Asymptotic stability of the solution of the $M/M^B/1$ queueing model

Abdukerim Haji$^{a,*,1}$, Agnes Radl$^b$

$^a$College of Mathematics and Systems Science, Xinjiang University, Urumqi 830046, China
$^b$Mathematisches Institut, Auf der Morgenstelle 10, D-72076 Tübingen, Germany

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Abstract

By using positive $C_0$-semigroup theory we study the asymptotic stability of the solution of a bulk queueing model $M/M^B/1$.

Keywords: $M/M^B/1$ queueing model; Positive $C_0$-semigroup; Dirichlet operator; Resolvent set; Eigenvalue

1. Introduction

The $M/M^B/1$ model is one important model in operations research. Many authors are studying it (see [1–4]). According to [1], The $M/M^B/1$ queueing model can be expressed as:

$$\frac{dp_{0,0}(t)}{dt} = -\lambda p_{0,0}(t) + \mu \int_0^\infty p_{0,1}(x, t)dx,$$

$$\frac{dp_{0,1}(x, t)}{dt} + \frac{dp_{0,1}(x, t)}{dx} = -(\lambda + \mu)p_{0,1}(x, t),$$

$$\frac{dp_{n,1}(x, t)}{dt} + \frac{dp_{n,1}(x, t)}{dx} = -(\lambda + \mu)p_{n,1}(x, t) + \lambda p_{n-1,1}(x, t), \quad n \geq 1,$$

$$p_{0,1}(0, t) = \sum_{k=1}^B \mu \int_0^\infty p_{k,1}(x, t)dx + \lambda p_{0,0}(t),$$

$$p_{n,1}(0, t) = \mu \int_0^\infty p_{n+B,1}(x, t)dx, \quad n \geq 1,$$

$$p_{0,0}(0) = 1, \quad p_{n,1}(x, 0) = 0, \quad n \geq 0.$$
queue [not in system (queue+service)] and the elapsed service time lies in \((x, x + dx]\), \(\mu\) is the mean service rate of the server, \(\lambda\) is the mean arrival rate of the customer, \(B\) represents the maximum size of service.

The book [1] established the mathematical model of the \(M/M^B/1\) queue and studied the static solution by using probability generating functions under following hypothesis:

**Hypothesis 1.** The \(M/M^B/1\) queueing model has a unique positive time-dependent solution \(p(x, t)\).

**Hypothesis 2.** The time-dependent solution \(p(x, t)\) converges to the static solution \(p(x)\) as time tends to infinite.

Here

\[
p(x, t) = (p_{0,0}(t), p_{0,1}(x, t), p_{1,1}(x, t), p_{2,1}(x, t), p_{3,1}(x, t), \ldots),
\]

\[
p(x) = (p_{0,0}, p_{0,1}(x), p_{1,1}(x), p_{2,1}(x), p_{3,1}(x), \ldots).
\]

In [2], Gupur converted this model into an abstract Cauchy problem on a suitable Banach space and then proved the existence of a unique positive time-dependent solution by using the theory of \(C_0\)-semigroups of linear operators. In other words, the author proved that the above Hypothesis 1 holds. In [3], Gupur proved that all points on the imaginary axis except for 0 belong to the resolvent set of the \(M/M^B/1\) operator \(A\), but until now the proof of the Hypothesis 2 has been still an open problem (see [3]).

In this paper, first we write Dirichlet operator; second by using Dirichlet operator we prove that 0 is an eigenvalue of the \(M/M^B/1\) operator \(A\); and third, we prove that all points on the imaginary axis except for 0 belong to the resolvent set of the \(M/M^B/1\) operator \(A\); fourth we show that the semigroup \(T(t)\) generated by \(A\) is irreducible; finally, by using those results and Theorem 2.1 we prove that the Hypothesis 2 holds.

2. The problem as an abstract Cauchy problem

We first reformulate the system (1)–(6) an abstract Cauchy problem with an operator \((A, D(A))\) on a suitable stated space. The stated space \(X\) is chosen as

\[
X = \left\{ p \in \mathbb{C} \times L^1[0, \infty) \times L^1[0, \infty) \times \cdots \mid \|p\| = |p_{0,0}| + \sum_{n=0}^{\infty} \|p_{n,1}\|_{L^1[0,\infty)} < \infty \right\}.
\]

It is obvious that \(X\) is a Banach space.

