Variational iteration method for solving a nonlinear system of second-order boundary value problems

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Abstract

The variational iteration method is introduced to solve a nonlinear system of second-order boundary value problems. Numerical results demonstrate that this method is promising and readily implemented.

Keywords: Variational iteration method; Second order boundary value problems; Nonlinear; Exact solution; Approximate solution

1. Introduction

Ordinary differential systems have been the focus of many studies due to their frequent appearance in various applications in physics, engineering, biology and other fields. Wazwaz applied the Adomian decomposition method [1] to solve singular initial value problems in the second-order ordinary differential equations [2]. Ramos proposed linearization techniques for solving singular initial value problems of ordinary differential equations [3], and there are other papers for solving second-order initial value problems. However, many classical numerical methods used to solve second-order initial value problems can’t be applied to second-order boundary value problems. For a nonlinear system of second-order boundary value problems, there are few valid methods to obtain numerical solutions. Many authors discussed the existence of solutions to second-order systems, including the approximation of solutions via finite difference method [4–10]. Recently, He proposed a variational approach to the sixth-order boundary value problems, and applied the homotopy perturbation method to solve boundary value problems and bifurcation of nonlinear problems [11–13], Tiryaki et al. considered the oscillation criteria of certain second-order nonlinear differential equations [14], Geng et al. presented a new method to solve a nonlinear system of second-order boundary value problems [15].

The variational iteration method, which was proposed originally by He [16] in 1999, has been proved by many authors to be a powerful mathematical tool for various types of nonlinear problem. It was successfully applied to Burger’s equation and coupled Burger’s equation [17], to generalized KdV and coupled Schrodinger-KdV [18], to delay differential equations [19], to the Duffing equation with nonlinearity of the fifth order and mathematical
Consider the following equations

\begin{align}
  u'' + a_1(x)u' + a_2(x)u + a_3(x)v'' + a_4(x)v' + a_5(x)v + N_1(u, v) &= f_1(x) \\
v'' + b_1(x)v' + b_2(x)v + b_3(x)u'' + b_4(x)u' + b_5(x)u + N_2(u, v) &= f_2(x)
\end{align}

subject to the boundary conditions

\[
u(0) = u(1) = 0, \quad v(0) = v(1) = 0
\]

where \(0 < x < 1\), \(N_1, N_2\) are nonlinear functions of \(u\) and \(v\), \(a_i(x), b_i(x), f_1(x)\) and \(f_2(x)\) are given functions, and \(a_i(x), b_i(x)\) are continuous, \(i = 1, 2, 3, 4, 5\).

3. Analysis of the variational iteration method

To illustrate its basic concepts of the variational iteration method [16,17], we consider the following differential equation

\[Lu + Nu = g(x)\]

where \(L\) is a linear operator, \(N\) is a nonlinear operator, and \(g(x)\) is an inhomogeneous term. Then, we can construct a correct functional as follows:

\[u_{n+1}(x) = u_n(x) + \int_0^x \lambda [Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi)]d\xi\]  

where \(\lambda\) is a general Lagrange multiplier [16,20,25], which can be identified optimally via variational theory. The second term on the right is called the correction and \(\tilde{u}_n\) is considered as a restricted variation, i.e. \(\delta\tilde{u}_n = 0\).

4. Numerical examples

In this section, we will apply the variational iteration method to solve some nonlinear second-order boundary value problems, and compare the approximate solutions with the exact solutions.

Example 1. Consider the following equations

\begin{align}
  u''(x) + xu(x) + xv(x) &= f_1(x) \\
v''(x) + 2xv(x) + 2xu(x) &= f_2(x)
\end{align}

subject to the boundary conditions

\[
u(0) = u(1) = 0, \quad v(0) = v(1) = 0
\]

where \(0 < x < 1\), \(f_1(x) = 2\) and \(f_2(x) = -2\). The exact solutions of (4) are \(u(x) = x^2 - x\) and \(v(x) = x - x^2\), respectively.
According to the variational iteration method, the correct functionals are given by
\[
\begin{align*}
    u_{n+1}(x) &= u_n(x) + \int_0^x \lambda_1 \{ u''_n(\xi) + \xi \tilde{u}_n(\xi) + \xi \tilde{v}_n(\xi) - f_1(\xi) \} \, d\xi \\
    v_{n+1}(x) &= v_n(x) + \int_0^x \lambda_2 \{ u''_n(\xi) + 2\xi \tilde{u}_n(\xi) + 2\xi \tilde{v}_n(\xi) - f_2(\xi) \} \, d\xi
\end{align*}
\]  
(5)

where \( \tilde{u}_n \) and \( \tilde{v}_n \) denote restricted variations, i.e. \( \delta \tilde{u}_n = 0, \delta \tilde{v}_n = 0; \lambda_1 \) and \( \lambda_2 \) are general Lagrange multipliers, and can be easily identified as
\[
\lambda_1 = \xi - x, \quad \lambda_2 = \xi - x
\]

