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Physics Letters B 652 (2007) 111–117
PHYSICS LETTERS B
www.elsevier.com/locate/physletb

On the moduli space of non-BPS attractors for $\mathcal{N}=2$ symmetric manifolds

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Received 30 June 2007; accepted 1 July 2007

Available online 6 July 2007

Editor: L. Alvarez-Gaumé

Abstract

We study the "flat" directions of non-BPS extremal black hole attractors for $\mathcal{N}=2$, $d=4$ supergravities whose vector multiplets' scalar manifold is endowed with homogeneous symmetric special Kähler geometry. The non-BPS attractors with non-vanishing central charge have a moduli space described by real special geometry (and thus related to the $d=5$ parent theory), whereas the moduli spaces of non-BPS attractors with vanishing central charge are certain Kähler homogeneous symmetric manifolds. The moduli spaces of the non-BPS attractors of the corresponding $\mathcal{N}=2$, $d=5$ theories are also indicated, and shown to be rank-1 homogeneous symmetric manifolds.

1. Introduction

The issue of the attractor mechanism in extremal black holes [1–5] has recently received much attention, and a number of interesting advances has been performed [6–35]. Among the others, we cite here the OSV conjecture [36] (see also [32] and references therein), relating black hole (BH) entropy to topological partition functions, and the entropy function formalism [6,8], which allows one to include the higher derivative (gravitational and electromagnetic) corrections to Maxwell–Einstein action (this is crucial specially for the so-called "small" BHs, with vanishing classical entropy). An important step has been the realization that the attractor mechanism allows for extremal non-BPS BH scalar configurations of different nature [5,17,21] (see also [18]).

The present investigation concerns the latter issue, and in particular the study of the "flat" directions of the Hessian matrix of the black hole potential V_{BH} at its critical points [10,23,34,37–40]. Beside considering the case of $\mathcal{N}=8$, $d=4,5$ supergravity, we will deal with $\mathcal{N}=2$, $d=4,5$ Maxwell–Einstein supergravity theories (which in the following treatment we will simply call "supergravities") whose vector multiplets' scalar manifold is homogeneous symmetric. Indeed, for such theories a rather general analysis can be performed, determining the moduli space of the various species of non-BPS critical points of V_{BH}, mainly by using group theoretical methods (see e.g. [41–43]). In fact such moduli spaces are closely related to the nature (of the stabilizer) of the "orbits" [33,44,47] of the background dyonic BH charge vector

$$\mathcal{Q}\equiv\left(m^{\Lambda},e_{\Lambda}\right) \tag{1.1}$$

which supports the considered attractor, where m^{Λ} and e_{Λ} respectively stand for the magnetic and electric BH charges, and $\Lambda=0,1,\ldots,n_V$, with n_V being the complex dimension of the special Kähler scalar manifold. In the case of the stu model [23,50,51], our results are in agreement with the ones obtained in [21,33,34].

The Letter is organized as follows.

In Section 2 we review the BPS and non-BPS critical points of $V_{\mathrm{BH},\mathcal{N}=2}$ for extremal BHs on homogeneous symmetric scalar manifolds, and the corresponding orbits of the supporting BH charges [21,33]. The resulting properties are summarized, in particular the existence of "flat" directions for the non-BPS

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0370-2693
doi:10.1016/j.physletb.2007.07.001

case, related to the rank of the Hessian matrix of $V_{\text{BH},\mathcal{N}=2}$ at the corresponding critical points of $V_{\text{BH},\mathcal{N}=2}$. Thence, in Section 3 we deal with the $\mathcal{N}=8$ theory, and derive the moduli spaces for non-singular[1] $\frac{1}{8}$-BPS and non-BPS critical points of $V_{\text{BH},\mathcal{N}=8}$. In Section 4 we do the same for the $\mathcal{N}=2$ supergravities considered in Section 2, by taking into account that in general non-BPS critical points of $V_{\text{BH},\mathcal{N}=2}$ can occur in two different species, depending on the vanishing of the $\mathcal{N}=2$ central charge Z. Thus, in Section 5 we consider the case $d=5$, in particular the $\mathcal{N}=8$ theory (having only an $\frac{1}{8}$ non-singular class of attractors) and the $\mathcal{N}=2$ homogeneous symmetric supergravities (having an unique non-BPS class at attractors). Finally, some outlooks are given in Section 6.

2. $\mathcal{N}=2$, $d=4$ homogeneous symmetric supergravities: Attractors and critical Hessian

The symmetric special Kähler manifolds $\frac{G_V}{H_0\otimes U(1)}$ of $\mathcal{N}=2$, $d=4$ supergravities have been classified in the literature [45,46]. With the exception of the family whose prepotential is quadratic, all such theories can be obtained by dimensional reduction of the $\mathcal{N}=2$, $d=5$ supergravities that were constructed in [52–54] (they will be treated in Section 5). The supergravities with symmetric manifolds that originate from 5 dimensions all have cubic prepotentials determined by the norm form of the Jordan algebra of degree 3 that defines them [52–54].