To define the operator \((A, D(A))\) we introduce a maximal operator \((A_m, D(A_m))\) on \(X\) as

\[
A_m = \begin{pmatrix}
-\lambda & \mu \psi & 0 & 0 & \cdots \\
0 & D & 0 & 0 & \cdots \\
0 & \lambda & D & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \lambda & D & \cdots
\end{pmatrix}
\]

\[
D(A_m) = \left\{ p \in X \mid \frac{dp_{n,1}(x)}{dx} \in L^1[0, \infty), \quad p_{n,1}(x) \text{ is absolutely continuous function } (n \geq 0) \right\}.
\]

Here and in the following \(\psi\) denotes the linear functional:

\[
\psi : L^1[0, \infty) \to \mathbb{C}, \quad f \mapsto \psi(f) := \int_{0}^{\infty} f(x)dx.
\]

Moreover, the operator \(D\) on \(W^{1,1}[0, \infty)\) are defined as

\[
D := -\frac{d}{dx} - (\lambda \eta_1 + \mu).
\]

To formulate the boundary conditions (4) and (5) we will use the following boundary operators \(L\) and \(\Phi\) mapping into the boundary space

\[
\partial X := l^1.
\]
The operator $L$ is defined as

\[
    L : D(A_m) \to \partial X, \quad \begin{pmatrix} p_0, 0 \\ p_0, 1 \\ p_{1.1} \\ p_{2.1} \\ \vdots \end{pmatrix} \mapsto L \begin{pmatrix} p_0, 0 \\ p_0, 1 \\ p_{1.1} \\ p_{2.1} \\ \vdots \end{pmatrix} := \begin{pmatrix} p_{0.1}(0) \\ p_{1.1}(0) \\ p_{2.1}(0) \\ \vdots \end{pmatrix},
\]

and the operator $\Phi \in L(X, \partial X)$ is defined as

\[
    \Phi = \begin{pmatrix} \lambda & 0 & \mu \psi & \mu \psi & \cdots & \mu \psi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \mu \psi & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \mu \psi & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.
\]

The operator $(A, D(A))$ on $X$ is given as

\[
    Ap := A_m p, \quad D(A) := \{ p \in D(A_m) \mid Lp = \Phi p \}.
\]

The above Eqs. (1)–(6) are equivalent to the abstract Cauchy problem

\[
    \begin{cases}
        \frac{dp(t)}{dt} = Ap(t), & t \in [0, \infty), \\
        p(0) = (1, 0, 0, \ldots) & \in X.
    \end{cases}
\]

In [3], Gupur obtained the following result: when the traffic intensity $\rho$ satisfies

\[
    \rho = \frac{\lambda}{\mu} < 1.
\]

**Theorem 2.1.** The operator $(A, D(A))$ generates a positive contraction $C_0$-semigroup $(T(t))_{t \geq 0}$.

### 3. Main results

In this section we investigate the spectrum $\sigma(A)$ of $A$ and then give our main result on the asymptotic behaviour of the solutions. We first characterize $\sigma(A)$ by the spectrum of an infinite scalar matrix, i.e. an operator on the boundary space $\partial X$. To do so we apply results from [5]. Therefore we need the operator $(A_0, D(A_0))$ defined by

\[
    D(A_0) := \{ p \in D(A_m) \mid Lp = 0 \}, \\
    A_0 p := A_m p.
\]

By [5, Lemma 1.2] we can decompose $D(A_m)$ for any $\gamma \in \rho(A_0)$ as

\[
    D(A_m) = D(A_0) \oplus \ker(\gamma - A_m).
\]

A simple calculation shows that $\ker(\gamma - A_m)$ has the form

\[
    \ker(\gamma - A_m) = \left\{ \begin{array}{l}
        p(x) = (p_{0.0}, p_{0.1}(x), p_{1.1}(x), p_{2.1}(x), \ldots); \\
        p_{0.0} = \frac{\mu c_1}{\mu + \gamma + \lambda}; \\
        p_{n.1}(x) = e^{-(\gamma + \lambda + \mu)x} \sum_{k=0}^{n} \frac{\lambda^k}{k!} c_{n+1-k}, \quad n \geq 0; \\
        \text{and} \quad (c_n)_{n \geq 1} \in l^1
    \end{array} \right\}
\]
Moreover, since $L$ is surjective, $L|_{\ker(\gamma - A_m)} : \ker(\gamma - A_m) \to \partial X$ is invertible for any $\gamma \in \rho(A_0)$, see [5, Lemma 1.2]. We denote its inverse by

$$D_\gamma := (L|_{\ker(\gamma - A_m)})^{-1} : \partial X \to \ker(\gamma - A_m),$$

and $D_\gamma$ will be called “Dirichlet operator”.

We now give the explicit form of $D_\gamma$.