Therefore, we have the following iteration formulae
\[
\begin{align*}
    u_{n+1}(x) &= u_n(x) + \int_0^x (\xi - x) \{ u''_n(\xi) + \xi u_n(\xi) + \xi v_n(\xi) - f_1(\xi) \} \, d\xi \\
    v_{n+1}(x) &= v_n(x) + \int_0^x (\xi - x) \{ v''_n(\xi) + 2\xi v_n(\xi) + 2\xi u_n(\xi) - f_2(\xi) \} \, d\xi
\end{align*}
\]  
(6)

By the above iteration formulae, begin with arbitrary initial approximations
\[
u_0(x) = Ax, \quad v_0(x) = Bx
\]
where \( A \) and \( B \) are constants to be determined, we obtain
\[
\begin{align*}
    u_1(x) &= Ax + \int_0^x (\xi - x) [(A + B)x^2 - 2] \, d\xi \\
    &= Ax + x^2 - \frac{1}{12} (A + B)x^4 \\
    v_1(x) &= Bx + \int_0^x (\xi - x) [2(A + B)x^2 + 2] \, d\xi \\
    &= Bx - x^2 - \frac{1}{6} (A + B)x^4
\end{align*}
\]

By imposing the boundary conditions at \( x = 0 \) and \( x = 1 \) yields \( A = -1 \) and \( B = 1 \), thus
\[
\begin{align*}
    u_1(x) &= x^2 - x, \quad v_1(x) = x - x^2
\end{align*}
\]  
(7)

are the exact solutions of \( u(x) \) and \( v(x) \), respectively. It is important to note that the initial approximations can be freely selected with unknown constants, which can be determined by the boundary conditions.

**Example 2.** We consider the following equations
\[
\begin{align*}
    u''(x) + (2x - 1)u'(x) + \cos(\pi x)u'(x) &= f_1(x) \\
    v''(x) + xu(x) &= f_2(x)
\end{align*}
\]  
(8)

subject to the boundary conditions
\[
u(0) = u(1) = 0, \quad v(0) = v(1) = 0
\]
where \( 0 < x < 1, f_1(x) = -\pi^2 \sin(\pi x) + (2x - 1)\pi \cos(\pi x) + (2x - 1) \cos(\pi x) \) and \( f_2(x) = 2 + x \sin(\pi x) \).

Similarly to the previous operation, we can obtain the following iteration formulae
\[
\begin{align*}
    u_{n+1}(x) &= u_n(x) + \int_0^x (\xi - x) \{ u''_n(\xi) + (2\xi - 1)u'_n(\xi) + \cos(\pi \xi)u'_n(\xi) - f_1(\xi) \} \, d\xi \\
    v_{n+1}(x) &= v_n(x) + \int_0^x (\xi - x) \{ v''_n(\xi) + \xi u_n(\xi) - f_2(\xi) \} \, d\xi
\end{align*}
\]  
(9)

We begin with the initial approximations
\[
u_0(x) = \sin(Ax), \quad v_0(x) = Bx,
\]
We deal with the following nonlinear second-order boundary value problem

\[ u''(x) + xu'(x) + \cos(\pi x)v'(x) = f_1(x) \]
\[ v''(x) + xu'(x) + N_2(u(x), v(x)) = f_2(x) \]

subject to the boundary conditions

\[ u(0) = u(1) = 0, \quad v(0) = v(1) = 0 \]

where \( A \) and \( B \) are constants to be determined. By the variational iteration formulae (9), we have

\[ u_1(x) = \sin(Ax) + \int_0^x (\xi - x)(-A^2 \sin(A\xi) + A(2\xi - 1) \cos(\pi \xi) - f_1(\xi))d\xi \]
\[ v_1(x) = Bx + \int_0^x (\xi - x)(\xi \sin(A\xi) - f_2(\xi))d\xi \]

By imposing the boundary conditions at \( x = 0 \) and \( x = 1 \), we obtain

\[ A = \pi, \quad B = -1 \]

and the corresponding approximate solutions are given by

\[ u_1(x) = \sin(\pi x) + \frac{4 \sin(\pi x) - 2\pi x \cos(\pi x) - 2\pi x}{\pi^3} \]
\[ v_1(x) = x^2 - x \]

Note that the exact solutions of (8) are \( u(x) = \sin(\pi x) \) and \( v(x) = x^2 - x \), respectively, we derive the approximate solutions by only one iteration step and the more steps iterated, the more accurate results can be obtained.

