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# Hecke operators on Drinfeld cusp forms ☆

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**Abstract**

In this paper, we study the Drinfeld cusp forms for $\Gamma_1(T)$ and $\Gamma(T)$ using Teitelbaum's interpretation as harmonic cocycles. We obtain explicit eigenvalues of Hecke operators associated to degree one prime ideals acting on the cusp forms for $\Gamma_1(T)$ of small weights and conclude that these Hecke operators are simultaneously diagonalizable. We also show that the Hecke operators are not diagonalizable in general for $\Gamma_1(T)$ of large weights, and not for $\Gamma(T)$ even of small weights. The Hecke eigenvalues on cusp forms for $\Gamma(T)$ with small weights are determined and the eigenspaces characterized.


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*Keywords:* Drinfeld cusp forms; Harmonic cocycles; Hecke operators

## 1. Introduction

Hecke operators played a crucial role in the study of the arithmetic of classical modular forms. Their actions on cusp forms are skew Hermitian with respect to the Petersson inner product, and

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hence they are diagonalizable. This property is fundamental in understanding the classical cusp forms.

The function field analogue of the Poincaré upper half plane is the Drinfeld upper half plane. Parallel to the classical modular forms, there are the Drinfeld modular forms introduced by Goss in [Gos80]. He also defined the Hecke operators in a similar way. While certain arithmetic properties are alike for classical and Drinfeld modular forms, there are also sharp differences. For instance, Böckle [Böc04] showed that the Eichler–Shimura correspondence over a function field associates a Drinfeld (cuspidal) common eigenform of Hecke operators to a degree one, instead of degree two as in the classical case, Galois representation, reflecting different multiplicative relations on Hecke operators. Moreover, since the domain and image of Drinfeld modular forms have the same positive characteristic, there is no adequate analog of the Petersson inner product. Hence the diagonalizability of the Hecke operators on Drinfeld forms still remains an open question.

Using the residue map, Teitelbaum [Tei91] in 1991 gave an interpretation of Drinfeld cusp forms as harmonic cocycles on the directed edges of a regular tree $\mathcal{T}$. The actions of the Hecke operators were carried over to harmonic cocycles by Böckle [Böc04]. Since the directed edges of $\mathcal{T}$ are parametrized by cosets of $\mathrm{PGL}_2$ over a local field $F$ modulo its Iwahori subgroup $\mathcal{I}$, the Drinfeld cusp forms for a congruence subgroup $\Gamma$ can then be regarded as vector-valued left $\Gamma$-equivariant functions on $\mathrm{PGL}_2(F)/\mathcal{I}$, and hence they are determined by the values on $\Gamma\backslash \mathrm{PGL}_2(F)/\mathcal{I}$. This viewpoint is quite helpful in computation when a fundamental domain is easily described. Another advantage is that, by means of the strong approximation theorem, the Drinfeld cusp forms can also be seen as equivariant functions in adelic setting. This approach appeared in Gekeler and Reversat [GR96] and also in Böckle [Böc04].

Let $K = \mathbb{F}_q(T)$ be the rational function field. The arithmetic of Drinfeld modular forms for the full modular group $\mathrm{GL}_2(\mathbb{F}_q[T])$ was studied extensively in [Gos80] and [Gek88]. Using geometric methods, Böckle and Pink investigated in [Böc04] the structure of double cusp forms for $\Gamma_1(T)$ with weight $k \leqslant q+2$. They also computed the Hecke eigenvalues for weight 4 double cusp forms.

The purpose of this paper is to study Drinfeld cusp and double cusp forms for the congruence subgroups $\Gamma_1(T)$ and $\Gamma(T)$ of $\mathrm{GL}_2(\mathbb{F}_q[T])$, with emphasis on the behavior of the Hecke operators. Working with harmonic cocycles, we determine the eigenvalues and the corresponding eigenspaces for Hecke operators at degree one places of $K$. As we shall see, the diagonalizability of the Hecke operators depends on the group and also the weight. More precisely, the Hecke operators on the space of cusp forms of $\Gamma_1(T)$ are diagonalizable for small weights $k \leqslant q$, but not for large weights $k > q$ in general. Further, as we pass from $\Gamma_1(T)$ to its subgroup $\Gamma(T)$, the distinct eigenvalues for Hecke operators on cusp forms with weights $k \leqslant q$ remain the same although the multiplicities may differ. We also characterize each eigenspace. Explicit computations show that the Hecke actions on the spaces of cusp forms and double cusp forms for $\Gamma(T)$ of small weights change from diagonalizable to not diagonalizable as the weight increases.

This paper is organized as follows. The Drinfeld cusp forms and properties of the tree are reviewed in Sections 2 and 3, respectively. Harmonic cocycles are recalled in Section 4. In Section 5 we summarize Teitelbaum's isomorphism between Drinfeld cusp forms and harmonic cocycles and describe Böckle's criterion of double cusp forms as harmonic cocycles. The actions of the Hecke operators on harmonic cocycles are introduced in Section 6. The body of this paper is Sections 7 and 8, dealing with cusp forms for $\Gamma_1(T)$ and $\Gamma(T)$, respectively. The final section gives examples of the Hecke actions on the cusp forms for $\Gamma(T)$ for weights $k = 3, 4$ and 5, making explicit the main results of the paper.

## 2. Drinfeld cusp forms

Let $K = \mathbb{F}(T)$ be the rational function field over the finite field $\mathbb{F}$ with $q$ elements. Write $\infty$ for the place of $K$ with $1/T$ as a uniformizer. Then $A = \mathbb{F}[T]$ is the ring of functions in $K$ regular outside $\infty$. Denote by $K_\infty$ the completion of $K$ at $\infty$, $\mathcal{O}_\infty$ its ring of integers, and $\mathcal{P}_\infty$ the maximal ideal in $\mathcal{O}_\infty$. Let $C = \widehat{\bar{K}}_\infty$ be the completion of an algebraic closure of $K_\infty$.

The Drinfeld upper half plane $\Omega = C \smallsetminus K_\infty$ is endowed with a rigid analytic structure, on which $\mathrm{GL}_2(K_\infty)$ acts by fractional linear transformations. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(K_\infty)$, $m, k \in \mathbb{Z}$ and $f : \Omega \to C$, define

$$(f \underset{k,m}{|} \gamma)(z) := f(\gamma z)(\det \gamma)^m (cz + d)^{-k}.$$

Let $\Gamma$ be a congruence subgroup of the modular group $\mathrm{GL}_2(A)$. It has finitely many cusps, represented by $\Gamma \backslash \mathbb{P}^1(K)$. A rigid analytic function $f : \Omega \to C$ is called a *Drinfeld cusp form for $\Gamma$ of weight $k$ and type $m$ for $\Gamma$* if it satisfies

(i) $f \underset{k,m}{|} \gamma = f$ for all $\gamma \in \Gamma$;
(ii) $f$ is holomorphic at all cusps;
(iii) $f$ vanishes at all cusps.

The cusp forms for $\Gamma$ of weight $k$ and type $m$ form a vector space $S_{k,m}(\Gamma)$ over $C$. It contains a subspace $S^2_{k,m}(\Gamma)$ of double cusp forms, which vanish at all cusps at least twice.

**Remark.** While the weight can be any integer, the possible type is an element in $\mathbb{Z}/(m_\Gamma)$, where $m_\Gamma$ is the order of $\det(\Gamma)$, a subgroup of $\mathbb{F}_q^\times$. Thus $S_{k,m}(\Gamma) \neq 0$ implies $k \equiv 2m \bmod (m_\Gamma)$. In particular, if $m_\Gamma = 1$, which is the case to be considered in this paper, then for fixed $k$, all $S_{k,m}(\Gamma)$ are identical, and the same holds for $S^2_{k,m}(\Gamma)$.

The following dimension formula for cusp forms was computed by Teitelbaum.

**Proposition 1.** *(See [Tei91].) Let $g_\Gamma$ be the genus of $\Gamma \backslash \bar{\Omega}$ and $h_\Gamma$ the number of cusps of $\Gamma \backslash \Omega$. If $\Gamma$ is $p'$-torsion free and $m_\Gamma = 1$, then*

$$\dim_C S_{k,m}(\Gamma) = (k-1)(g_\Gamma + h_\Gamma - 1).$$

## 3. The tree $\mathcal{T}$

The coset space $\mathrm{PGL}_2(K_\infty)/\mathrm{PGL}_2(\mathcal{O}_\infty) =: \mathcal{T}$ may be interpreted as a $(q+1)$-regular tree on which the group $\mathrm{GL}_2(K_\infty)$ acts by left translations. The vertices of $\mathcal{T}$ are the cosets $\mathrm{PGL}_2(K_\infty)/\mathrm{PGL}_2(\mathcal{O}_\infty)$, while the directed edges of $\mathcal{T}$ are parametrized by $\mathrm{PGL}_2(K_\infty)/\mathfrak{I}_\infty$, where

$$\mathfrak{I}_\infty = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_\infty) \colon c \in \mathcal{P}_\infty \right\} \Big/ \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_\infty) \right\}$$

is the Iwahori subgroup of $\mathrm{PGL}_2(\mathcal{O}_\infty)$. The edge represented by $g \in \mathrm{GL}_2(K_\infty)$ will be abbreviated as $\langle g \rangle$.

As in Serre [Ser80], a vertex or edge of $\mathcal{T}$ is called *$\Gamma$-stable* if its stabilizer in $\Gamma$ is trivial; otherwise it is $\Gamma$-unstable. Let $\mathcal{T}_\infty$ be the subgraph of $\mathcal{T}$ consisting of unstable vertices and edges. Then $S_0 = \mathrm{Vert}(\mathcal{T}) \smallsetminus \mathrm{Vert}(\mathcal{T}_\infty)$ is the set of stable vertices and $S_1 = [\mathrm{Edge}(\mathcal{T}) \smallsetminus \mathrm{Edge}(\mathcal{T}_\infty)]/\pm$ is the set of non-oriented stable edges.

Two infinite paths in $\mathcal{T}$ are considered equivalent if they differ at only finitely many edges. An *end* of $\mathcal{T}$ is an equivalence class of infinite paths $\{e_1, e_2, \ldots\}$. There is a canonical bijection between the set of ends and $\mathbb{P}^1(K_\infty)$, the boundary of $\Omega$; the rational ends are $\mathbb{P}^1(K)$, corresponding to the cusps. The stabilizer of an unstable vertex $v$ fixes a unique rational end, and similarly for an unstable edge $e$; denote them by $b(v)$ and $b(e)$, respectively. An edge $w$ of $\mathcal{T}$ is a *source* of an unstable edge $e$ if $w$ has the same orientation as $e$ and there exists an unstable boundary vertex $v$ of $w$ such that the path from $v$ to its end $b(v)$ passes through $e$. If $e$ is stable, then it is its own source. Denote by $\mathrm{src}(e)$ the set of all sources of $e$. There are certain inaccuracies in [Tei91] concerning the sources of an edge. We thank the referee for pointing them out.