**Lemma 3.1.** For every $\gamma \in \rho(A_0)$, the operator $D_\gamma$ has the form

$$D_\gamma = \begin{pmatrix}
\mu & 0 & 0 & 0 & \ldots \\
(\gamma + \lambda)(\gamma + \lambda + \mu) & 0 & 0 & 0 & \ldots \\
e^{-(\gamma + \lambda + \mu)x} & & & & \\
\lambda x e^{-(\gamma + \lambda + \mu)x} & e^{-(\gamma + \lambda + \mu)x} & 0 & 0 & \ldots \\
\frac{\lambda^2}{2!} x^2 e^{-(\gamma + \lambda + \mu)x} & \lambda x e^{-(\gamma + \lambda + \mu)x} & e^{-(\gamma + \lambda + \mu)x} & 0 & \ldots \\
\frac{\lambda^3}{3!} x^3 e^{-(\gamma + \lambda + \mu)x} & \frac{\lambda^2}{2!} x^2 e^{-(\gamma + \lambda + \mu)x} & \lambda x e^{-(\gamma + \lambda + \mu)x} & e^{-(\gamma + \lambda + \mu)x} & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.$$

At first we prove the following lemmas in order to obtain the main result in this paper. With the help of the operators $D_\gamma$ and $\Phi$ we now characterise the spectrum $\sigma(A)$ and the point spectrum $\sigma_p(A)$ of $A$:

**Lemma 3.2.** Let $\gamma \in \rho(A_0)$. Then the following holds.

1. $\gamma \in \sigma(A) \iff 1 \in \sigma(D_\gamma \Phi) \iff 1 \in \sigma(\Phi D_\gamma)$.
2. $\gamma \in \sigma_p(A) \iff 1 \in \sigma_p(D_\gamma \Phi) \iff 1 \in \sigma_p(\Phi D_\gamma)$.

**Proof.** By [5, Lemma 1.4], for any $\gamma \in \rho(A_0)$ we have

$$\gamma - A = (\gamma - A_0)(\text{Id} - D_\gamma \Phi).$$

This shows the first equivalence in (1) and (2).

It follows from this that

$$\gamma - A \text{ is injective} \iff \text{Id} - D_\gamma \Phi \text{ is injective},$$

hence

$$\gamma \in \sigma_p(A) \iff 1 \in \sigma_p(D_\gamma \Phi).$$

In the following we prove that

$$1 \in \sigma_p(D_\gamma \Phi) \iff 1 \in \sigma_p(\Phi D_\gamma).$$

If we suppose that $\text{Id} - \Phi D_\gamma$ is injective, then there exists an operator $B$ such that $B(\text{Id} - \Phi D_\gamma) = \text{Id}$. Since

$$(\text{Id} + D_\gamma B \Phi)(\text{Id} - D_\gamma \Phi) = \text{Id} - D_\gamma \Phi + D_\gamma B(\text{Id} - \Phi D_\gamma) \Phi$$
$$= \text{Id} - D_\gamma \Phi + D_\gamma \Phi = \text{Id},$$

the operator $\text{Id} - D_\gamma \Phi$ is injective.

If we suppose that $\text{Id} - D_\gamma \Phi$ is injective, then there exists an operator $B'$ such that $B'(\text{Id} - \Phi) = \text{Id}$. Since

$$(\text{Id} + \Phi B'D_\gamma)(\text{Id} - \Phi D_\gamma) = \text{Id} - \Phi D_\gamma + \Phi B'(\text{Id} - D_\gamma \Phi) D_\gamma$$
$$= \text{Id} - \Phi D_\gamma + \Phi D_\gamma = \text{Id},$$

the operator $\text{Id} - \Phi D_\gamma$ is injective. It follows from this that:

$$1 \in \sigma_p(D_\gamma \Phi) \iff 1 \in \sigma_p(\Phi D_\gamma).$$
A similar computation as above shows that for $1 \in \rho(D_0)$ also

$$(\text{Id} - \Phi D_\gamma)(\text{Id} + \Phi' B D_\gamma) = \text{Id}$$

holds and for $1 \in \rho(\Phi D_\gamma)$

$$(\text{Id} - D_\gamma \Phi)(\text{Id} + D_\gamma B \Phi) = \text{Id}.$$ 

Hence,

$$1 \in \sigma(D_\gamma \Phi) \iff 1 \in \sigma(\Phi D_\gamma).$$

\[ \square \]

**Remark 3.3.** For $\gamma \in \rho(A_0)$ the operator $\Phi D_\gamma$ is represented by the following matrix:

$$\Phi D_\gamma = \begin{pmatrix}
\frac{\mu \lambda}{(\lambda + \mu) I} + \sum_{k=0}^{B-1} \frac{\mu \lambda^k}{I^{k+1}} & \sum_{k=0}^{B-2} \frac{\mu \lambda^k}{I^{k+1}} & \cdots & \frac{\mu}{I} & \frac{\mu}{I^2} & \frac{\mu}{I^3} & \frac{\mu}{I^4} & \cdots \\
\frac{\mu \lambda B}{I^B+1} & \frac{\mu \lambda B}{I^B+1} & \cdots & \frac{\mu \lambda B}{I^B+1} & \frac{\mu \lambda B}{I^B+1} & \frac{\mu \lambda B}{I^B+1} & \frac{\mu \lambda B}{I^B+1} & \cdots \\
\frac{\mu \lambda B}{I^B+1} & \frac{\mu \lambda B}{I^B+1} & \cdots & \frac{\mu \lambda B}{I^B+1} & \frac{\mu \lambda B}{I^B+1} & \frac{\mu \lambda B}{I^B+1} & \frac{\mu \lambda B}{I^B+1} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
$$

where $I = \gamma + \lambda + \mu$.