The vector multiplets' scalar manifolds of homogeneous symmetric $\mathcal{N}=2$, $d=4$ supergravities are given in Table 1.

The irreducible sequence in the second row of Table 1 has quadratic prepotentials (and thus $C_{ijk}=0$). On the other hand, the reducible sequence in the third row, usually referred to as the *generic Jordan family*, has a 5-dim. origin, and it is related tot the sequence $\mathbb{R}\oplus\Gamma_n$ of reducible Euclidean Jordan algebras of degree 3. Here $\mathbb{R}$ denotes the 1-dim. Jordan algebra and Γ_n denotes the Jordan algebra of degree 2 associated with a quadratic form of Lorentzian signature (see[2] e.g. Table 4 of [21], and references therein).

Beside the generic Jordan family, there exist four other supergravities defined by simple Jordan algebras of degree 3. They are called *magic*, since their symmetry groups are the groups of the famous *Magic Square* of Freudenthal, Rozenfeld and Tits associated with some remarkable geometries [60,61]. $J_3^{\mathbb{O}}$, $J_3^{\mathbb{H}}$, $J_3^{\mathbb{C}}$ and $J_3^{\mathbb{R}}$ denote the four simple Jordan algebras of degree 3 with irreducible norm forms, namely by the Jordan algebras of Hermitian 3×3 matrices over the four division algebras, i.e. respectively over $\mathbb{A}=\mathbb{O}$ (octonions), $\mathbb{A}=\mathbb{H}$ (quaternions), $\mathbb{A}=\mathbb{C}$ (complex numbers) and $\mathbb{A}=\mathbb{R}$ (real numbers) [52–59]. By defining $A\equiv\dim_{\mathbb{R}}\mathbb{A}$ $(=8,4,2,1$ for $\mathbb{A}=\mathbb{O},\mathbb{H},\mathbb{C},\mathbb{R}$, respectively), Table 1 yields that the complex dimension of the scalar manifolds of the magic $\mathcal{N}=2$, $d=4$ supergravities is $3(A+1)$. Beside the analysis performed in [21], Jordan algebras have been recently studied (and related to extremal BHs) also in [62] and [63].

As found in [47], the $\frac{1}{2}$-BPS supporting charge orbit is $\frac{G_V}{H_0}$. By denoting by $\tilde{H}$ and $\hat{H}$ two non-compact forms of H_0, in [21] it was found that the non-BPS $Z=0$ and non-BPS $Z\neq 0$ supporting BH charge orbits respectively are the cosets $\frac{G_V}{\tilde{H}}$ and $\frac{G_V}{\hat{H}}$. Due to the compact nature of H_0, the symmetry group of the $\frac{1}{2}$-BPS critical points is the whole H_0, whereas the symmetry group of the non-BPS $Z=0$ and non-BPS $Z\neq 0$ critical points respectively is the *maximal compact subgroup* (m.c.s.) of $\tilde{H}$ and $\hat{H}$, in turn denoted by $\tilde{h}$ and $\hat{h}$ (actually, in the non-BPS $Z=0$ case, the symmetry is $\tilde{h}'\equiv\frac{\tilde{h}}{U(1)}$; see [21] for further details).

The data of all the $\mathcal{N}=2$, $d=4$ homogeneous symmetric supergravities are given in Tables 3 and 8 of [21].

In the following treatment we will denote by $\mathfrak{r}$ the rank of the $2n_V\times 2n_V$ Hessian matrix $\mathbf{H}$ of V_{BH}. Since in $\mathcal{N}=2$, $d=4$ supergravity the $\frac{1}{2}$-BPS critical points of V_{BH} are stable, and $\mathbf{H}_{\frac{1}{2}\text{-BPS}}$ has no massless modes [5], it holds that the rank is maximal: $\mathfrak{r}_{\frac{1}{2}\text{-BPS}}=2n_V$. On the other hand, from the analysis performed in [21] for homogeneous symmetric $\mathcal{N}=2$, $d=4$ supergravities, it follows that $\mathfrak{r}_{\text{non-BPS}}$ is model-dependent, and it also depends on the vanishing of the $\mathcal{N}=2$ central charge Z.

In the quadratic sequence $\frac{SU(1,n)}{U(1)\otimes SU(n)}$ $(n\in\mathbb{N})$, only non-BPS critical points with $Z=0$ exist. In this case, $\mathfrak{r}_{\text{non-BPS},Z=0}=2$, and $\mathbf{H}_{\text{non-BPS},Z=0}$ has $2n-2=2n_V-2$ massless modes.

