\Delta}+\delta(N)\bigr)\hat{Z}\bigr),\end{aligned} \tag{71}$$

where $\hat{Z} = \exp(F^*_{N+1} - \mathrm{Re}(F_{N+1}))$ and the last line is valid if $(N^{-\Delta}+\delta(N))\hat{Z} = o(1)$.

Applying (61) to the definition of $F_j(z)$, we find that

$$F_{N+1} = \Theta_1 + \Theta_2 + O\bigl(N^{-\Delta}\bigr),$$

from which it follows that $\hat{Z}$ has the value in the theorem statement to within a multiplied constant. Also note that $\hat{Z} \geqslant 1$, which is easiest to see by noting that the argument of the exponential is a non-negative quadratic form for each j. The theorem now follows from (71). □

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