With the help of the above “characteristic equations” we investigate the boundary spectrum of $A$ in more detail. Since by Theorem 2.1 the semigroup is bounded, the spectral bound $s(A)$ of $A$ is not greater than 0. It indeed coincides with 0 as we can conclude from the following lemma:

**Lemma 3.4.** 0 is an eigenvalue of $A$, i.e. $0 \in \rho(A)$.

**Proof.** By Lemma 3.2 it suffices to prove that $1 \in \sigma_\rho(\Phi D_0)$. Since $\Phi D_0 : l^1 \to l^1$, and

$$\Phi D_0 = \begin{pmatrix}
\sum_{k=0}^{B} pq^k & \sum_{k=0}^{B-1} pq^k & \sum_{k=0}^{B-2} pq^k & \cdots & p + pq & p & 0 & 0 & \cdots \\
pq^{B+1} & pq^B & pq^{B-1} & \cdots & pq^2 & pq & p & 0 & \cdots \\
pq^{B+2} & pq^{B+1} & pq^B & \cdots & pq^3 & pq^2 & pq & p & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
$$

where, $p = \frac{\mu}{\mu + \lambda}$, $q = \frac{\lambda}{\mu + \lambda}$.

The equation $\Phi D_0 c = c$ is equivalent to the following system of equations:

$$\begin{align*}
\left(\sum_{k=0}^{B} pq^k\right) c_1 + \left(\sum_{k=0}^{B-1} pq^k\right) c_2 + \left(\sum_{k=0}^{B-2} pq^k\right) c_3 + \cdots + (p + pq)c_B + pc_B+1 &= c_1 \\
pq^{B+1} c_1 + pq^B c_2 + pq^{B-1} c_3 + \cdots + pq^2 c_B + pc_B+1 + pc_B+2 &= c_2 \\
pq^{B+2} c_1 + pq^{B+1} c_2 + pq^B c_3 + \cdots + pq^3 c_B + pc_B+1 + pc_B+2 + pc_B+3 &= c_3 \\
pq^{B+3} c_1 + pq^{B+2} c_2 + pq^{B+1} c_3 + \cdots + pq^4 c_B + pc_B+1 + pc^2 c_B+2 + pc^2 c_B+3 + pc_B+4 &= c_4 \\
pq^{B+4} c_1 + pq^{B+3} c_2 + pq^{B+2} c_3 + \cdots + pq^5 c_B + pc^3 c_B+1 + pc^3 c_B+2 + pc^3 c_B+3 + pc_B+4 &= c_5 \\
& \vdots \\
\end{align*}
$$

From this system of equations we obtain
\[ c_{B+n+1} = \frac{c_{n+1} - q^n c_n}{1 - q}, \quad n \geq 2. \] (*)

We now define the function
\[ f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto f(x) := q^{(B+1)x} - q^{(B+1)x+1} - q^x + q. \]

Clearly, \( f \) is continuously differentiable and
\[ f'(x) = (B + 1)(1 - q) \ln q e^{(B+1)x}q - \ln q e^{xq}. \]

Since the traffic intensity \( \rho = \frac{\lambda}{\mu + \lambda} < 1 \), it follows that \( q = \frac{\lambda}{\mu + \lambda} < \frac{1}{2} \) and thus \( (B + 1)(1 - q) > 1 \). Hence we can estimate
\[ f'(0) = (B + 1)(1 - q) \ln q - \ln q < 0 \]

therefore there exists \( x_0 > 0 \) such that \( f'(x) < 0 \) for all \( x \in (0, x_0) \), this means that \( f(x) \) is decreasing on \( (0, x_0) \), since \( \lim_{x \to +\infty} f(x) = q \), \( f(0) = 0 \) and \( f(x) < 0 \) for all \( x \in (0, x_0) \), hence there exists \( a \in \mathbb{R} \) such that \( f(a) = 0 \).