For the generic Jordan family $\frac{SU(1,1)}{U(1)}\otimes\frac{SO(2,n)}{SO(2)\otimes SO(n)}$ $(n\in\mathbb{N})$, it holds that $\mathfrak{r}_{\text{non-BPS},Z\neq 0}=n+2=n_V+1$ ($\mathbf{H}_{\text{non-BPS},Z\neq 0}$ has $n=n_V-1$ massless modes), whereas $\mathfrak{r}_{\text{non-BPS},Z=0}=6$ ($\mathbf{H}_{\text{non-BPS},Z=0}$ has $2n-4=2n_V-6$ massless modes).

Concerning the magic models, it holds that $\mathfrak{r}_{\text{non-BPS},Z\neq 0}=3A+4=n_V+1$ ($\mathbf{H}_{\text{non-BPS},Z\neq 0}$ has $3A+2=n_V-1$ massless modes), whereas $\mathfrak{r}_{\text{non-BPS},Z=0}=2A+6$ ($\mathbf{H}_{\text{non-BPS},Z=0}$ has $4A$ massless modes).

Thus, the above findings match the result found by Tripathy and Trivedi in [10] for a generic special Kähler d-geometry[3] of complex dimension n_V: $\mathfrak{r}_{\text{non-BPS},Z\neq 0}=n_V+1$, i.e. $\mathbf{H}_{\text{non-BPS},Z\neq 0}$ has n_V-1 massless modes.

3. $\mathcal{N}=8$, $d=4$ supergravity: Attractors and their moduli spaces

In order to understand the moduli spaces of the two classes ($Z\neq 0$ and $Z=0$) of non-BPS attractors of homogeneous symmetric $\mathcal{N}=2$, $d=4$ supergravities, it is instructive to consider $\mathcal{N}=8$, $d=4$ supergravity, based on the real 70-dim. homogeneous symmetric manifold $\frac{G_8}{H_8}=\frac{E_{7(7)}}{SU(8)}$.

From the analysis performed in [19,47,49] it holds that only two non-singular classes of critical points of $V_{\text{BH},\mathcal{N}=8}$ exist (see

[1] We will consider only *non-singular* critical points of V_{BH}, i.e. solutions of the criticality conditions $\partial_i V_{\text{BH}}=0\ \forall i$, such that $V_{\text{BH}}|_{\partial V_{\text{BH}}=0}\neq 0$.

[2] In order to make contact with the notation used in the present Letter, with respect to the notation used in [21] one has to shift $n+1\to n$ (and thus $n\in\mathbb{N}$) for the quadratic sequence, and $n+2\to n$ (and thus $n\in\mathbb{N}$) for the cubic sequence.

[3] Following the notation of [46], by d-geometry we mean a special Kähler geometry based on an holomorphic prepotential function of the cubic form $F(X)=d_{ABC}\frac{X^AX^BX^C}{X^0}$ $(A,B,C=0,1,\dots,n_V)$.

Table 1
$\mathcal{N}=2$, $d=4$ homogeneous symmetric special Kähler manifolds

	$\frac{G_V}{H_V}$	r	$\dim_{\mathbb{C}} \equiv n_V$
Quadratic sequence			
$n \in \mathbb{N}$	$\frac{SU(1,n)}{U(1)\otimes SU(n)}$	1	n
$\mathbb{R} \oplus \Gamma_n, n \in \mathbb{N}$	$\frac{SU(1,1)}{U(1)} \otimes \frac{SO(2,n)}{SO(2)\otimes SO(n)}$	2 ($n=1$) 3 ($n \geqslant 2$)	$n+1$
$J_3^{\mathbb{O}}$	$\frac{E_{7(-25)}}{E_{6(-78)}\otimes U(1)}$	3	27
$J_3^{\mathbb{H}}$	$\frac{SO^*(12)}{U(6)}$	3	15
$J_3^{\mathbb{C}}$	$\frac{SU(3,3)}{S(U(3)\otimes U(3))} = \frac{SU(3,3)}{SU(3)\otimes SU(3)\otimes U(1)}$	3	9
$J_3^{\mathbb{R}}$	$\frac{Sp(6,\mathbb{R})}{U(3)}$	3	6

also [33]): the $\frac{1}{8}$-BPS class, supported by the BH charge orbit $\mathcal{O}_{\frac{1}{8}\text{-BPS},\mathcal{N}=8} \equiv \frac{G_8}{\mathcal{H}_0} = \frac{E_{7(7)}}{E_{6(2)}}$, and the non-BPS class, supported by the BH charge orbit $\mathcal{O}_{\text{non-BPS},\mathcal{N}=8} \equiv \frac{G_8}{\hat{\mathcal{H}}_0} = \frac{E_{7(7)}}{E_{6(6)}}$. Thus, the $\frac{1}{8}$-BPS and non-BPS orbits respectively correspond to the maximal (non-compact) subgroup of $E_{7(7)}$ to be $E_{6(2)} \otimes U(1)$ and $E_{6(6)} \otimes SO(1,1)$, where $E_{6(2)}$ and $E_{6(6)}$ are two non-compact forms of the exceptional group $E_6 \equiv E_{6(-78)}$ [42]. The 70×70 $\frac{1}{8}$-BPS Hessian $\mathbf{H}_{\frac{1}{8}\text{-BPS},\mathcal{N}=8}$ has rank 30, with 40 massless modes [39] sitting in the representation $(\mathbf{20},\mathbf{2})$ of the enhanced $\frac{1}{8}$-BPS symmetry group $SU(6) \otimes SU(2) = \text{m.c.s.}(\mathcal{H}_0)$ [33]. On the other hand, the 70×70 non-BPS Hessian $\mathbf{H}_{\text{non-BPS},\mathcal{N}=8}$ has rank 28, with 42 massless modes sitting in the representation $\mathbf{42}$ of the enhanced non-BPS symmetry group $USp(8) = \text{m.c.s.}(\hat{\mathcal{H}}_0)$ [33].