## 4. Harmonic cocycles

For $k \geqslant 0$ and $m \in \mathbb{Z}$, let $V(k,m)$ be the $(k-1)$-dimensional vector space over $C$ with a basis $\{X^j Y^{k-2-j}\colon 0 \leqslant j \leqslant k-2\}$ endowed with the action of $\mathrm{GL}_2(K_\infty)$ given by

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : X^j Y^{k-2-j} \mapsto (\det\gamma)^{m-1}(dX - bY)^j(-cX + aY)^{k-2-j}$$

for all $0 \leqslant j \leqslant k-2$. This then induces the action of $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(K_\infty)$ on the dual space $\mathrm{Hom}(V(k,m), C)$ by sending $w \in \mathrm{Hom}(V(k,m), C)$ to

$$(\gamma w)\big(X^j Y^{k-2-j}\big) = (\det\gamma)^{1-m} w\big((aX + bY)^j (cX + dY)^{k-2-j}\big)$$

for $0 \leqslant j \leqslant k-2$.

A *harmonic cocycle of weight $k$ and type $m$ for $\Gamma$* is a function $\mathbf{c}$ from the set of directed edges of $\mathcal{T}$ to $\mathrm{Hom}(V(k,m), C)$ satisfying

(a) for all vertices $v$ of $\mathcal{T}$,

$$\sum_{e \mapsto v} \mathbf{c}(e) = 0,$$

where $e$ runs through all edges in $\mathcal{T}$ with terminal vertex $v$;

(b) for all edges $e$ of $\mathcal{T}$, $\mathbf{c}(\bar{e}) = -\mathbf{c}(e)$, where $\bar{e}$ denotes $e$ with reversed orientation;

(c) it is $\Gamma$-equivariant, namely, for all edges $e$ and elements $\gamma \in \Gamma$,

$$\mathbf{c}(\gamma e) = \gamma\big(\mathbf{c}(e)\big).$$

The last condition means

$$\mathbf{c}(\gamma e)\big(X^j Y^{k-2-j}\big) = \big(\gamma\mathbf{c}(e)\big)\big(X^j Y^{k-2-j}\big) = (\det\gamma)^{1-m}\mathbf{c}(e)\big((aX + bY)^j (cX + dY)^{k-2-j}\big)$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and $0 \leqslant j \leqslant k-2$. Let $H_{k,m}(\Gamma)$ denote the space of harmonic cocycles of weight $k$ and type $m$ for $\Gamma$.

As observed by Teitelbaum [Tei91], the value of a cocycle $\mathbf{c} \in H_{k,m}(\Gamma)$ at a directed edge $e$ is the sum of $\mathbf{c}$ evaluated at the source of $e$. Consequently, cocycles in $H_{k,m}(\Gamma)$ are determined by their values on $\Gamma \backslash S_1$.

## 5. Cusp forms and harmonic cocycles

There is a building map from $\Omega$ to $\mathcal{T}$ commuting with the action of $\mathrm{GL}_2(K_\infty)$ (cf. [Fv04] and [Tei91]). Using it one can define, for any $C$-valued holomorphic 1-form $f(z)\,dz$ on $\Omega$, the residue $\mathrm{Res}_e\, f(z)\,dz$ at any directed edge $e$ of $\mathcal{T}$. This in turn gives a way to associate harmonic cocycles to cusp forms. More precisely, for each cusp form $f \in S_{k,m}(\Gamma)$, define the function $\mathrm{Res}(f)$ from the directed edges of $\mathcal{T}$ to $\mathrm{Hom}(V(k,m), C)$ by assigning, for any directed edge $e$, the values of $\mathrm{Res}(f)(e)$ at the basis elements $X^j Y^{k-2-j}$ to be

$$\mathrm{Res}(f)(e)\big(X^j Y^{k-2-j}\big) = \mathrm{Res}_e\, z^j f(z)\,dz \tag{5.1}$$

for all $0 \leqslant j \leqslant k-2$. Then properties (a) and (b) follow from the rigid analytic residue theorem, and (c) from the modularity of $f$. Therefore $\mathrm{Res}(f)$ lies in $H_{k,m}(\Gamma)$.

**Theorem 2.** *(See Teitelbaum [Tei91].) The residue map* $\mathrm{Res}: S_{k,m}(\Gamma) \to H_{k,m}(\Gamma)$ *is an isomorphism.*

Thus we identify cusp forms with harmonic cocycles. This allows us to view cusp forms for $\Gamma$ as vector valued left $\Gamma$-equivariant functions on $\mathrm{PGL}_2(K_\infty)/\mathfrak{I}_\infty$, or left $\mathrm{GL}_2(K)$-equivariant functions on the adelic group $\mathrm{GL}_2(A_K)$ by applying the strong approximation theorem (cf. [GR96] and [Rev00]). When $k=2$, such functions are $C$-valued and $\Gamma$-equivariance becomes $\Gamma$-invariance. Indeed, some harmonic cocycles can be lifted to $\mathbb{Z}$-valued functions on $\mathrm{GL}_2(K)\backslash \mathrm{GL}_2(\mathbb{A}_K)$, as remarked in [GR96,Rev00] and [Böc04].

Denote by $H^2_{k,m}(\Gamma)$ the image of $S^2_{k,m}(\Gamma)$ under the Res map. To describe double cusp forms as cocycles, we define the source of an end $[s]$ to be

$$\mathrm{src}\big([s]\big) := \big\{e\colon\ e \text{ is stable}, t(e) \text{ is unstable and } b\big(t(e)\big) = [s]\big\},$$

where $t(e)$ denotes the terminal vertex of $e$. The following result of Böckle characterizes the image of double cusp forms under the residue map.

**Theorem 3.** *(See Böckle [Böc04].) Let $\Gamma_{[s]}$ denote the $\Gamma$-stabilizer of an end $[s]$ representing a cusp of $\Gamma$. Then*:

(a) *The subspace of $V(k,m)$ stabilized by $\Gamma_{[s]}$, denoted $V(k,m)^{\Gamma_{[s]}}$, is one-dimensional.*
(b) *$\Gamma_{[s]}$ acts freely on* $\mathrm{src}([s])$ *with finitely many orbits, represented by edges $e_1^{[s]}, \ldots, e_{l_s}^{[s]}$.*
(c) *Let $f \in S_{k,m}(\Gamma)$ and* $\mathbf{c} = \mathrm{Res}(f)$. *Then $f$ is a double cusp form if and only if for any cusp $[s]$, $\sum_{i=1}^{l_s} \mathbf{c}(e_i^{[s]})(g_s) = 0$ for any generator $g_s$ of $V(k,m)^{\Gamma_{[s]}}$.*

Combined with Proposition 1, one obtains the dimension formula for the space of double cusp forms:

**Proposition 4.** *(See Böckle [Böc04].) Let $g_\Gamma$ be the genus of $\Gamma\backslash\bar{\Omega}$ and $h_\Gamma$ the number of cusps of $\Gamma\backslash\Omega$. If $\Gamma$ is $p'$-torsion free and $m_\Gamma = 1$, then*

$$\dim_C S^2_{k,m}(\Gamma) = \begin{cases} g_\Gamma & \text{if } k = 2; \\ (k-2)(g_\Gamma + h_\Gamma - 1) + g_\Gamma - 1 & \text{if } k > 2. \end{cases}$$

## 6. Hecke operators

We shall focus on the congruence groups $\Gamma = \Gamma_1(T)$ and $\Gamma(T)$ defined as

$$\Gamma_1(T) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(A)\colon\ a \equiv d \equiv 1 \text{ and } c \equiv 0 \bmod T \right\}$$

and

$$\Gamma(T) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(A)\colon\ a \equiv d \equiv 1 \text{ and } b \equiv c \equiv 0 \bmod T \right\}.$$

They are $p'$-torsion free. Let $\mathfrak{P} \neq (T)$ be a maximal ideal of $A$; choose the generator $P$ to be the irreducible polynomial in $\mathfrak{P}$ satisfying $P(0) = 1$. Suppose $\deg P = d$. Then

$$\Gamma(T)\begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}\Gamma(T) = \Gamma(T)\begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} \sqcup \bigsqcup_{b\in A,\,\deg b<d} \Gamma(T)\begin{pmatrix} 1 & b(1-P) \\ 0 & P \end{pmatrix}.$$

The *Hecke operator* at $\mathfrak{P}$ is defined using the coset representatives of this double coset:

$$T_\mathfrak{P} = P^{k-m-1}\left[\begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} + \sum_{b\in A,\,\deg b<d} \begin{pmatrix} 1 & b(1-P) \\ 0 & P \end{pmatrix}\right],$$

which acts on a holomorphic function $f$ on $\Omega$ via $\underset{k,m}{|}\, T_\mathfrak{P}$. That is,

$$\begin{aligned} T_\mathfrak{P} f(z) &= \big(f \underset{k,m}{|}\, T_\mathfrak{P}\big)(z) \\ &= P^{k-m-1}\left[f \underset{k,m}{|} \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}(z) + \sum_{b\in A,\,\deg b<d} f \underset{k,m}{|} \begin{pmatrix} 1 & b(1-P) \\ 0 & P \end{pmatrix}(z)\right]. \end{aligned}$$

The generator $P$ is chosen in order to avoid the use of characters. Here we have followed the normalization in Böckle [Böc04], which is a constant multiple of that defined by Goss [Gos80]. It is easy to check that $T_\mathfrak{P}$ sends $S_{k,m}(\Gamma)$ to itself and preserves the double cusp forms. For two prime ideals $\mathfrak{P}$ and $\mathfrak{Q}$ not equal to $(T)$, $T_\mathfrak{P}$ commutes with $T_\mathfrak{Q}$.

The action of the Hecke operator $T_\mathfrak{P}$ can be transported to harmonic cocycles by means of the residue map. This was carried out in [Böc04]. Precisely, $T_\mathfrak{P}$ sends $\mathbf{c} \in H_{k,m}(\Gamma)$ to a harmonic cocycle whose value at a directed edge $e$ of $\mathcal{T}$ is

$$T_{\mathfrak{P}}\mathbf{c}(e) = P^{k-m-1}\Bigg(\begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}^{-1} \mathbf{c}\Bigg(\begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} e\Bigg)$$

$$+ \sum_{b\in A,\, \deg b<d} \begin{pmatrix} 1 & b(1-P) \\ 0 & P \end{pmatrix}^{-1} \mathbf{c}\Bigg(\begin{pmatrix} 1 & b(1-P) \\ 0 & P \end{pmatrix} e\Bigg)\Bigg). \tag{6.1}$$

This formula will be used to compute the eigenvalues and eigenfunctions of Hecke operators. As we shall see from the cases $\Gamma = \Gamma_1(T)$ and $\Gamma(T)$, the Hecke operators are sometimes diagonalizable and sometimes not, depending on the group and the weight.