We take \( c_n = q^{na} \), \( n \geq 2 \) and substitute \( c_n = q^{na}, n \geq 2 \) into the Eq. (\( \ast \)), then
\[ q^{(B+n+1)a} = \frac{q^{(n+1)a} - q^{na}}{1 - q} \]
\[ \implies q^{(B+1)a} - q^{(B+1)a+1} - q^a + q = 0 \]
i.e., \( c_n = q^{na}, n \geq 2 \) is a solution of the Eq. (\( \ast \)). Substituting \( c_n = q^{na}, n \geq 2 \) into the first equation in the above system of equations we obtain that
\[ q^{B+1}c_1 = \left( \sum_{k=0}^{B-1} pq^k \right) q^{2a} + \left( \sum_{k=0}^{B-2} pq^k \right) q^{3a} + \cdots + (p + pq)q^{Ba} + pq^{(B+1)a} \]
\[ = pq^{2a} \sum_{k=0}^{B-1} q^k + pq^{3a} \sum_{k=0}^{B-2} q^k + \cdots + (p + pq)q^{Ba} + pq^{(B+1)a} \]
\[ = pq^{2a} \sum_{k=0}^{B-1} \frac{1 - q^B}{1 - q} + pq^{3a} \sum_{k=0}^{B-2} \frac{1 - q^{B-1}}{1 - q} + \cdots + pq^{Ba} \sum_{k=0}^{1} \frac{1 - q^a}{1 - q} + pq^{(B+1)a} \]
\[ = q^{2a} \left( 1 + q^a + q^{2a} + \cdots + q^{(B-1)a} \right) - q^{2a} \sum_{k=0}^{B-1} \frac{1 - q^a}{1 - q^a} + \cdots + q^{(B-1)(a-1)} \]
\[ = q^{2a} \left( 1 - q^{Ba} \right) (1 - q^{a-1}) \left( 1 - q^a \right) \]
\[ \implies c_1 = q^{2a-B-1} \frac{1 - q^{Ba}(1 - q^{a-1}) - (q^B - q^{Ba})(1 - q^a)}{(1 - q^a)(1 - q^{a-1})}. \]

Since
\[ \sum_{n=1}^{\infty} |c_n| = q^{2a-B-1} \frac{1 - q^{Ba}(1 - q^{a-1}) - (q^B - q^{Ba})(1 - q^a)}{(1 - q^a)(1 - q^{a-1})} + \sum_{n=2}^{\infty} q^{na} \]
\[ = q^{2a-B-1} \frac{1 - q^{Ba}(1 - q^{a-1}) - (q^B - q^{Ba})(1 - q^a)}{(1 - q^a)(1 - q^{a-1})} + q^{2a} (1 + q^a + q^{2a} + \cdots) \]
\[ = q^{2a-B-1} \frac{1 - q^{Ba}(1 - q^{a-1}) - (q^B - q^{Ba})(1 - q^a)}{(1 - q^a)(1 - q^{a-1})} + \frac{q^{2a}}{1 - q^a} < +\infty. \]

Obviously, \( c = (c_1, c_2, c_3, \ldots) \in l^1 \) and \( c \) is a fixed point of the operator \( \Phi D_0 \), it follows from this that \( 1 \in \sigma_p(\Phi D_0) \). By Lemma 3.2 we conclude that \( 0 \in \sigma_p(A) \). \( \Box \)
0 is the only spectral value of $A$ on the imaginary axis as the following lemma shows.

**Lemma 3.5.**

\[ \sigma(A) \cap i\mathbb{R} = \{0\}. \]

**Proof.** Since for any $\gamma = ai, a \in i\mathbb{R}$, $a \neq 0$, we have

\[
\Phi \gamma = R(\gamma) = \begin{pmatrix}
\frac{\mu \lambda}{(\lambda + \mu)\Gamma} + \sum_{k=1}^{B} \frac{\mu \lambda^k}{\Gamma^{k+1}} + \sum_{k=0}^{B-1} \frac{\mu \lambda^k}{\Gamma^{k+1}} + \sum_{k=0}^{B-2} \frac{\mu \lambda^k}{\Gamma^{k+1}} \cdots \frac{\mu \lambda}{\Gamma^2} \frac{\mu}{\Gamma} 0 0 \\
\frac{\mu \lambda B^2 + 1}{\Gamma^{B+2}} \frac{\mu \lambda^B}{\Gamma^{B+1}} \frac{\mu \lambda B^2}{\Gamma^{B+1}} \cdots \frac{\mu \lambda^2}{\Gamma^3} \frac{\mu \lambda}{\Gamma^2} 0 \\
\frac{\mu \lambda (B+1)}{\Gamma^{B+3}} \frac{\mu \lambda B}{\Gamma^{B+2}} \frac{\mu \lambda B}{\Gamma^{B+2}} \cdots \frac{\mu \lambda^2}{\Gamma^3} \frac{\mu \lambda^2}{\Gamma^2} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}
\]

where $\Gamma = \gamma + \lambda + \mu$ and for $j \geq B + 1$ we have

\[
\sum_{j=1}^{\infty} |a_{ij}| = \frac{|\mu|}{|\Gamma|} + \frac{|\mu \lambda|}{|\Gamma^2|} + \frac{|\mu \lambda^2|}{|\Gamma^3|} + \frac{|\mu \lambda^3|}{|\Gamma^4|} + \cdots \leq \frac{\mu}{|\Gamma|} \sum_{k=0}^{\infty} \left( \frac{|\lambda|}{|\Gamma|} \right)^k = \frac{\mu}{|\Gamma| - \lambda} < 1.
\]