As it will be evident from the reasoning performed below, the massless modes of the Hessian of $V_{\text{BH},\mathcal{N}=8}$ at its non-singular $\frac{1}{8}$-BPS and non-BPS critical points actually are "flat" directions of $V_{\text{BH},\mathcal{N}=8}$ at the corresponding critical points. Such "flat" directions span the following real homogeneous symmetric sub-manifolds of $\frac{E_{7(7)}}{SU(8)}$:

$\frac{1}{8}$-BPS moduli space:

$$\frac{\mathcal{H}_0}{\text{m.c.s.}(\mathcal{H}_0)} = \frac{E_{6(2)}}{SU(6)\otimes SU(2)}, \qquad \dim_{\mathbb{R}} = 40;$$

non-BPS moduli space:

$$\frac{\hat{\mathcal{H}}_0}{\text{m.c.s.}(\hat{\mathcal{H}}_0)} = \frac{E_{6(6)}}{USp(8)}, \qquad \dim_{\mathbb{R}} = 42. \tag{3.1}$$

Both moduli spaces $\frac{E_{6(2)}}{SU(6)\otimes SU(2)}$ and $\frac{E_{6(6)}}{USp(8)}$ share the same structure: they are the coset of the (non-compact) stabilizer of the corresponding supporting BH charge orbit and of its m.c.s. As yielded by the analysis performed in Section 4, this is also the structure of the moduli spaces of the two classes of non-BPS attractors of the homogeneous symmetric $\mathcal{N}=2$, $d=4$ supergravities.

Remarkably, $\frac{E_{6(6)}}{USp(8)}$ is the real manifold on which $\mathcal{N}=8$, $d=5$ supergravity is based. Such a relation with the $d=5$ parent theory is exhibited also by non-BPS $Z \neq 0$ moduli spaces of the homogeneous symmetric $\mathcal{N}=2$, $d=4$ supergravities; see Section 4.

In order to understand that the "flat" directions of the Hessian of

$$V_{\text{BH},\mathcal{N}=8} \equiv \frac{1}{2} Z_{AB}(\phi,Q)\bar{Z}^{AB}(\phi,Q) \tag{3.2}$$

at its critical points actually span a moduli space, it is useful to recall that the $\mathcal{N}=8$ central charge matrix $Z_{AB}(\phi,Q)$ can be rewritten as [48]

$$Z_{AB}(\phi,Q) = \left(Q^T L(\phi)\right)_{AB} = \left(Q^T\right)_\Lambda L^\Lambda_{AB}(\phi), \tag{3.3}$$

where ϕ denote the 70 real scalar fields parameterizing the coset $\frac{G_8}{H_8} = \frac{E_{7(7)}}{SU(8)}$, Q is the $\mathcal{N}=8$ charge vector, and $L^\Lambda_{AB}(\phi) \in G_8$ is the field-dependent coset representative, i.e. a local section of the principal bundle G_8 over $\frac{G_8}{H_8}$ with structure group H_8. Thus, it follows that

$$V_{\text{BH},\mathcal{N}=8}(\phi,Q) = V_{\text{BH},\mathcal{N}=8}\left(\phi_g, Q^g\right) = V_{\text{BH},\mathcal{N}=8}\left(\phi_g, \left(g^{-1}\right)^T Q\right), \tag{3.4}$$

which shows that $V_{\text{BH},\mathcal{N}=8}$ is not G_8-invariant, because its coefficients (given by the components of Q) do not in general remain the same.