## 7. Cusp forms for $\Gamma_1(T)$

In this section we consider cusp forms and double cusp forms for $\Gamma_1(T)$. We may choose as a fundamental domain of $\Gamma_1(T)\backslash\mathcal{T}$ the path connecting the cusp $[\infty] = \binom{1}{0}$ and cusp $[0] = \binom{0}{1}$, as shown below. Recall from Section 3 that $\langle g\rangle$ denotes the directed edge represented by $g$.

$$\begin{array}{ccccccccccc}
 & & & \langle(\begin{smallmatrix} 0 & 1 \\ T & 0 \end{smallmatrix})\rangle & & \gamma_0 = \langle(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})\rangle & & \langle(\begin{smallmatrix} 0 & T \\ 1 & 0 \end{smallmatrix})\rangle & & & \\
 & & & \longleftarrow & & \longleftarrow & & \longleftarrow & & & \\
[\infty] & \cdots & \leftrightarrows (\begin{smallmatrix} 1 & 0 \\ 0 & T^2 \end{smallmatrix}) & & (\begin{smallmatrix} 1 & 0 \\ 0 & T \end{smallmatrix}) & & (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}) & & (\begin{smallmatrix} T & 0 \\ 0 & 1 \end{smallmatrix}) \leftrightarrows & \cdots & [0] \\
 & & & \longrightarrow & & \longrightarrow & & \longrightarrow & & & \\
 & & & \langle(\begin{smallmatrix} 1 & 0 \\ 0 & T^2 \end{smallmatrix})\rangle & & \bar\gamma_0 = \langle(\begin{smallmatrix} 1 & 0 \\ 0 & T \end{smallmatrix})\rangle & & \langle(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})\rangle & & &
\end{array}$$

It contains no stable vertices and one stable edge $\langle(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})\rangle$, denoted by $\gamma_0$. Then $g_{\Gamma_1(T)} = 0$ so that $\dim_C S_{k,m}(\Gamma_1(T)) = k-1$ by Proposition 1, and $\dim_C S^2_{2,m}(\Gamma_1(T)) = 0$ and $\dim_C S^2_{k,m}(\Gamma_1(T)) = k-3$ for $k \geqslant 3$ by Proposition 4. Theorem 3 of [Tei91] implies that any harmonic cocycle $\mathbf{c}$ for $\Gamma_1(T)$ automatically vanishes on all edges of the fundamental domain except $\gamma_0$ and its two neighboring edges up to orientation. Further, the value of $\mathbf{c}$ at $\gamma_0$ determines its values at the two neighboring edges by harmonicity. Therefore to determine a harmonic cocycle for $\Gamma_1(T)$, it suffices to first know its value in $\mathrm{Hom}(V(k,m),C)$ at $\gamma_0$, and then extend to other edges by $\Gamma_1(T)$-equivariancy and harmonicity. This is the strategy we shall use to compute the action of the Hecke operators.

The stabilizers of the cusps $[\infty]$ and $[0]$ are $(\Gamma_1(T))_{[\infty]} = \{(\begin{smallmatrix} 1 & c \\ 0 & 1 \end{smallmatrix})\colon c \equiv 0 \bmod T\}$ and $(\Gamma_1(T))_{[0]} = \{(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix})\colon c \equiv 0 \bmod T\}$, respectively. Thus $V(k,m)^{(\Gamma_1(T))_{[\infty]}}$ and $V(k,m)^{(\Gamma_1(T))_{[0]}}$ are generated by $Y^{k-2}$ and $X^{k-2}$, respectively. Also, $(\Gamma_1(T))_{[\infty]}\backslash \mathrm{src}([\infty]) = \{\bar\gamma_0 = \langle(\begin{smallmatrix} 1 & 0 \\ 0 & T \end{smallmatrix})\rangle\}$ and $(\Gamma_1(T))_{[0]}\backslash \mathrm{src}([0]) = \{\gamma_0\}$. Recall that $\bar\gamma_0$ is the opposite of $\gamma_0$. Hence by Theorem 3, we have

**Proposition 5.** $S^2_{k,m}(\Gamma_1(T)) = \{\mathbf{c} \in S_{k,m}(\Gamma_1(T))\colon \mathbf{c}(\gamma_0)(Y^{k-2}) = \mathbf{c}(\gamma_0)(X^{k-2}) = 0\}$.

Now we study the action of the Hecke operators $T_{\mathfrak{P}}$ on $S_{k,m}(\Gamma_1(T))$, where $\mathfrak{P}$ is generated by $P = 1+\alpha T$. Using Eq. (6.1), harmonicity and $\Gamma_1(T)$-equivariancy, and noting $q$ is the cardinality of the field $\mathbb{F}$, we get, for $0 \leqslant j \leqslant k-2$,

$$T_{\mathfrak{P}}\mathbf{c}(\gamma_0)\big(X^j Y^{k-2-j}\big)$$

$$= \mathbf{c}(\gamma_0)\Bigg(X^j (PY)^{k-2-j} + \sum_{m=0}^{\lfloor\frac{j}{q-1}\rfloor}\Bigg(\sum_{l=0}^{j-m(q-1)} \binom{j}{l+m(q-1)}\binom{k-2-j}{l}(1-P)^l$$

$$- P^{k-2-j}\binom{j}{m(q-1)}\Bigg)\Bigg)X^{j-m(q-1)}Y^{(k-2-j)+m(q-1)}$$

$$+ \sum_{n=1}^{\lfloor \frac{k-2-j}{q-1}\rfloor}\Bigg(\sum_{l=n(q-1)}^{k-2-j}\binom{j}{l-n(q-1)}\binom{k-2-j}{l}(1-P)^l$$

$$- P^j\binom{k-2-j}{n(q-1)}T^{n(q-1)}\Bigg)X^{j+n(q-1)}Y^{(k-2-j)-n(q-1)}\Bigg). \tag{7.1}$$

For each $0 \leqslant j \leqslant k-2$, define the harmonic cocycle $\mathbf{c}_j$ by specifying its value at $\gamma_0$ by

$$\mathbf{c}_j(\gamma_0)\big(X^jY^{k-2-j}\big) = 1 \quad \text{and} \quad \mathbf{c}_j(\gamma_0)\big(X^lY^{k-2-l}\big) = 0 \quad \text{for } l \neq j. \tag{7.2}$$

Further, put, for $0 \leqslant j \leqslant k-2$ and a degree one polynomial $Q = 1 + \beta T$, the polynomial

$$\lambda_j(Q) = \sum_{l=0}^{j}\binom{j}{l}\binom{k-2-j}{l}(1-Q)^l = \sum_{l=0}^{\min\{j,k-2-j\}}\binom{j}{l}\binom{k-2-j}{l}(-\beta T)^l. \tag{7.3}$$

Then $\lambda_j(Q)$ has degree at most $\min\{j, k-2-j\}$. Note that $\lambda_0(Q) = \lambda_{k-2}(Q) = 1$ and $\lambda_j(Q) = \lambda_{k-2-j}(Q)$ for all $0 \leqslant j \leqslant k-2$.

To see the behavior of the Hecke operators, we distinguish two cases, according to the weight being small or large. First assume $q \geqslant k \geqslant 2$. In this case (7.1) is reduced to

$$T_{\mathfrak{P}}\mathbf{c}(\gamma_0)\big(X^jY^{k-2-j}\big) = \lambda_j(P)\mathbf{c}(\gamma_0)\big(X^jY^{k-2-j}\big). \tag{7.4}$$

Therefore each $\mathbf{c}_j$ is an eigenfunction of $T_{\mathfrak{P}}$ with eigenvalue $\lambda_j(P)$. We have shown

**Theorem 6.** *Let $\mathfrak{P}$ be a prime ideal of $A$ generated by $P$ with $P(0) = 1$ and* $\deg P = 1$. *Suppose $q \geqslant k \geqslant 2$. Then*

(1) *each $\mathbf{c}_j$, $0 \leqslant j \leqslant k-2$, is an eigenfunction of the Hecke operator $T_{\mathfrak{P}}$ with eigenvalue $\lambda_j(P)$; and*
(2) *the Hecke operators at the ideals of degree one are simultaneously diagonalized on $H_{k,m}(\Gamma_1(T))$ with respect to the basis $\mathbf{c}_j$, $0 \leqslant j \leqslant k-2$.*

It is natural to ask if the $\mathbf{c}_j$, $0 \leqslant j \leqslant k-2$, are also common eigenfunctions of the Hecke operators $T_{\mathfrak{P}}$ for prime ideals $\mathfrak{P}$ of degree $d > 1$; and if so, find the eigenvalues. Our computations lead to the following

**Conjecture.** Let $\mathfrak{P}$ be a prime ideal of $A$ generated by $P$ with $P(0) = 1$ and $\deg P = d \geqslant 1$. Suppose $q \geqslant k \geqslant 2$. Let $\theta$ be a root of $P$. Then each $\mathbf{c}_j$, $0 \leqslant j \leqslant k-2$, is an eigenfunction of the Hecke operator $T_{\mathfrak{P}}$ with eigenvalue $\lambda_j(P) := \prod_{i=0}^{d-1}\lambda_j(1-\theta^{-q^i}T)$. Consequently, the Hecke operators are simultaneously diagonalized on $S_{k,m}(\Gamma_1(T))$.

This conjecture is verified for $d \leqslant 2$. Another evidence is for the case $k=4$ and all $d$, provided by Proposition 15.6 in [Böc04]. It would be nice if the method there could be extended to settle the conjecture.

**Remark.** If we factor the polynomial $\lambda_j(1+T) = \prod_{s=1}^{\deg \lambda_j(1+T)}(1+\delta_s T)$, then the eigenvalue $\lambda_j(P)$ above can also be expressed as $\prod_{s=1}^{\deg \lambda_j(1+T)} P(\delta_s T)$.

It is worth pointing out that the degree of $\lambda_j(P)$ above is at most $d(k-2)/2$. This may be regarded as the Ramanujan conjecture on Drinfeld cusp forms. A similar observation on weights can be found in [Böc04], above Corollary 15.5.

Notice that for $k \leqslant q+2$ and $1 \leqslant j \leqslant q-2$, Eq. (7.1) is easily reduced to (7.4) as well. Therefore for $\mathfrak{P}$ of degree 1, $k=q+1$ and $1 \leqslant j \leqslant k-3$, one gets

$$T_{\mathfrak{P}}\mathbf{c}_j = \lambda_j(P)\mathbf{c}_j.$$

Recall that a double cusp form $\mathbf{c}$ for $\Gamma_1(T)$ satisfies $\mathbf{c}(\gamma_0)(Y^{k-2}) = \mathbf{c}(\gamma_0)(X^{k-2}) = 0$. Therefore $\mathbf{c}_1, \ldots, \mathbf{c}_{k-3}$ form a basis of the subspace of double cusp forms, on which a similar result holds but with a slightly extended range for $k$.