For $1 \leq j < B + 1$,

\[
\sum_{j=1}^{\infty} |a_{ij}| \leq \left| \frac{\mu \lambda}{(\lambda + \mu)\Gamma} \right| + \left| \frac{\mu \lambda^2}{\Gamma^2} \right| + \left| \frac{\mu \lambda^2}{\Gamma^3} \right| + \left| \frac{\mu \lambda^3}{\Gamma^4} \right| + \cdots \leq \frac{\mu}{|\Gamma|} \sum_{k=1}^{\infty} \left( \frac{|\lambda|}{|\Gamma|} \right)^k \leq \frac{\mu}{|\Gamma|} \sum_{k=0}^{\infty} \left( \frac{|\lambda|}{|\Gamma|} \right)^k < 1
\]

Therefore $\|\Phi \gamma\| < 1$ for all $\gamma = ai, a \in R, a \neq 0$, namely $\|r(\Phi \gamma)\| \leq \|\Phi \gamma\| < 1$, it follows from this that $1 \notin \sigma(\Phi \gamma)$. By Lemma 3.2 we obtain that $\gamma \notin \sigma(A)$, i.e. $\sigma(A) \cap iR = \{0\}$. \qed

In the next step we determine the resolvent of $A$ in terms of the resolvent of $A_0$, the Dirichlet operator $D_\gamma$ and the boundary operator $\Phi$.

**Lemma 3.6.** For any $\gamma \in \rho(A_0) \cap \rho(A)$ one has

\[ R(\gamma, A) = R(\gamma, A_0) + D_\gamma (\text{Id} - \Phi D_\gamma)^{-1} \Phi R(\gamma, A_0). \]

**Proof.** If $\gamma \in \rho(A_0) \cap \rho(A)$ then by Lemma 3.2 we have $1 \in \rho(\Phi D_\gamma) \cap \rho(D_\gamma \Phi)$. Hence, we can compute

\[
\text{Id} = \text{Id} - D_\gamma \Phi + D_\gamma \Phi
\]

\[ = \text{Id} - D_\gamma \Phi + D_\gamma (\text{Id} - \Phi D_\gamma)^{-1} \Phi
\]

\[ = \text{Id} - D_\gamma \Phi + D_\gamma (\text{Id} - \Phi D_\gamma)^{-1} \Phi - D_\gamma \Phi D_\gamma (\text{Id} - \Phi D_\gamma)^{-1} \Phi
\]

\[ = \text{Id} - D_\gamma \Phi + (\text{Id} - D_\gamma \Phi) D_\gamma (\text{Id} - \Phi D_\gamma)^{-1} \Phi.
\]
Multiplying both sides by \((\text{Id} - D_{\gamma} \Phi)^{-1}\) yields
\[
(\text{Id} - D_{\gamma} \Phi)^{-1} = \text{Id} + D_{\gamma} (\text{Id} - \Phi D_{\gamma})^{-1} \Phi.
\] (8)

By (7) the resolvent of \(A\) in \(\gamma\) is given by
\[
R(\gamma, A) = (\text{Id} - D_{\gamma} \Phi)^{-1} R(\gamma, A_0).
\] (9)

Putting (8) in the formula (9) for the resolvent we finally obtain
\[
R(\gamma, A) = R(\gamma, A_0) + D_{\gamma} (\text{Id} - \Phi D_{\gamma})^{-1} \Phi R(\gamma, A_0).
\] \(\Box\)

We can compute the resolvent of \(A_0\) explicitly applying the formula for the inverse of operator matrices, see [6, Thm. 2.4].

For \(\gamma \in \{\gamma \in \mathbb{C} | \gamma \neq -\lambda\text{ and } \text{Re} \gamma > -(\lambda + \mu)\}\) the resolvent of \(A_0\) is obtained as
\[
R(\gamma, A_0) = \begin{pmatrix}
\frac{1}{\gamma + \lambda} & \frac{\mu}{\gamma + \lambda} & \int_0^\infty R(\gamma, D) \, dx & 0 & 0 & \cdots \\
0 & R(\gamma, D) & 0 & 0 & \cdots \\
0 & \lambda R^2(\gamma, D) & R(\gamma, D) & 0 & \cdots \\
0 & (\lambda)^2 R^3(\gamma, D) & \lambda R(\gamma, D) & R(\gamma, D) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}
\]

where
\[
D := -\frac{d}{dx} - (\lambda + \mu), \quad (R(\gamma, D)p)(x) = e^{-(\gamma + \lambda + \mu)x} \int_0^x e^{(\gamma + \lambda + \mu)s} p(s) \, ds
\]
for \(p \in L^1[0, \infty)\).