Now, if we take $g \equiv g_Q \in H_Q$, where H_Q is the stabilizer of one of the orbits $\frac{G_8}{H_Q}$ spanned by the charge vector Q, then $Q^{g_Q} = Q$, and thus:

$$V_{\text{BH},\mathcal{N}=8}(\phi,Q) = V_{\text{BH},\mathcal{N}=8}(\phi_{g_Q},Q). \tag{3.5}$$

Let us now split the fields ϕ into $\phi_Q \in \frac{H_Q}{h_Q}$ (where $h_Q \equiv \text{m.c.s.}(H_Q)$) and into the remaining $\hat{\phi}_Q$, paremeterizing the complement of $\frac{H_Q}{h_Q}$ in $\frac{G_8}{H_Q}$. By defining

$$V_{\text{BH},\mathcal{N}=8,\text{crit}}(\phi_Q,Q) \equiv V_{\text{BH},\mathcal{N}=8}(\phi,Q)\Big|_{\frac{\partial V_{\text{BH},\mathcal{N}=8}}{\partial \hat{\phi}_Q}=0}, \tag{3.6}$$

Eq. (3.5) yields the invariance of $V_{\text{BH},\mathcal{N}=8,\text{crit}}(\phi_Q,Q)$ under H_Q:

$$V_{\text{BH},\mathcal{N}=8,\text{crit}}\left((\phi_Q)_{g_Q},Q\right) = V_{\text{BH},\mathcal{N}=8,\text{crit}}(\phi_Q,Q). \tag{3.7}$$

Since H_Q is a non-compact group, this implies $V_{\text{BH},\mathcal{N}=8}$ to be independent *at its critical points* on the fields ϕ_Q parameterizing the coset $\frac{H_Q}{h_Q}$. In other words, the (covariant) derivatives of $V_{\mathrm{BH},\mathcal{N}=8}$, when evaluated *at its critical points* and with all indices spanning the coset $\frac{H_Q}{h_Q}$, vanish at *all* orders.

It is easy to realize that such a reasoning can be performed for all supergravities with $\mathcal{N} \geqslant 1$ based on homogeneous (not necessarily symmetric) manifolds[4] $\frac{G_{\mathcal{N}}}{H_{\mathcal{N}}}$, also in presence of matter multiplets (and thus of matter charges). Indeed, such arguments also apply to a generic, not necessarily supersymmetric, Maxwell–Einstein system with an homogeneous (not necessarily symmetric) scalar manifold.

By choosing Q belonging to an orbit of the representation R_V of $G_{\mathcal{N}}$ which supports critical points of $V_{\mathrm{BH},\mathcal{N}}$, the previous reasoning yields the interesting result that, *up to "flat" directions (at all orders in covariant differentiation of* $V_{\mathrm{BH},\mathcal{N}}$*)*, *all* critical points of $V_{\mathrm{BH},\mathcal{N}}$ in *all* $\mathcal{N} \geqslant 0$ Maxwell–Einstein (super)gravities with an homogeneous (not necessarily symmetric) scalar manifold (also in presence of matter multiplets) are *stable*, and thus they are *attractors* in a generalized sense.

4. $\mathcal{N} = 2$, $d = 4$ symmetric supergravities: Attractors and their moduli spaces

By using the arguments of the previous section, we now determine the moduli spaces of non-BPS critical points of $V_{\mathrm{BH},\mathcal{N}=2}$ (with $Z \neq 0$ and $Z = 0$) for all $\mathcal{N} = 2$, $d = 4$ homogeneous symmetric supergravities.

As previously noticed, $\mathcal{N} = 2$ $\frac{1}{2}$-BPS critical points are *stable*, and at such points all the scalars are stabilized by the classical attractor mechanism, because $\mathbf{H}_{\frac{1}{2}\text{-BPS}}$ has no massless modes at all [5]; thus, there is no $\frac{1}{2}$-BPS moduli space for all $\mathcal{N} = 2$, $d = 4$ supergravities (as far as the metric of the scalar manifold is non-singular and positive-definite). This is qualitatively different from the previously considered case of $\mathcal{N} = 8$ $\frac{1}{8}$-BPS critical points.

In the framework of $\mathcal{N} = 2$, $d = 4$ homogeneous symmetric supergravities, such a difference can be traced back to the fact that the stabilizer of the $\mathcal{N} = 2$ charge orbit $\mathcal{O}_{\frac{1}{2}\text{-BPS},\mathcal{N}=2}$ is *compact* (see Tables 3 and 8 of [21]).

In general, such a difference can be explained by noticing that for $\mathcal{N} = 2$ the $\frac{1}{\mathcal{N}} = \frac{1}{2}$-BPS configurations are the *maximally supersymmetric* ones, i.e. they preserve the maximum number of supersymmetries out of the ones related to the asymptotically flat BH background. For $2 < \mathcal{N} \leqslant 8$ the $\frac{1}{\mathcal{N}}$-BPS configurations are *not maximally supersymmetric*, and the configurations preserving the maximum number of supersymmetries have vanishing classical BH entropy.

It is now possible to determine the moduli spaces of non-BPS critical points of $V_{\mathrm{BH},\mathcal{N}=2}$ (with $Z \neq 0$ and $Z = 0$) for all $\mathcal{N} = 2$, $d = 4$ homogeneous symmetric supergravities (which match the results about the rank of the Hessian reported in Section 2). Consistently with the notation introduced in Section 2, the $\mathcal{N} = 2$ non-BPS $Z \neq 0$ moduli space is the coset $\frac{\hat{H}}{\hat{h}}$, whereas the $\mathcal{N} = 2$ non-BPS $Z = 0$ moduli space is the coset $\frac{\tilde{H}}{\tilde{h}} = \frac{\tilde{H}}{\tilde{h}' \otimes U(1)}$ (see [21] for further details on notation). They are respectively given by Tables 2 and 3.