**Proposition 7.** *Let $\mathfrak{P}$ be a prime ideal of $A$ generated by the polynomial $P$ of degree* 1 *with $P(0)=1$. If $q+2 \geqslant k \geqslant 4$, then for all $\mathbf{c} \in S^2_{k,m}(\Gamma_1(T))$ and $1 \leqslant j \leqslant k-3$, one has*

$$T_{\mathfrak{P}}\mathbf{c}(\gamma_0)\big(X^jY^{k-2-j}\big) = \lambda_j(P)\mathbf{c}(\gamma_0)\big(X^jY^{k-2-j}\big).$$

**Proof.** It remains to prove the proposition for the case $k=q+2$, and $j=1$ or $k-3$. In this case, Eq. (7.1) gives, for $\mathbf{c} \in S^2_{k,m}(\Gamma_1(T))$,

$$T_{\mathfrak{P}}\mathbf{c}(\gamma_0)\big(XY^{k-3}\big) = \lambda_1(P)\mathbf{c}(\gamma_0)\big(XY^{k-3}\big)$$

and

$$T_{\mathfrak{P}}\mathbf{c}(\gamma_0)\big(X^{k-3}Y\big) = \lambda_{k-3}(P)\mathbf{c}(\gamma_0)\big(X^{k-3}Y\big)$$

since $\mathbf{c}(\gamma_0)(Y^{k-2}) = \mathbf{c}(\gamma_0)(X^{k-2}) = 0$. $\square$

**Corollary 8.** *Let $\mathfrak{P}$ be a degree one prime ideal of $A$ generated by the polynomial $P$ with $P(0)=1$. If $q+2 \geqslant k \geqslant 4$, then $\mathbf{c}_j$, $1 \leqslant j \leqslant k-3$, are eigenfunctions of the Hecke operator $T_{\mathfrak{P}}$ with eigenvalue $\lambda_j(P)$. Further, the Hecke operators for degree one prime ideals are simultaneously diagonalized on $S^2_{k,m}(\Gamma_1(T))$ with respect to the basis $\mathbf{c}_j$, $1 \leqslant j \leqslant k-3$.*

Note that there are no nonzero double cusp forms for weight $k<4$. The above result for $k=4$ is Proposition 15.6 of [Böc04], proved by Böckle and Pink.

We now consider the case of general weight $k$. Assume $\mathfrak{P} = (P)$, where $\deg P = 1$ and $P(0)=1$. Again, we appeal to (7.1). For $i = 0, 1, \ldots, q-2$ and $m_i = 0, 1, \ldots, \lfloor \frac{k-2-i}{q-1} \rfloor$, we have

$$
\begin{aligned}
&T_{\mathfrak{P}}\mathbf{c}(\gamma_0)\big(X^{i+m_i(q-1)}Y^{k-2-(i+m_i(q-1))}\big)\\
&\quad=\mathbf{c}(\gamma_0)\Bigg(X^{i+m_i(q-1)}(PY)^{k-2-(i+m_i(q-1))}\\
&\qquad+\sum_{m=0}^{m_i}\Bigg(\sum_{l=0}^{i+(m_i-m)(q-1)}\binom{i+m_i(q-1)}{l+m(q-1)}\binom{k-2-(i+m_i(q-1))}{l}(1-P)^l\\
&\qquad-P^{k-2-(i+m_i(q-1))}\binom{i+m_i(q-1)}{m(q-1)}\Bigg)X^{i+(m_i-m)(q-1)}Y^{k-2-(i+(m_i-m)(q-1))}\\
&\qquad+\sum_{n=1}^{\lfloor\frac{k-2-i}{q-1}\rfloor-m_i}\Bigg(\sum_{l=n(q-1)}^{k-2-(i+m_i(q-1))}\binom{i+m_i(q-1)}{l-n(q-1)}\binom{k-2-(i+m_i(q-1))}{l}(1-P)^l\\
&\qquad-P^{i+m_i(q-1)}\binom{k-2-(i+m_i(q-1))}{n(q-1)}T^{n(q-1)}\Bigg)X^{i+(m_i+n)(q-1)}Y^{k-2-(i+(m_i+n)(q-1))}\Bigg).
\end{aligned}
$$

Recall the function $\mathbf{c}_j$ defined by (7.2). For $i=0,1,\ldots,q-2$, denote by $S_{k,m}(\Gamma_1(T))_i$ the subspace of $S_{k,m}(\Gamma_1(T))$ generated by $\{\mathbf{c}_i,\mathbf{c}_{i+(q-1)},\ldots,\mathbf{c}_{i+\lfloor\frac{k-2-i}{q-1}\rfloor(q-1)}\}$ so that $S_{k,m}(\Gamma_1(T))=\bigoplus_{i=0}^{q-2}S_{k,m}(\Gamma_1(T))_i$. The above calculation proves the following

**Theorem 9.** *Let $\mathfrak{P}=(P)$, where $\deg P=1$ and $P(0)=1$. Then for each $i=0,1,\ldots,q-2$, $S_{k,m}(\Gamma_1(T))_i$ is invariant under $T_{\mathfrak{P}}$. The action of $T_{\mathfrak{P}}$ restricted to $S_{k,m}(\Gamma_1(T))_i$ with respect to the basis $\{\mathbf{c}_i,\mathbf{c}_{i+(q-1)},\ldots,\mathbf{c}_{i+\lfloor\frac{k-2-i}{q-1}\rfloor(q-1)}\}$ is represented by the matrix $[T_{\mathfrak{P}}]_i=$*

$$
\begin{pmatrix}
\alpha_{0,0}^{(i)}+P^{k-2-i} & \beta_{0,1}^{(i)} & \beta_{0,2}^{(i)} & \cdots & \beta_{0,\lfloor\frac{k-2-i}{q-1}\rfloor}^{(i)}\\
\alpha_{1,1}^{(i)} & \alpha_{1,0}^{(i)}+P^{k-2-(i+(q-1))} & \beta_{1,1}^{(i)} & \cdots & \beta_{1,\lfloor\frac{k-2-i}{q-1}\rfloor-1}^{(i)}\\
\alpha_{2,2}^{(i)} & \alpha_{2,1}^{(i)} & \alpha_{2,0}^{(i)}+P^{k-2-(i+2(q-1))} & \cdots & \beta_{2,\lfloor\frac{k-2-i}{q-1}\rfloor-2}^{(i)}\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
\alpha^{(i)}_{\lfloor\frac{k-2-i}{q-1}\rfloor,\lfloor\frac{k-2-i}{q-1}\rfloor} & \alpha^{(i)}_{\lfloor\frac{k-2-i}{q-1}\rfloor,\lfloor\frac{k-2-i}{q-1}\rfloor-1} & \alpha^{(i)}_{\lfloor\frac{k-2-i}{q-1}\rfloor,\lfloor\frac{k-2-i}{q-1}\rfloor-2} & \cdots & \alpha^{(i)}_{\lfloor\frac{k-2-i}{q-1}\rfloor,0}+P^{k-2-(i+\lfloor\frac{k-2-i}{q-1}\rfloor(q-1))}
\end{pmatrix}
$$

*where*

$$
\begin{aligned}
\alpha_{m_i,m'}^{(i)}={}&\sum_{l=0}^{i+(m_i-m')(q-1)}\binom{i+m_i(q-1)}{l+m'(q-1)}\binom{k-2-(i+m_i(q-1))}{l}(1-P)^l\\
&-P^{k-2-(i+m_i(q-1))}\binom{i+m_i(q-1)}{m'(q-1)}
\end{aligned}
$$

*and*

$$\beta_{m_i,n}^{(i)} = \sum_{l=n(q-1)}^{k-2-(i+m_i(q-1))} \binom{i+m_i(q-1)}{l-n(q-1)} \binom{k-2-(i+m_i(q-1))}{l} (1-P)^l - P^{i+m_i(q-1)} \binom{k-2-(i+m_i(q-1))}{n(q-1)} T^{n(q-1)}$$

*for* $m_i = 0, 1, \ldots, \lfloor \frac{k-2-i}{q-1} \rfloor$, $0 \leqslant m' \leqslant m_i$ *and* $1 \leqslant n \leqslant \lfloor \frac{k-2-i}{q-1} \rfloor - m_i$.

Using geometric arguments, Böckle and Pink computed the above structures for the space of double cusp forms of $k = 5, q = 2$ and $k = 6, q = 3$ in Proposition 15.3 of [Böc04]. To illustrate the above theorem, we give two examples of cusp forms with weights $k > q$; in the first each Hecke action is diagonalizable, while in the second it is not.

**Example 10.** $q = 3$, $k = 7$ and $P = 1 + T$. There are two invariant subspaces under $T_{\mathfrak{P}}$, namely, $S_{7,m}(\Gamma_1(T))_0$ and $S_{7,m}(\Gamma_1(T))_1$ spanned by $\{\mathbf{c}_0, \mathbf{c}_2, \mathbf{c}_4\}$ and $\{\mathbf{c}_1, \mathbf{c}_3, \mathbf{c}_5\}$, respectively. With respect to these bases, we have

$$[T_{\mathfrak{P}}]_0 = \begin{pmatrix} 1 & 0 & 0 \\ 2T^3 & 1 & T^3 \\ 2T & 2T & 1+2T \end{pmatrix} \quad \text{and} \quad [T_{\mathfrak{P}}]_1 = \begin{pmatrix} 1+2T & 2T^3 & 2T^4 \\ T & 1 & 2T^5 \\ 0 & 0 & 1 \end{pmatrix}.$$

Both matrices have the same distinct eigenvalues $1$, $1 + T + T\sqrt{1-T^2}$ and $1 + T - T\sqrt{1-T^2}$. Thus $[T_{\mathfrak{P}}]_0$ and $[T_{\mathfrak{P}}]_1$ are diagonalizable, and hence so is $T_{\mathfrak{P}}$.

**Example 11.** $q = 2$ and $k = 5$. There is only one polynomial $P = 1 + T$ to consider. Further there is only one residue class mod $q - 1$ given by $i = 0$, so one has

$$[T_{\mathfrak{P}}]_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ T^2 & 1 & T^2 & T^3 \\ T & T & 1 & T^3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus $T_{\mathfrak{P}}$ has the eigenvalue 1 of multiplicity two with two linearly independent eigenfunctions $\mathbf{c}_0$ and $\mathbf{c}_3$, and the eigenvalue $1 + T^{3/2}$ of multiplicity two with only one linearly independent eigenfunction $T^{1/2}\mathbf{c}_1 + \mathbf{c}_2$. Hence $T_{\mathfrak{P}}$ is not diagonalizable on $S_{5,m}(\Gamma_1(T))$. Further, since $\mathbf{c}_1$ and $\mathbf{c}_2$ span the space of the double cusp forms $S_{5,m}^2(\Gamma_1(T))$, this shows that the Hecke operator $T_{\mathfrak{P}}$ is not diagonalizable on $S_{5,m}^2(\Gamma_1(T))$ either.