The above representation for the resolvent in particular shows that it is a positive operator for \(\gamma > 0\). We need this property in the following lemma to prove the irreducibility of the semigroup generated by \(A\):

**Lemma 3.7.** The semigroup \((T(t))_{t \geq 0}\) generated by \((A, D(A))\) is irreducible.

**Proof.** It suffices to show that there exists \(\gamma > 0\) such that \(0 \leq p \in X\) implies \(R(\gamma, A)p \gg 0\), see [7, Def. C-III 3.1]. By Lemma 3.6 we have to prove that there exists \(\gamma > 0\) such that \(0 \leq p \in X\) implies
\[
R(\gamma, A_0)p + D_{\gamma}(\text{Id}_{\delta X} - \Phi D_{\gamma})^{-1} \Phi R(\gamma, A_0)p \gg 0.
\]

Suppose that \(\gamma > 0\) and \(0 \leq p \in X\). Then also \(R(\gamma, A_0)p \geq 0\) and \(\Phi R(\gamma, A_0)p \geq 0\). Since \(\|\Phi D_{\gamma}\| < 1\) for any \(\gamma > 0\), the inverse of \(\text{Id}_{\delta X} - \Phi D_{\gamma}\) is given by the Neumann series
\[
(\text{Id}_{\delta X} - \Phi D_{\gamma})^{-1} = \sum_{n=0}^\infty (\Phi D_{\gamma})^n.
\]

We know from the form of \(\Phi D_{\gamma}\) that for every \(i = 1, 2, \ldots\) there exists \(k \in \mathbb{N}\) such that the real number \(((\Phi D_{\gamma})^k \Phi R(\gamma, A_0)p)_i > 0\), i.e. \((\text{Id}_{\delta X} - \Phi D_{\gamma})^{-1} \Phi R(\gamma, A_0)p \gg 0\), and by the form of \(D_{\gamma}\) we have
\[
D_{\gamma}(\text{Id}_{\delta X} - \Phi D_{\gamma})^{-1} \Phi R(\gamma, A_0)p \gg 0.
\]

This implies
\[
R(\gamma, A)p \gg 0.
\]

Therefore the semigroup \((T(t))_{t \geq 0}\) is irreducible. \(\Box\)

**Lemma 3.8.** The set \(\{T(t) : t \geq 0\} \subseteq \mathcal{L}(X)\) is relatively compact for the weak operator topology.
**Proof.** Since $0 \in \sigma_p(A)$, by [8, Cor. IV. 3.8] we know that there exists $0 \neq p \in \text{fix}(T(t))_{t \geq 0}$. By the positivity of the semigroup we have

$$|p| = |T(t)p| \leq T(t)|p| \quad \text{for all } t \geq 0.$$  \hfill (10)

Suppose that $|p| < T(t)|p|$. By Theorem 2.1, $(T(t))_{t \geq 0}$ is a contraction semigroup and the norm on $X$ is strictly monotone, we obtain

$$\|p\| < \|T(t)|p|| \leq \|p\|$$

which is a contradiction, thus in (10),

$$|p| = T(t)|p|$$

holds and we can assume in the following without loss of generality that $p > 0$. Let $n \in \mathbb{N}$ and take $w \in [-nu, nu]$, i.e. $-nu \leq w \leq nu$, then

$$-np = -nT(t)p \leq T(t)w \leq nT(t)p = np \quad \text{for any } t \geq 0.$$  

Since the order interval $[-np, np]$ is weakly compact in $X$, the orbit $\{T(t)w : t \geq 0\}$ is relatively weakly compact in $X$. Since $(T(t))_{t \geq 0}$ is irreducible, we obtain from [7, Prop. C-III 3.5 (a)] that $p$ is a quasi-interior point of $X$ which implies that

$$X_p := \bigcup_{n \geq 1} [-np, np]$$

is dense in $X$. We have shown that $(T(t)w : t \geq 0) \subseteq X_p$ and $(T(t)w : t \geq 0)$ is relatively weakly compact. Since the semigroup $(T(t))_{t \geq 0}$ is bounded, we know from [8, Lemma V. 2.7] that $\{T(t) : t \geq 0\} \subseteq \mathcal{L}(X)$ is relatively weakly compact. \hfill $\square$

The mean ergodicity of the semigroup allows a decomposition of $X$ into the direct sum of $\ker A$ and $\overline{\text{rg}(A)}$. Combining Lemmas 3.4, 3.5, 3.7 and 3.8 with Theorem 2.1 we obtain the following result:

**Theorem 3.9.** $X$ can be decomposed into the direct sum

$$X = X_1 \oplus X_2$$

where $X_1 = \text{fix}(T(t))_{t \geq 0} = \ker A$ is one-dimensional and spanned by a strictly positive eigenvector $\tilde{p} \in \ker A$ of $A$ and $(T(t)|X_2)_{t \geq 0}$ is strongly stable.