Example 3. We deal with the following nonlinear second-order boundary value problem

\[ u''(x) + xu'(x) + \cos(\pi x)v'(x) = f_1(x) \]
\[ v''(x) + xu'(x) + N_2(u(x), v(x)) = f_2(x) \]

subject to the boundary conditions

\[ u(0) = u(1) = 0, \quad v(0) = v(1) = 0 \]

where \( 0 < x < 1, N_2(u(x), v(x)) = xu^2(x), f_1(x) = \sin(x) + (x^2 - x + 2) \cos(x) + (1 - 2x) \cos(\pi x) \) and \( f_2(x) = -2 + x \sin(x) + x(x - 1)^2 \sin^2(x) + (x^2 - x) \cos(x) \).

According to the variational iteration method, the corresponding correct functionals can be constructed, and the Lagrange multipliers \( \lambda_1, \lambda_2 \) can be readily identified as \( \lambda_1 = \xi - x \) and \( \lambda_2 = \xi - x \), respectively. As a result, we have the following iteration formulae

\[ u_{n+1}(x) = u_n(x) + \int_0^x (\xi - x)(u_n''(\xi) + \xi u_n'(\xi) + \cos(\pi \xi)v_n'(\xi) - f_1(\xi))d\xi \]
\[ v_{n+1}(x) = v_n(x) + \int_0^x (\xi - x)(\xi u_n'(\xi) + \xi u_n^2(\xi) - f_2(\xi))d\xi \]

Table 1
Solutions by VIM and comparisons

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<th>( u_1 )</th>
<th>Absolute error</th>
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<td>0.0</td>
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Table 2
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Begin with $u_0(x) = (x - 1) \sin(Ax)$, $v_0(x) = Bx$, by the above iteration formulae, we obtain

$$
\begin{align*}
\ u_1(x) & = (x - 1) \sin(Ax) + \int_0^x (\xi - x)((\xi - A^2\xi + A^2) \sin(A\xi)) \\
& \quad + (A\xi^2 - A\xi + 2A) \cos(A\xi) + B \cos(\pi \xi) - f_1(\xi))d\xi \\
\ v_{n+1}(x) & = v_n(x) + \int_0^x (\xi - x)(\xi(\xi - 1)^2 \sin^2(A\xi) + \xi \sin(A\xi) + A(\xi^2 - \xi) \cos(A\xi) - f_2(\xi))d\xi
\end{align*}
$$

By imposing the boundary conditions on the above formulae, we can obtain the values of $A$ and $B$ as follows

$$A = 1, \quad B = 1$$

Therefore,

$$
\begin{align*}
\ u_1(x) & = (x - 1) \sin(x) - \frac{4 \sin(\pi x) - 2 \pi x \cos(\pi x) - 2 \pi x}{\pi^3} \\
\ v_1(x) & = x - x^2.
\end{align*}
$$

Note that we derive the approximate solutions by one iteration step, and $v_1(x)$ is the exact solution. Compared with the method presented to solve the linear and nonlinear second-order boundary value problems [15], the variational iteration method is not involved in the complicated computation of the reproducing kernel; therefore, it is readily implemented to obtain the approximate solutions to ordinary differential equations in practice.

Now, we compare the results obtained by the variational iteration method for $u = u_1$ with the exact solution $u(x) = (x - 1) \sin(x)$ in Table 2.

5. Conclusions

In this paper, we have demonstrated the applicability of the variational iteration method for solving a nonlinear system of second-order boundary value problems with the help of some concrete examples. The numerical results show that:

1. The variational iteration method is a new approach to provide an analytical approximation to both linear and nonlinear second-order boundary value problems without linearization and discretization;
2. The initial approximations can be freely selected with unknown constants, which can be determined by the boundary conditions;
3. It is possible to derive the exact solutions by using one iteration only, and this method is also valid for large coefficients. Moreover, compared with the method proposed in [15], the variational iteration method is promising and readily implemented.
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References