**Remark.** In both examples, unlike the case $k \leqslant q + 2$, there are irrational eigenvalues. Our computations seem to suggest that the nondiagonalizability results from inseparable eigenvalues. It would be interesting to know if it could occur with separable eigenvalues.

## 8. Cusp forms for $\boldsymbol{\Gamma(T)}$

In this section, we work with $\Gamma = \Gamma(T)$, the group of matrices in $\mathrm{GL}_2(A)$ congruent to the identity matrix modulo $T$. A fundamental domain of $\Gamma(T)\backslash\mathcal{T}$ contains $q + 1$ rays, corresponding to the cusps $[\infty] = \binom{1}{0}$ and $[r] = \binom{r}{1}$, $r \in \mathbb{F}$, one stable vertex $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $q + 1$ stable edges

$\gamma_r := \langle\left(\begin{smallmatrix} r & 1 \\ 1 & 0 \end{smallmatrix}\right)\rangle$, $r \in \mathbb{F}$, and $\gamma_\infty := \langle\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\rangle$. Thus $g_{\Gamma(T)} = 0$ so that $\dim_C S_{k,m}(\Gamma(T)) = (k-1)q$ by Proposition 1, and $\dim_C S^2_{2,m}(\Gamma(T)) = 0$ and $\dim_C S^2_{k,m}(\Gamma(T)) = (k-2)q - 1$ for $k \geqslant 3$ by Proposition 4. To determine a harmonic cocycle for $\Gamma(T)$, as noted in Section 4, one needs to know only its values at $\gamma_r$, $r \in \mathbb{F}$, and its value at $\gamma_\infty$ is determined by the harmonicity condition $\mathbf{c}(\gamma_\infty) + \sum_{r\in\mathbb{F}} \mathbf{c}(\gamma_r) = 0$. The stabilizer of the cusp $[\infty]$ (respectively $[r]$, $r \in \mathbb{F}$) is $\Gamma_{[\infty]} = \{\left(\begin{smallmatrix} 1 & c \\ 0 & 1 \end{smallmatrix}\right)\colon\ c \equiv 0 \bmod T\}$ (respectively $\Gamma_{[r]} = \{\left(\begin{smallmatrix} 1+rc & -r^2c \\ c & 1-rc \end{smallmatrix}\right)\colon\ c \equiv 0 \bmod T\}$) so that $V(k,m)^{\Gamma_{[\infty]}}$ (respectively $V(k,m)^{\Gamma_{[r]}}$) is spanned by $Y^{k-2}$ (respectively $(X-rY)^{k-2}$). Moreover, $\Gamma_{[\infty]}\backslash \operatorname{src}([\infty]) = \{\gamma_\infty\}$ and $\Gamma_{[r]}\backslash \operatorname{src}([r]) = \{\gamma_r\}$, $r \in \mathbb{F}$. Thus by Theorem 3, the double cusp forms can be described as follows.

**Proposition 12.** *A harmonic cocycle* $\mathbf{c} \in H_{k,m}(\Gamma(T))$ *lies in* $H^2_{k,m}(\Gamma(T))$ *if and only if* $\mathbf{c}(\gamma_\infty)(Y^{k-2}) = 0$ *and* $\mathbf{c}(\gamma_r)((X-rY)^{k-2}) = 0$ *for all* $r \in \mathbb{F}$.

Next we study the action of the Hecke operator $T_{\mathfrak{P}}$ at $\mathbf{c} \in H_{k,m}(\Gamma(T))$. Recall that a harmonic cocycle takes values in $\operatorname{Hom}(V(k,m), C)$. In view of the above proposition, it turns out that the action is best described if, for all $r \in \mathbb{F}$, the basis $(X-rY)^jY^{k-2-j}$, $0 \leqslant j \leqslant k-2$, of $V(k,m)$ is used when we discuss the values of a harmonic cocycle at the directed edge $\gamma_r$. Therefore we shall describe the action using such bases. To ease our notation, for $\mathbf{c} \in H_{k,m}(\Gamma(T))$, $r \in \mathbb{F}$, and $0 \leqslant j \leqslant k-2$, let

$$Z(\mathbf{c}, r, j) = \mathbf{c}(\gamma_r)\big((X-rY)^jY^{k-2-j}\big). \tag{8.1}$$

Assume that $\mathfrak{P}$ is generated by $P = 1 + \alpha T$ with $\alpha \in \mathbb{F}^\times$. Again, we use (6.1), harmonicity and $\Gamma(T)$-equivariancy to arrive at the main identity of the Hecke action:

$$\begin{aligned} &Z(T_{\mathfrak{P}}\mathbf{c}, r, j) \\ &\quad = P^{k-2-j} Z(\mathbf{c}, r, j) - P^j \sum_{n=1}^{\lfloor\frac{k-2-j}{q-1}\rfloor} \binom{k-2-j}{n(q-1)} T^{n(q-1)} Z\big(\mathbf{c}, r, j+n(q-1)\big) \\ &\qquad + \sum_{b\neq r}\Bigg[\sum_{u=0}^{j}(b-r)^{j-u}\Bigg(P^{k-2-j}\binom{j}{u} - \sum_{l=0}^{u}\binom{j}{u-l}\binom{k-2-j}{l}(1-P)^l\Bigg)Z(\mathbf{c}, b, u) \\ &\qquad - \sum_{u=j+1}^{k-2}\sum_{l=u-j}^{k-2-j}\binom{j}{u-l}\binom{k-2-j}{l}(1-P)^l(b-r)^{j-u}Z(\mathbf{c}, b, u)\Bigg]. \end{aligned} \tag{8.2}$$

Notice that when $j = k-2$, (8.2) becomes

$$Z(T_{\mathfrak{P}}\mathbf{c}, r, k-2) = Z(\mathbf{c}, r, k-2) \tag{8.3}$$

for all $r \in \mathbb{F}$. Moreover, for $j = 0$ and $r \in \mathbb{F}$ we have

$$Z(T_{\mathfrak{P}}\mathbf{c}, r, 0) = P^{k-2}Z(\mathbf{c}, r, 0) - \sum_{n=1}^{\lfloor\frac{k-2}{q-1}\rfloor}\binom{k-2}{n(q-1)}T^{n(q-1)}Z\big(\mathbf{c}, r, n(q-1)\big)$$

$$+\sum_{b\neq r}\left(\left(P^{k-2}-1\right)Z(\mathbf{c},b,0)-\sum_{u=1}^{k-2}(1-P)^u(b-r)^{-u}Z(\mathbf{c},b,u)\right).$$

Summing over all $r\in\mathbb{F}$ and using harmonicity, we get

$$-T_{\mathfrak{P}}\mathbf{c}(\gamma_\infty)\left(Y^{k-2}\right)=\mathrm{I}+\mathrm{II},$$

where

$$\mathrm{I}=\sum_{r\in\mathbb{F}}\left(P^{k-2}Z(\mathbf{c},r,0)-\sum_{n=1}^{\lfloor\frac{k-2}{q-1}\rfloor}\binom{k-2}{n(q-1)}T^{n(q-1)}Z\big(\mathbf{c},r,n(q-1)\big)\right)$$

and

$$\begin{aligned}\mathrm{II}&=\sum_{b\in\mathbb{F}}\left(\sum_{r\neq b}\left(P^{k-2}-1\right)Z(\mathbf{c},b,0)-\sum_{u=1}^{k-2}(1-P)^u\sum_{r\neq b}(b-r)^{-u}Z(\mathbf{c},b,u)\right)\\&=\sum_{b\in\mathbb{F}}\left(-\left(P^{k-2}-1\right)Z(\mathbf{c},b,0)+\sum_{n=1}^{\lfloor\frac{k-2}{q-1}\rfloor}(1-P)^{n(q-1)}Z\big(\mathbf{c},b,n(q-1)\big)\right).\end{aligned}$$

Combined, this gives

$$T_{\mathfrak{P}}\mathbf{c}(\gamma_\infty)\left(Y^{k-2}\right)=\mathbf{c}(\gamma_\infty)\left(Y^{k-2}\right).\tag{8.4}$$

Eqs. (8.3) and (8.4) then imply

**Proposition 13.** *Let $\mathbf{c}\in S_{k,m}(\Gamma(T))$ be an eigenfunction of $T_{\mathfrak{P}}$, where $\mathfrak{P}\neq(T)$ has degree* 1. *If it is not a double cusp form, then the eigenvalue is* 1.

Assume further that $q\geqslant k\geqslant 2$. In this case (8.2) is reduced to

$$\begin{aligned}Z(T_{\mathfrak{P}}\mathbf{c},r,j)&=\sum_{u=0}^{j-1}\sum_{b\in\mathbb{F}}(b-r)^{j-u}\left(P^{k-2-j}\binom{j}{u}-\sum_{l=0}^{u}\binom{j}{u-l}\binom{k-2-j}{l}(1-P)^l\right)\\&\quad\times Z(\mathbf{c},b,u)+\left[P^{k-2-j}-\lambda_j(P)\right]\sum_{b\in\mathbb{F}}Z(\mathbf{c},b,j)+\lambda_j(P)Z(\mathbf{c},r,j)\\&\quad-\sum_{u=j+1}^{k-2}\sum_{l=u-j}^{k-2-j}\binom{j}{u-l}\binom{k-2-j}{l}(1-P)^l\sum_{b\neq r}(b-r)^{j-u}Z(\mathbf{c},b,u)\\&=\sum_{u=0}^{j-1}\alpha_u(j,P)\sum_{b\in\mathbb{F}}(b-r)^{j-u}Z(\mathbf{c},b,u)+\left[P^{k-2-j}-\lambda_j(P)\right]\sum_{b\in\mathbb{F}}Z(\mathbf{c},b,j)\\&\quad+\lambda_j(P)Z(\mathbf{c},r,j)-\sum_{u=j+1}^{k-2}\beta_u(j,P)\sum_{b\neq r}(b-r)^{j-u}Z(\mathbf{c},b,u),\end{aligned}\tag{8.5}$$

where $\lambda_j(P)=\sum_{l=0}^{j}\binom{j}{l}\binom{k-2-j}{l}(1-P)^l$ is given by (7.3),

$$\alpha_u(j,P)=P^{k-2-j}\binom{j}{u}-\sum_{l=0}^{u}\binom{j}{u-l}\binom{k-2-j}{l}(1-P)^l \quad \text{for } 0\leqslant u\leqslant j-1,$$

and

$$\beta_u(j,P)=\sum_{l=u-j}^{k-2-j}\binom{j}{u-l}\binom{k-2-j}{l}(1-P)^l \quad \text{for } j+1\leqslant u\leqslant k-2.$$

For $r\in\mathbb{F}$ and $0\leqslant j\leqslant k-2$, denote by $\mathbf{c}_j^{(r)}$ the function

$$\mathbf{c}_j^{(r)}(\gamma_r)\big((X-rY)^jY^{k-2-j}\big)=1 \quad \text{and} \quad \mathbf{c}_j^{(r)}(\gamma_s)\big((X-rY)^lY^{k-2-l}\big)=0 \quad \text{if } s\neq r \text{ or } l\neq j.$$

Let $\mathbf{c}_j=\sum_{r\in\mathbb{F}}\mathbf{c}_j^{(r)}$. Then $T_{\mathfrak{P}}\mathbf{c}_j=\lambda_j(P)\mathbf{c}_j$, that is, $\mathbf{c}_j$ is an eigenfunction of $T_{\mathfrak{P}}$ with eigenvalue $\lambda_j(P)$. Observe that $\mathbf{c}_j$ are liftings of the eigenfunctions of $S_{k,m}(\Gamma_1(T))$.