**Proof.** By Lemma 3.7, $(T(t) : t \geq 0) \subseteq \mathcal{L}(X)$ is relatively weakly compact, hence we know from [8, Cor. V. 4.6] that $(T(t))_{t \geq 0}$ is mean ergodic, therefore it follows from [8, Lemma V. 4.6] that the space $X$ can be decomposed into

$$X = \ker A \oplus \overline{\text{rg}(A)} := X_1 \oplus X_2$$

where $\ker A = \text{fix}(T(t))_{t \geq 0}$. Since $0 \in \sigma_p(A)$, as in the proof of Lemma 3.8 we can show that there exists $\tilde{p} \in \ker A$ such that $\tilde{p} > 0$. Moreover, we find by the same construction as in the proof of [8, Lemma V. 2.20 (i)] $p' \in X'$ such that $p' > 0$ and $A'p' = 0$. Thus, by [7, Prop. C-III 3.5] we obtain that

$$\dim \ker A = 1$$

and that $\tilde{p}$ is strictly positive, i.e. $\tilde{p} \gg 0$.

Both spaces $X_1$ and $X_2$ are invariant under $(T(t))_{t \geq 0}$. Now we consider the restricted semigroup $(T_2(t))_{t \geq 0}$ where $T_2(t) := T(t)|X_2$. Its generator $(A_2, D(A_2))$ is given by

$$A_2v = Av, \quad D(A_2) = D(A) \cap X_2.$$  

In the next step we show that $\sigma_p(A_2') \cap i\mathbb{R} = \emptyset$. From [8, Prop. IV. 1.12] we have

$$\sigma_p(A_2') = \sigma_r(A_2)$$

where $\sigma_r(A) = \{ \gamma \in C : \text{rg}(\gamma - A) \}$ is not dense in $X_2$ denotes the residual spectrum. Since $\sigma(A_2) \subseteq \sigma(A)$ and $\sigma(A) \cap i\mathbb{R} = \{0\}$, we only have to prove that $0 \notin \sigma_p(A_2') = \sigma_r(A_2)$. Since $(T(t))_{t \geq 0}$ is a mean ergodic bounded
semigroup on $X$, it follows from this that $(T_2(t))_{t \geq 0}$ is a mean ergodic bounded semigroup on $X_2$. By [8, Thm. V. 4.5], ker $A_2$ separates ker $A_2'$. But ker $A_2 = \{0\}$, and thus ker $A_2' = \{0\}$. It follows that $\sigma_p(A_2') = \emptyset$, i.e. $\sigma_p(A_2') \cap i\mathbb{R} = \emptyset$. By [8, Thm. 2.21], we obtain that $(T(t))_{t \geq 0}$ is strongly stable. □

By Theorem 3.9 we obtain the main result in this paper.

**Theorem 3.10.** For all $p \in X$ there exists $\alpha \in \mathbb{R}$ such that

$$\lim_{t \to \infty} T(t)p = \alpha \tilde{p},$$

where ker $A = \langle \tilde{p} \rangle$, $\tilde{p} \gg 0$.

**Proof.** By Theorem 3.9, we have

$$X = X_1 \oplus X_2$$

where $X_1 = \text{fix}(T(t))_{t \geq 0} = \text{ker} A$ is one-dimensional and spanned by a strictly positive eigenvector $\tilde{p} \in \text{ker} A$, and $(T(t)|_{X_2})_{t \geq 0}$ is strongly stable, i.e. for any $p \in X$ there exists $\alpha \in \mathbb{R}$ and $p_2 \in X_2$ such that

$$p = \alpha \tilde{p} + p_2 \quad \text{and} \quad \lim_{t \to \infty} T(t)p_2 = 0.$$ 

Since $T(t)p = T(t)(\tilde{p} + p_2) = \alpha T(t)\tilde{p} + T(t)p_2 = \alpha \tilde{p} + T(t)p_2$, it follows that $\lim_{t \to \infty} T(t)p = \alpha \tilde{p}$. □

By Theorem 3.10 we obtain asymptotic stability of the solution of $M/M^B/1$ model.

**Theorem 3.11.** The time-dependent solution of the system (1)–(6) converges strongly to the steady-state solution as time tends to infinite, that is, $\lim_{t \to \infty} p(\cdot, t) = \alpha \tilde{p}$, where $\alpha > 0$ and $\tilde{p}$ as in Theorem 3.10.

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**References**