Table 2
Moduli spaces of non-BPS $Z \neq 0$ critical points of $V_{\mathrm{BH},\mathcal{N}=2}$ in $\mathcal{N} = 2$, $d = 4$ homogeneous symmetric supergravities. They are the $\mathcal{N} = 2$, $d = 5$ homogeneous symmetric real special manifolds

	$\frac{\hat{H}}{\hat{h}}$	r	$\dim_{\mathbb{R}}$
$\mathbb{R} \oplus \Gamma_n$, $n \in \mathbb{N}$	$SO(1,1) \otimes \frac{SO(1,n-1)}{SO(n-1)}$	1 $(n = 1)$ 2 $(n \geqslant 2)$	n
$J_3^{\mathbb{O}}$	$\frac{E_{6(-26)}}{F_{4(-52)}}$	2	26
$J_3^{\mathbb{H}}$	$\frac{SU^*(6)}{USp(6)}$	2	14
$J_3^{\mathbb{C}}$	$\frac{SL(3,\mathbb{C})}{SU(3)}$	2	8
$J_3^{\mathbb{R}}$	$\frac{SL(3,\mathbb{R})}{SO(3)}$	2	5

Table 3
Moduli spaces of non-BPS $Z = 0$ critical points of $V_{\mathrm{BH},\mathcal{N}=2}$ in $\mathcal{N} = 2$, $d = 4$ homogeneous symmetric supergravities. They are (non-special) homogeneous symmetric Kähler manifolds

	$\frac{\tilde{H}}{\tilde{h}} = \frac{\tilde{H}}{\tilde{h}' \otimes U(1)}$	r	$\dim_{\mathbb{C}}$
Quadratic sequence $n \in \mathbb{N}$	$\frac{SU(1,n-1)}{U(1) \otimes SU(n-1)}$	1	$n-1$
$\mathbb{R} \oplus \Gamma_n$, $n \in \mathbb{N}$	$\frac{SO(2,n-2)}{SO(2) \otimes SO(n-2)}$, $n \geqslant 3$	1 $(n = 3)$ 2 $(n \geqslant 4)$	$n-2$
$J_3^{\mathbb{O}}$	$\frac{E_{6(-14)}}{SO(10) \otimes U(1)}$	2	16
$J_3^{\mathbb{H}}$	$\frac{SU(4,2)}{SU(4) \otimes SU(2) \otimes U(1)}$	2	8
$J_3^{\mathbb{C}}$	$\frac{SU(2,1)}{SU(2) \otimes U(1)} \otimes \frac{SU(1,2)}{SU(2) \otimes U(1)}$	2	4
$J_3^{\mathbb{R}}$	$\frac{SU(2,1)}{SU(2) \otimes U(1)}$	1	2

Remarkably, the moduli spaces of non-BPS $Z \neq 0$ critical points are nothing but the $\mathcal{N} = 2$, $d = 5$ homogeneous symmetric real special manifolds, i.e. the scalar manifolds of the $d = 5$ parents of the considered theories. Their real dimension $\dim_{\mathbb{R}}$ (rank r) is the complex dimension $\dim_{\mathbb{C}}$ (rank r) of the $\mathcal{N} = 2$, $d = 4$ symmetric special Kähler manifolds listed in Table 1, minus one. With the exception of the st^2 model ($n = 1$ element of the generic Jordan family) having $\frac{\hat{H}}{\hat{h}} = SO(1,1)$ with rank $r = 1$, all such moduli spaces have rank $r = 2$. The results of Table 2 are consistent with the non-BPS $Z \neq 0$ "$n_V + 1$ / $n_V - 1$" mass degeneracy splitting found by Tripathy and Trivedi in [10] (and confirmed in [21,33,34]) for a generic special Kähler d-geometry of complex dimension n_V.

Concerning the moduli spaces of non-BPS $Z = 0$ critical points, they are homogeneous symmetric (not special) Kähler manifolds. In the models st^2 and stu ($n = 1$ and $n = 2$ elements of the generic Jordan family) there are no non-BPS $Z = 0$ "flat" directions at all (see Appendix II of [21]). By recalling that $A \equiv \dim_{\mathbb{R}} \mathbb{A}$, Table 3 yields that the moduli spaces of non-BPS $Z = 0$ critical points of $V_{\mathrm{BH},\mathcal{N}=2}$ in magic $\mathcal{N} = 2$, $d = 4$ supergravities have complex dimension $2A$. Interestingly, for the $\mathcal{N} = 2$, $d = 4$ magic supergravity associated to $J_3^{\mathbb{O}}$, the non-BPS $Z = 0$ moduli space is the manifold $\frac{E_{6(-14)}}{SO(10) \otimes U(1)}$, which

[4] This is actually always the case for $\mathcal{N} \geqslant 3$ (see e.g. [37]).

is related to another exceptional Jordan triple system over $\mathbb{O}$, as found long time ago by Günaydin, Sierra and Townsend [52,53].