Our next goal is to show that $\lambda_j(P)$ are the eigenvalues for the Hecke operator $T_{\mathfrak{P}}$ on $S_{k,m}(\Gamma(T))$ when $q\geqslant k$. For this, we need

**Lemma 14.** *Suppose that* $\mathbf{c}$ *is an eigenfunction of the Hecke operator* $T_{\mathfrak{P}}$ *on* $S_{k,m}(\Gamma(T))$ *with eigenvalue* $\lambda\neq\lambda_n(P)$ *for all* $0\leqslant n\leqslant k-2$. *Then for each* $0\leqslant n\leqslant k-2$ *and each* $r\in\mathbb{F}$, *there are constants* $A_u^{(n)}\in\mathbb{F}(T)$ *for* $n+1\leqslant u\leqslant k-2$ *such that*

$$\big(\lambda-\lambda_n(P)\big)Z(\mathbf{c},r,n)=\sum_{u=n+1}^{k-2}A_u^{(n)}\sum_{b\neq r}(b-r)^{n-u}Z(\mathbf{c},b,u). \tag{8.6$_n$}$$

Grant this lemma. By applying $(8.6)_n$ repeatedly from $n=k-2$ down to $n=0$, we deduce that $\mathbf{c}=0$. This proves

**Theorem 15.** *Let* $\mathfrak{P}=(P)\neq(T)$ *be a degree one prime ideal of* $A$. *For* $q\geqslant k\geqslant 2$ *the distinct eigenvalues for the Hecke operator* $T_{\mathfrak{P}}$ *on* $S_{k,m}(\Gamma(T))$ *are the distinct* $\lambda_j(P)$, $0\leqslant j\leqslant k-2$.

Let $\mathbf{c}$ be an eigenfunction of $T_{\mathfrak{P}}$ with eigenvalue $\lambda$. Then (8.5) gives rise to

$$\begin{aligned}\big(\lambda-\lambda_j(P)\big)Z(\mathbf{c},r,j)={}&\sum_{u=0}^{j-1}\alpha_u(j,P)\sum_{b\in\mathbb{F}}(b-r)^{j-u}Z(\mathbf{c},b,u)\\&+\sum_{b\in\mathbb{F}}\big[P^{k-2-j}-\lambda_j(P)\big]Z(\mathbf{c},b,j)\\&-\sum_{u=j+1}^{k-2}\beta_u(j,P)\sum_{b\neq r}(b-r)^{j-u}Z(\mathbf{c},b,u)\end{aligned}$$

for all $0 \leqslant j \leqslant k-2$ and $r \in \mathbb{F}$. Summing over all $r \in \mathbb{F}$, we get, for each $0 \leqslant j \leqslant k-2$,

$$\big(\lambda - \lambda_j(P)\big) \sum_{r\in\mathbb{F}} Z(\mathbf{c}, r, j) = 0. \tag{8.7}$$

Hence if $\lambda \neq \lambda_j(P)$, then $\sum_{r\in\mathbb{F}} Z(\mathbf{c}, r, j) = 0$ so that

$$\begin{aligned}\big(\lambda - \lambda_j(P)\big) Z(\mathbf{c}, r, j) &= \sum_{u=0}^{j-1} \alpha_u(j, P) \sum_{b\in\mathbb{F}} (b-r)^{j-u} Z(\mathbf{c}, b, u) \\ &\quad - \sum_{u=j+1}^{k-2} \beta_u(j, P) \sum_{b\neq r} (b-r)^{j-u} Z(\mathbf{c}, b, u)\end{aligned} \tag*{$(8.8)_j$}$$

for all $0 \leqslant j \leqslant k-2$ and $r \in \mathbb{F}$. When $j = 0$, the first sum on the right side is void and hence $(8.6)_0$ holds with $A_u^{(0)} = \beta_u(0, P)$ for $1 \leqslant u \leqslant k-2$. We shall prove Lemma 14 by induction on $n$. To proceed, we prove an identity which will be used repeatedly in the computations to follow.

**Proposition 16.** *For $1 \leqslant l, t \leqslant k-2 \leqslant q-2$ and any $C$-valued function $X(s)$ on $\mathbb{F}$, we have*

$$\sum_{b\in\mathbb{F}} \sum_{\substack{s\in\mathbb{F}\\ s\neq b}} \frac{(b-r)^t}{(s-b)^l} X(s) = \begin{cases} \sum_{s\in\mathbb{F}} (-1)^{l+1} \binom{t}{l} (s-r)^{t-l} X(s) & \text{if } t > l; \\ \sum_{s\in\mathbb{F}} (-1)^{l+1} X(s) & \text{if } t = l; \\ 0 & \text{if } t \leqslant l. \end{cases}$$

**Proof.** Let $1 \leqslant l, t \leqslant k-2 \leqslant q-2$. Then

$$\begin{aligned}\sum_{b\in\mathbb{F}} \sum_{\substack{s\in\mathbb{F}\\ s\neq b}} \frac{(b-r)^t}{(s-b)^l} X(s) &= \sum_{s\in\mathbb{F}} \sum_{\substack{b\in\mathbb{F}\\ b\neq s}} \frac{((b-s)+(s-r))^t}{(s-b)^l} X(s) \\ &= \sum_{s\in\mathbb{F}} \sum_{i=0}^{t} (-1)^l \binom{t}{i} (s-r)^{t-i} \sum_{b\neq s} (b-s)^{i-l} X(s).\end{aligned}$$

Since $1 \leqslant l, t \leqslant k-2 \leqslant q-2$, $\sum_{\substack{b\in\mathbb{F}\\ b\neq s}} (b-s)^{i-l}$ vanishes unless $i = l$ in which case it is $-1$, so

$$\sum_{b\in\mathbb{F}} \sum_{\substack{s\in\mathbb{F}\\ s\neq b}} \frac{(b-r)^t}{(s-b)^l} X(s) = \begin{cases} \sum_{s\in\mathbb{F}} (-1)^{l+1} \binom{t}{l} (s-r)^{t-l} X(s), & \text{if } t > l; \\ 0, & \text{if } t < l. \end{cases}$$

If $t = l$, then

$$\sum_{b\in\mathbb{F}} \sum_{\substack{s\in\mathbb{F}\\ s\neq b}} \left(\frac{b-r}{s-b}\right)^t X(s) = \sum_{s\in\mathbb{F}} \sum_{\substack{b\in\mathbb{F}\\ b\neq s}} \left(\frac{s-r}{s-b} - 1\right)^t X(s)$$

$$
\begin{aligned}
&= \sum_{s \in \mathbb{F}} \sum_{b \neq s} \left( \sum_{i=0}^{t-1} \binom{t}{i} (-1)^i \left( \frac{s-r}{s-b} \right)^{t-i} + (-1)^t \right) X(s) \\
&= (-1)^{t+1} \sum_{s \in \mathbb{F}} X(s).
\end{aligned}
$$

This proves the proposition. □

**Proof of Lemma 14.** We shall apply Proposition 16 to $X(s) = Z(\mathbf{c}, s, j)$, in which case the sum is equal to 0 when $t = l$ because of (8.7) and the assumption $\lambda \neq \lambda_j(P)$ for all $j$. Assume that the statement is valid up to $n$, where $0 \leqslant n < k-2$. That is, for all $0 \leqslant j \leqslant n$ and $b \in \mathbb{F}$, we have

$$
Z(\mathbf{c}, b, j) = \frac{1}{\lambda - \lambda_j(P)} \sum_{u=j+1}^{k-2} A_u^{(j)} \sum_{s \neq b} (s-b)^{j-u} Z(\mathbf{c}, s, u). \tag{8.9$_j$}
$$

Substituting $(8.9)_0$ into $(8.8)_{n+1}$, we get

$$
\begin{aligned}
&\big(\lambda - \lambda_{n+1}(P)\big) Z(\mathbf{c}, r, n+1) \\
&\quad = \sum_{u=0}^{n} \alpha_u(n+1, P) \sum_{b \in \mathbb{F}} (b-r)^{(n+1)-u} Z(\mathbf{c}, b, u) \\
&\qquad - \sum_{u=n+2}^{k-2} \beta_u(n+1, P) \sum_{b \neq r} (b-r)^{(n+1)-u} Z(\mathbf{c}, b, u) \\
&\quad = \sum_{b \in \mathbb{F}} \frac{\alpha_0(n+1, P)}{\lambda - \lambda_0(P)} \sum_{u=1}^{k-2} A_u^{(0)} \sum_{s \neq b} \frac{(b-r)^{n+1}}{(s-b)^u} Z(\mathbf{c}, s, u) \\
&\qquad + \sum_{u=1}^{n} A_u^{(n+1),0} \sum_{b \in \mathbb{F}} (b-r)^{(n+1)-u} Z(\mathbf{c}, b, u) \\
&\qquad + \sum_{u=n+2}^{k-2} A_u^{(n+1)} \sum_{b \neq r} (b-r)^{(n+1)-u} Z(\mathbf{c}, b, u).
\end{aligned}
$$

Here $A_u^{(n+1),0} = \alpha_u(n+1, P)$, $1 \leqslant u \leqslant n+1$, depend only on $u$ and $n$. By Proposition 16, the first triple sum of the right-hand side is equal to

$$
\sum_{u=1}^{n} \frac{\alpha_0(n+1, P)}{\lambda - \lambda_0(P)} \sum_{s \in \mathbb{F}} (-1)^{u+1} \binom{n+1}{u} (s-r)^{(n+1)-u} Z(\mathbf{c}, s, u),
$$

which can be combined with the middle double sum of the right-hand side to bring the above identity to the following form:

$$\begin{aligned}&\big(\lambda-\lambda_{n+1}(P)\big)Z(\mathbf{c},r,n+1)\\&\quad=\sum_{u=1}^{n}A_u^{(n+1),1}\sum_{b\in\mathbb{F}}(b-r)^{(n+1)-u}Z(\mathbf{c},b,u)+\sum_{u=n+2}^{k-2}A_u^{(n+1)}\sum_{b\neq r}(b-r)^{(n+1)-u}Z(\mathbf{c},b,u).\end{aligned}$$