As mentioned in the Introduction, all this is consistent with the results about the *stu* model [23,50,51] obtained in [21,33,34]: for such a model ($n = 2$ element of the generic Jordan family) there are 2 non-BPS $Z \neq 0$ "flat" directions (spanning the manifold $(SO(1,1))^2$, as yielded by Table 2) and no non-BPS $Z = 0$ "flat" directions.

5. $d = 5, \mathcal{N} = 8$ and $\mathcal{N} = 2$ symmetric supergravities: attractors and their moduli spaces

$\mathcal{N} = 8$, $d = 5$ supergravity, based on the homogeneous symmetric real manifold $\frac{E_{6(6)}}{USp(8)}$ ($\dim_{\mathbb{R}} = 42$), has only one non-singular (i.e. with non-vanishing cubic invariant I_3) charge orbit, namely the $\frac{1}{8}$-BPS one [44,47,49]:

$$\frac{E_{6(6)}}{F_{4(4)}}. \tag{5.1}$$

The $d = 5$ supersymmetry reduction $\mathcal{N} = 8 \rightarrow \mathcal{N} = 2$ gives 14 vector multiplets and 7 hypermultiplets [64] corresponding to the two "extremal" (in the sense of having the maximum number of vector multiplets or hypermultiplets) truncations [64]:

$$(n_V, n_H) = (14, 0): \quad \frac{SU^*(6)}{USp(6)} \text{ real special;}$$
$$(n_V, n_H) = (0, 7): \quad \frac{F_{4(4)}}{USp(6) \otimes USp(2)} \text{ quaternionic Kähler,} \tag{5.2}$$

yielding 14 massive and 28 massless modes of $\mathbf{H}_{\frac{1}{8}\text{-BPS},\mathcal{N}=8,d=5}$. Thus, the moduli space of the non-singular $\frac{1}{8}$-BPS critical points of $V_{\mathrm{BH},\mathcal{N}=8}$ in $\mathcal{N} = 8, d = 5$ supergravity is given by the quaternionic Kähler manifold

$$\frac{F_{4(4)}}{USp(6) \otimes USp(2)}. \tag{5.3}$$

Considering now the case $\mathcal{N} = 2$, the manifolds of the homogeneous symmetric $\mathcal{N} = 2$, $d = 5$ supergravities are given by Table 2. As shown in [44], the $\frac{1}{2}$-BPS critical points are stable already at the Hessian level, as in the $d = 4$ case. There is an unique class of non-singular non-BPS critical points; by slightly modifying the notation introduced in [44], we denote by $\tilde{H}_5$ and $\tilde{K}_5$ the (non-compact) stabilizer of the corresponding non-BPS charge orbits and its m.c.s., respectively. It then follows that the moduli space of the unique class of non-singular non-BPS critical points of $V_{\mathrm{BH},\mathcal{N}=2}$ in homogeneous symmetric $\mathcal{N} = 2$, $d = 5$ supergravities is given by the homogeneous symmetric manifold

$$\frac{\tilde{H}_5}{\tilde{K}_5}. \tag{5.4}$$

The explicit form of $\frac{\tilde{H}_5}{\tilde{K}_5}$ and its data for all homogeneous symmetric $\mathcal{N} = 2$, $d = 5$ supergravities is given in Table 4. Such a table yields that the moduli spaces of non-singular non-BPS critical points of $V_{\mathrm{BH},\mathcal{N}=2}$ in magic $\mathcal{N} = 2$, $d = 5$ supergravities have real dimension $2A$. Their stabilizer contains the group spin$(1 + A)$. Here we just point out that, unlike the case $d = 4$ [10,34], an explicit calculation of the "flat" directions of non-BPS critical points of $V_{\mathrm{BH},\mathcal{N}=2}$ in $d = 5$, despite some recent works on attractor mechanism and entropy function formalism in $d = 5$ supergravities (see e.g. [66–68], and references therein), is missing at the present time.

Table 4
Moduli spaces of non-BPS critical points of $V_{\mathrm{BH},\mathcal{N}=2}$ in $\mathcal{N} = 2$, $d = 5$ homogeneous symmetric supergravities

	$\frac{\tilde{H}_5}{\tilde{K}_5}$	r	$\dim_{\mathbb{R}}$
$\mathbb{R} \oplus \Gamma_n, n \in \mathbb{N}$	$\frac{SO(1,n-2)}{SO(n-2)}, n \geqslant 3$	1 $(n \geqslant 3)$	$n - 2$
$J_3^{\mathbb{O}}$	$\frac{F_{4(-20)}}{SO(9)}$	1	16
$J_3^{\mathbb{H}}$	$\frac{USp(4,2)}{USp(4) \otimes USp(2)}$	1	8
$J_3^{\mathbb{C}}$	$\frac{SU(2,1)}{SU(2) \otimes U(1)}$	1	4
$J_3^{\mathbb{R}}$	$\frac{SL(2,\mathbb{R})}{SO(2)}$	1	2

6. Conclusion

In the present investigation we have extended the analysis performed in [21] and [33] about the spectrum of non-BPS critical points of $V_{\mathrm{BH},\mathcal{N}=2}$, their degeneracy and stability. For the case of d-geometries [10,34], and in particular for homogeneous symmetric special Kähler geometries [21,33], the Hessian matrix of $V_{\mathrm{BH},\mathcal{N}=2}$ at its non-BPS critical points generally has some strictly positive eigenvalues and some vanishing eigenvalues, corresponding to "flat" directions. For the non-BPS $Z \neq 0$ case, our analysis generalizes the findings of [34].

One should not be surprised by our result, because the existence of "flat" directions in the Hessian of V_{BH} was pointed out also at BPS critical points (preserving 4 supersymmetries) in the framework of $\mathcal{N} > 2$, $d = 4$ extended supergravities [33,39], the "flat" directions being associated to hypermultiplets' scalar degrees of freedom in the supersymmetry reduction $\mathcal{N} > 2 \rightarrow \mathcal{N} = 2$ of the considered theory [33,38–40] (see [37] for an introduction to the attractor mechanism in $\mathcal{N} \geqslant 2$-extended supergravities).

We have shown that the geometrical structure of the non-BPS moduli spaces depends on the vanishing of the $\mathcal{N} = 2$ central charge Z. As previously mentioned, for $Z \neq 0$ our results are in agreement with the ones of [10] and [34].

It is easy to realize that our results extend also to the case of homogeneous non-symmetric special Kähler geometries. Clearly, in such a framework the classification of the charge orbits supporting non-singular critical points might be different from the symmetric case. Actually, as mentioned above, our results also hold for a generic, not necessarily supersymmetric, Maxwell–Einstein system with an homogeneous (not necessarily symmetric) scalar manifold.

For generic, non-homogeneous special Kähler d-geometries, the U-duality group has no longer a transitive action on the representation space of the BH charges, and the analysis is more complicated, and it might yield different results about stability. However, the non-BPS moduli spaces are still present at least in some particular cases, e.g. in the model called Kaluza–Klein BH (in M-theory language) [49] or D0–D6 system (in type IIA Calabi–Yau compactifications in the language of superstring theory) [10,34], in which the only non-vanishing charges are p^0 and q_0. In this case, the moduli space is the corresponding real special manifold.

The existence of moduli spaces clarifies the issue of classical stability of non-BPS critical points of $V_{BH,\mathcal{N}=2}$, at least for the analyzed case of homogeneous symmetric vector multiplets' scalar manifolds. *All* such non-BPS critical points are *stable*, with a certain number of "flat" directions, which however do not enter into the classical Bekenstein–Hawking [65] BH entropy S_{BH}, whose U-invariant expression in the considered framework in $d = 4$ reads [39]

$$S_{BH}(Q) = \pi \left| I_2(Q) \right| \quad \text{for quadratic models;}$$
$$S_{BH}(Q) = \pi \sqrt{\left| I_4(Q) \right|} \quad \text{for cubic models,} \tag{6.1}$$

$I_2(Q)$ and $I_4(Q)$ being the unique invariant (quadratic and quartic in the BH charges, respectively) of the representation R_V of the U-duality group in which the charge vector sits.

It is conceivable that most of the "flat" directions will be removed by quantum effects, i.e. by higher-derivative corrections to the classical BH potential V_{BH}. However, this might not be the case for $\mathcal{N} = 8$ BHs.

We conclude by saying that for the cases considered in the present investigation the existence of "flat" directions is closely related to the Lorentzian signature of the BH charge orbits supporting non-BPS critical points of $V_{BH,\mathcal{N}=2}$, i.e. to the fact that the corresponding stabilizer is a non-compact group. The same phenomenon already happened for $\mathcal{N} > 2$ also at non-singular BPS critical points [38–40].

Acknowledgements

The work of S.F. has been supported in part by European Community Human Potential Program under contract MRTN-CT-2004-005104 *"Constituents, fundamental forces and symmetries of the universe"* and the contract MRTN-CT-2004-503369 *"The quest for unification: Theory Confronts Experiments"*, in association with INFN Frascati National Laboratories and by D.O.E. grant DE-FG03-91ER40662, Task C.

The work of A.M. has been supported by a Junior Grant of the *"Enrico Fermi"* Center, Rome, in association with INFN Frascati National Laboratories, and in part by D.O.E. grant DE-FG03-91ER40662, Task C.

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