Next we replace $Z(\mathbf{c},b,1)$ above by $(8.9)_1$ and use Proposition 16 to express $(\lambda-\lambda_{n+1}(P))$ times $Z(\mathbf{c},r,n+1)$ as a linear combination of $\sum_{b\in\mathbb{F}}(b-r)^{n+1-u}Z(\mathbf{c},b,u)$ for $2\leqslant u\leqslant n$ and $\sum_{b\neq r}(b-r)^{n+1-u}Z(\mathbf{c},b,u)$ for $n+2\leqslant u\leqslant k-2$ with coefficients $A_u^{(n+1),2}$ depending only on $n$ and $u$. Repeat this procedure. After $n-1$ iterations, we arrive at

$$\begin{aligned}&\big(\lambda-\lambda_{n+1}(P)\big)Z(\mathbf{c},r,n+1)\\&\quad=A_n^{(n+1),n}\sum_{b\in\mathbb{F}}(b-r)Z(\mathbf{c},b,n)+\sum_{u=n+2}^{k-2}A_u^{(n+1)}\sum_{b\neq r}(b-r)^{(n+1)-u}Z(\mathbf{c},b,u).\end{aligned}$$

For the final calculation, use $(8.9)_n$ to get

$$\begin{aligned}&\big(\lambda-\lambda_{n+1}(P)\big)Z(\mathbf{c},r,n+1)\\&\quad=\frac{A_n^{(n+1),n}}{\lambda-\lambda_n(P)}\sum_{b\in\mathbb{F}}\sum_{u=n+1}^{k-2}A_u^{(n)}\sum_{s\neq b}\frac{b-r}{(s-b)^{u-n}}Z(\mathbf{c},s,u)\\&\qquad+\sum_{u=n+2}^{k-2}A_u^{(n+1)}\sum_{b\neq r}(b-r)^{(n+1)-u}Z(\mathbf{c},b,u)\\&\quad=\sum_{u=n+2}^{k-2}A_u^{(n+1)}\sum_{b\neq r}(b-r)^{(n+1)-u}Z(\mathbf{c},b,u).\end{aligned}$$

Hence Lemma 14 follows by induction. □

The techniques used to prove Lemma 14 can be extended to describe the eigenspaces of $T_{\mathfrak{P}}$. Let $\mathbf{c}$ be an eigenfunction of $T_{\mathfrak{P}}$ with eigenvalue $\lambda_n(P)$. The relations among $Z(\mathbf{c},r,j)$ for $r\in\mathbb{F}$ and $0\leqslant j\leqslant k-2$ are distinguished by two cases, according to $\lambda_j(P)$ equal to $\lambda_n(P)$ or not.

For those $l$ with $\lambda_l(P)\neq\lambda_n(P)$, Eq. $(8.8)_l$ gives

$$\begin{aligned}Z(\mathbf{c},b,l)=\frac{1}{\lambda_n(P)-\lambda_l(P)}\Bigg[&\sum_{u=0}^{l-1}\alpha_u(l,P)\sum_{s\in\mathbb{F}}(s-b)^{l-u}Z(\mathbf{c},s,u)\\&-\sum_{u=l+1}^{k-2}\beta_u(l,P)\sum_{s\neq b}(s-b)^{l-u}Z(\mathbf{c},s,u)\Bigg]\end{aligned}$$

for all $b\in\mathbb{F}$. Further, for such $l$ we have $\sum_{b\in\mathbb{F}}Z(\mathbf{c},b,l)=0$ by (8.7). Let $l_0<l_1<\cdots<l_t$ be the distinct $l$'s such that $\lambda_l(P)\neq\lambda_n(P)$. Then the same inductive procedure as in the proof of Lemma 14 yields, for each $l_v$, $0\leqslant v\leqslant t$,

$$Z(\mathbf{c},b,l_v)=\sum_{\substack{0\leqslant u<l_v\\ \lambda_u(P)=\lambda_n(P)}} A_u^{(l_v)}\sum_{s\in\mathbb{F}}(s-b)^{l_v-u}Z(\mathbf{c},s,u)$$
$$+\sum_{u=l_v+1}^{k-2} A_u^{(l_v)}\sum_{s\neq b}(s-b)^{l_v-u}Z(\mathbf{c},s,u) \qquad (8.10)_{l_v}$$

for some explicitly determined elements $A_u^{(l_v)}$ in $\mathbb{F}(T)$ depending only on $u$ and $P$.

Let $i$ be an index such that $\lambda_i(P)=\lambda_n(P)$. The Hecke action (8.5) gives rise to

$$0=\sum_{u=0}^{i-1}\alpha_u(i,P)\sum_{b\in\mathbb{F}}(b-r)^{i-u}Z(\mathbf{c},b,u)+\left[P^{k-2-i}-\lambda_i(P)\right]\sum_{b\in\mathbb{F}}Z(\mathbf{c},b,i)$$
$$-\sum_{u=i+1}^{k-2}\beta_u(i,P)\sum_{b\neq r}(b-r)^{i-u}Z(\mathbf{c},b,u). \qquad (8.11)$$

By successively substituting $(8.10)_{l_v}$ into (8.11), starting with $v=0$ and ending with $v=t$, and simplifying the expression using Proposition 16 at each step, we eliminate all $Z(\mathbf{c},b,l)$'s in Eq. (8.11) with $\lambda_l(P)\neq\lambda_n(P)$ and arrive at an identity of the form

$$0=\sum_{\substack{0\leqslant u\leqslant k-2\\ \lambda_u(P)=\lambda_n(P)}} C_u(i,P)\sum_{b\neq r}(b-r)^{i-u}Z(\mathbf{c},b,u) \qquad (8.12)_{i,r}$$

for some explicitly determined elements $C_u(i,P)$ in $\mathbb{F}(T)$ depending only on $i,u$ and $P$. We have shown

**Theorem 17.** *Suppose $q\geqslant k\geqslant 2$. Let $\mathfrak{P}=(P)$, where $P\in\mathbb{F}[T]$ has degree one and $P(0)=1$. Then $\lambda_i(P)$, $0\leqslant i\leqslant k-2$, with suitable multiplicities are the eigenvalues of the Hecke operator $T_{\mathfrak{P}}$ on $S_{k,m}(\Gamma(T))$. For $0\leqslant n\leqslant k-2$, set $A_n=\{i\colon 0\leqslant i\leqslant k-2$ and $\lambda_i(P)=\lambda_n(P)\}$ and denote the integers in $[0,k-2]\setminus A_n$ by $l_0<\cdots<l_t$. Let $\mathbf{c}$ be an eigenfunction in $H_{k,m}(\Gamma(T))$ with eigenvalue $\lambda_n(P)$. Then $\mathbf{c}$ is determined by $Z(\mathbf{c},b,u)$ with $u\in A_n$ and $b\in\mathbb{F}$ subject to the conditions $(8.12)_{i,r}$ for $i\in A_n$ and $r\in\mathbb{F}$. The remaining $Z(\mathbf{c},b,l)$'s are determined recursively by $(8.10)_{l_v}$ from $v=t$ to $v=0$.*

## 9. Examples

To illustrate Theorem 17, we compute the action of $T_{\mathfrak{P}}$ on $H_{k,m}(\Gamma)$ for small weights $k=3,4,5$. None of these are diagonalizable with respect to the Hecke operator. Let $\mathbf{c}$ be an eigenfunction.

(i) $q\geqslant k=3$. Here $\lambda_0(P)=\lambda_1(P)=1$. It follows from (8.5) that

$$\sum_{b\in\mathbb{F}}Z(\mathbf{c},b,0)=\sum_{b\neq r}\frac{Z(\mathbf{c},b,1)}{r-b}$$

for all $r \in \mathbb{F}$. We shall solve this linear system. Fix a generator $a$ of $\mathbb{F}^\times$ and arrange the elements of $\mathbb{F}$ in the order $0, a, a^2, \ldots, a^{q-1}$. Express the above system in matrix form

$$\begin{pmatrix} 0 & -\frac{1}{a} & -\frac{1}{a^2} & -\frac{1}{a^3} & \cdots & -\frac{1}{a^{q-1}} \\ \frac{1}{a} & 0 & \frac{1}{a-a^2} & \frac{1}{a-a^3} & \cdots & \frac{1}{a-a^{q-1}} \\ \frac{1}{a^2} & \frac{1}{a^2-a} & 0 & \frac{1}{a^2-a^3} & \cdots & \frac{1}{a^2-a^{q-1}} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{a^{q-1}} & \frac{1}{a^{q-1}-a} & \frac{1}{a^{q-1}-a^2} & \frac{1}{a^{q-1}-a^3} & \cdots & 0 \end{pmatrix} \begin{pmatrix} Z(\mathbf{c},0,1) \\ Z(\mathbf{c},a,1) \\ Z(\mathbf{c},a^2,1) \\ \vdots \\ Z(\mathbf{c},a^{q-1},1) \end{pmatrix} = \begin{pmatrix} c \\ c \\ c \\ \vdots \\ c \end{pmatrix}, \tag{9.1}$$

where $c = \sum_{b\in\mathbb{F}} Z(\mathbf{c}, b, 0)$. We determine the nullity of the coefficient matrix $M$. Write

$$M = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & \frac{1}{a} & 0 & \ldots & 0 \\ 0 & 0 & \frac{1}{a^2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \frac{1}{a^{q-1}} \end{pmatrix} \begin{pmatrix} 0 & -\frac{1}{a} & -\frac{1}{a^2} & -\frac{1}{a^3} & \cdots & -\frac{1}{a^{q-1}} \\ 1 & 0 & \frac{1}{1-a} & \frac{1}{1-a^2} & \cdots & \frac{1}{1-a^{q-2}} \\ 1 & \frac{1}{1-a^{q-2}} & 0 & \frac{1}{1-a} & \cdots & \frac{1}{1-a^{q-3}} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{1}{1-a} & \frac{1}{1-a^2} & \frac{1}{1-a^3} & \cdots & 0 \end{pmatrix}.$$

Call the second matrix on the right-hand side $C$. Note that $\mathrm{Nul}(M) = \mathrm{Nul}(C)$. Consider the submatrix obtained from $C$ by deleting the first row and the first column

$$C' = \begin{pmatrix} 0 & \frac{1}{1-a} & \frac{1}{1-a^2} & \cdots & \frac{1}{1-a^{q-2}} \\ \frac{1}{1-a^{q-2}} & 0 & \frac{1}{1-a} & \cdots & \frac{1}{1-a^{q-3}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{1-a} & \frac{1}{1-a^2} & \frac{1}{1-a^3} & \cdots & 0 \end{pmatrix},$$

which is a $(q-1)\times(q-1)$ circulant matrix. Then

$$\mathbf{v}'_j = \begin{pmatrix} 1 \\ a^j \\ a^{2j} \\ \vdots \\ a^{(q-2)j} \end{pmatrix}, \quad j = 1, 2, \ldots, q-1,$$

are $q-1$ linearly independent eigenvectors of $C'$ with eigenvalue

$$\frac{a^j}{1-a} + \frac{a^{2j}}{1-a^2} + \cdots + \frac{a^{(q-2)j}}{1-a^{q-2}} = j,$$

as a consequence of the following lemma.

**Lemma 18.** *For $j = 1, 2, \ldots, q-1$ and $l \geqslant 1$, we have*

$$\sum_{n=1}^{q-2} \frac{a^{jn}}{(1-a^n)^l} = (-1)^{l-1}\binom{j}{l}.$$

**Proof.** We shall prove this lemma by induction on $l$. For $l = 1$, we compute

$$\sum_{n=1}^{q-2} \frac{a^{jn}}{1-a^n} = \sum_{n=1}^{q-2} \frac{a^{jn}-1+1}{1-a^n} = \sum_{n=1}^{q-2} \left[ -\left(1+a^n+\cdots+a^{(j-1)n}\right) + \frac{1}{1-a^n} \right].$$

Since $a$ has order $q-1$ and $1 \leqslant j \leqslant q-1$, $\sum_{n=1}^{q-2} a^{in} = -a^{i(q-1)} = -1$ for all $i = 1, \ldots, j-1$. As $\sum_{n=1}^{q-2} \frac{1}{1-a^n} = -1$, the above sum is equal to

$$\sum_{n=1}^{q-2} \frac{a^{jn}}{1-a^n} = -(q-2) + (j-1) - 1 = j.$$

Next, we assume that $\sum_{n=1}^{q-2} \frac{a^{jn}}{(1-a^n)^l} = (-1)^{l-1}\binom{j}{l}$ for all $j = 1, \ldots, q-1$. Then

$$\begin{aligned} \sum_{n=1}^{q-2} \frac{a^{jn}}{(1-a^n)^{l+1}} &= \sum_{n=1}^{q-2} \frac{a^{jn}-1+1}{(1-a^n)^{l+1}} = \sum_{n=1}^{q-2} \left[ -\frac{1+a+\cdots+a^{(j-1)n}}{(1-a^n)^l} + \frac{1}{(1-a^n)^{l+1}} \right] \\ &= -\left[ -1 + (-1)^{l-1}\binom{1}{l} + \cdots + (-1)^{l-1}\binom{j-1}{l} \right] - 1 = (-1)^l \binom{j}{l+1} \end{aligned}$$

by the Pascal's triangle identity $\sum_{i=1}^{m} \binom{i}{l} = \binom{m+1}{l+1}$. The lemma follows by induction. $\square$

Back to the matrix $C$. The vectors

$$\mathbf{v}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_j = \begin{pmatrix} 0 \\ 1 \\ a^j \\ \vdots \\ a^{(q-2)j} \end{pmatrix}, \quad j = 1, \ldots, q-1,$$

are $q$ linearly independent eigenvectors of $C$ with the eigenvalues 0 and $j$, respectively. Since our field has characteristic $p$, this shows that the nullity$(C) = q/p$. If $c = \sum_{b \in \mathbb{F}} Z(\mathbf{c}, b, 0) = 0$, then we obtain $(q-1) + q/p$ linearly independent eigenvectors for $T_{\mathfrak{P}}$. When $c \neq 0$, note that

$$\mathbf{v} = \begin{pmatrix} 0 \\ ca \\ ca^2 \\ \vdots \\ ca^{q-1} \end{pmatrix}$$

is a solution of (9.1). Together with the homogeneous ones, we have $q + q/p$ linearly independent eigenvectors for $T_{\mathfrak{P}}$, all with eigenvalue 1. Since 1 is the only eigenvalue of $T_{\mathfrak{P}}$, its total multiplicity $2q$, thus $T_{\mathfrak{P}}$ is not diagonalizable. We record this result in

**Proposition 19.** *Suppose $\mathbb{F}$ has cardinality $q \geqslant 3$ and characteristic $p$. For a maximal degree one ideal $\mathfrak{P} \neq (T)$, $1$ is the only eigenvalue of the Hecke operator $T_{\mathfrak{P}}$ on $S_{3,m}(\Gamma(T))$. The eigenspace of $T_{\mathfrak{P}}$ has dimension $q + q/p$, while the space $S_{3,m}(\Gamma(T))$ has dimension $2q$. Consequently, $T_{\mathfrak{P}}$ is not diagonalizable on $S_{3,m}(\Gamma(T))$.*

As the dimension of the 1-eigenspace of $T_{\mathfrak{P}}$ on $S^2_{3,m}(\Gamma(T))$ is $q-1$, which is $\dim_C S^2_{3,m}(\Gamma(T))$, so $T_{\mathfrak{P}}$ is diagonalizable on $S^2_{3,m}(\Gamma(T))$.

(ii) $q \geqslant k = 4$. In this case $\lambda_0(P) = \lambda_2(P) = 1$ and $\lambda_1(P) = -P + 2$. A similar computation yields

**Proposition 20.** *Suppose $\mathbb{F}$ has cardinality $q \geqslant 4$ and characteristic $p$. For a maximal degree one ideal $\mathfrak{P} \neq (T)$, $1$ and $2 - P$ are the two distinct eigenvalues of the Hecke operator $T_{\mathfrak{P}}$ on $S_{4,m}(\Gamma(T))$. The 1-eigenspace has dimension $q + 2q/p$ if $p > 2$ and dimension $q + q/p$ if $p = 2$. The $(2 - P)$-eigenspace has dimension $q$. Moreover, $T_{\mathfrak{P}}$ is not diagonalizable on $S_{4,m}(\Gamma(T))$.*

One checks that $T_{\mathfrak{P}}$ on $S^2_{4,m}(\Gamma(T))$ is diagonalizable since $\dim_C S^2_{4,m}(\Gamma(T)) = 2q - 1$, the 1-eigenspace is $(q-1)$-dimensional, and the $(2-P)$-eigenspace has dimension $q$.

(iii) $q \geqslant k = 5$. In this case $\lambda_0(P) = \lambda_3(P) = 1$ and $\lambda_1(P) = \lambda_2(P) = -2P + 3$. First we assume $p > 2$ so that $1 \neq -2P + 3$. To determine the 1-eigenspace, consider the equations from (8.5) with $j = 0, 1, 2$:

$$\begin{aligned} 0 = {} & \big(P^2 + P + 1\big) \sum_{b \in \mathbb{F}} Z(\mathbf{c}, b, 0) + 3 \sum_{b \neq r} \frac{Z(\mathbf{c}, b, 1)}{b - r} - 3(P - 1) \sum_{b \neq r} \frac{Z(\mathbf{c}, b, 2)}{(b - r)^2} \\ & + (P - 1)^2 \sum_{b \neq r} \frac{Z(\mathbf{c}, b, 3)}{(b - r)^3}, \end{aligned} \tag{9.2}$$

$$\begin{aligned} 2Z(\mathbf{c}, r, 1) = {} & (P + 1) \sum_{b \in \mathbb{F}} (b - r) Z(\mathbf{c}, b, 0) + (P + 3) \sum_{b \in \mathbb{F}} Z(\mathbf{c}, b, 1) \\ & - (P - 3) \sum_{b \neq r} \frac{Z(\mathbf{c}, b, 2)}{b - r} - (P - 1) \sum_{b \neq r} \frac{Z(\mathbf{c}, b, 3)}{(b - r)^2}, \end{aligned}$$

and

$$\begin{aligned} 2Z(\mathbf{c}, r, 2) = {} & \sum_{b \in \mathbb{F}} (b - r)^2 Z(\mathbf{c}, b, 0) + 3 \sum_{b \in \mathbb{F}} (b - r) Z(\mathbf{c}, b, 1) \\ & + 3 \sum_{b \in \mathbb{F}} Z(\mathbf{c}, b, 2) + \sum_{b \neq r} \frac{Z(\mathbf{c}, b, 3)}{b - r} \end{aligned}$$

for all $r \in \mathbb{F}$. Summing the second and third equations over all $r$, we get $\sum_{r \in \mathbb{F}} Z(\mathbf{c}, r, 1) = 0$ and $\sum_{r \in \mathbb{F}} Z(\mathbf{c}, r, 2) = 0$, which lead to the following simplifications of the second and third equations:

$$Z(\mathbf{c}, r, 1) = \frac{P+1}{2}\sum_{b\in\mathbb{F}}(b-r)Z(\mathbf{c}, b, 0) - \frac{P-3}{2}\sum_{b\neq r}\frac{Z(\mathbf{c}, b, 2)}{b-r} - \frac{P-1}{2}\sum_{b\neq r}\frac{Z(\mathbf{c}, b, 3)}{(b-r)^2}, \tag{9.3}$$

and

$$Z(\mathbf{c}, r, 2) = \frac{1}{2}\sum_{b\in\mathbb{F}}(b-r)^2 Z(\mathbf{c}, b, 0) + \frac{3}{2}\sum_{b\in\mathbb{F}}(b-r)Z(\mathbf{c}, b, 1) + \frac{1}{2}\sum_{b\neq r}\frac{Z(\mathbf{c}, b, 3)}{b-r} \tag{9.4}$$

for all $r \in \mathbb{F}$. Plugging (9.3) into (9.4) and using Proposition 16 to simplify, we get

$$Z(\mathbf{c}, r, 2) = \frac{1}{2}\sum_{b\in\mathbb{F}}(b-r)^2 Z(\mathbf{c}, b, 0) + \frac{1}{2}\sum_{b\neq r}\frac{Z(\mathbf{c}, b, 3)}{b-r} \tag{9.5}$$

for all $r \in \mathbb{F}$. Substituting (9.3) and (9.5) into (9.2) and simplifying the result using Proposition 16, we obtain

$$\sum_{b\in\mathbb{F}} Z(\mathbf{c}, b, 0) = \sum_{b\neq r}\frac{Z(\mathbf{c}, b, 3)}{(r-b)^3} \tag{9.6}$$

for all $r \in \mathbb{F}$. To solve the above linear system, we employ the same method as in case (i), that is, computing the nullity of

$$C = \begin{pmatrix} 0 & -\frac{1}{a^3} & -\frac{1}{a^6} & -\frac{1}{a^9} & \cdots & -\frac{1}{a^{3(q-1)}} \\ 1 & 0 & \frac{1}{(1-a)^3} & \frac{1}{(1-a^2)^3} & \cdots & \frac{1}{(1-a^{q-2})^3} \\ 1 & \frac{1}{(1-a^{q-2})^3} & 0 & \frac{1}{(1-a)^3} & \cdots & \frac{1}{(1-a^{q-3})^3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{1}{(1-a)^3} & \frac{1}{(1-a^2)^3} & \frac{1}{(1-a^3)^3} & \cdots & 0 \end{pmatrix}.$$

By Lemma 18, the vectors

$$\mathbf{v}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_j = \begin{pmatrix} 0 \\ 1 \\ a^j \\ \vdots \\ a^{(q-2)j} \end{pmatrix}, \quad j = 1, \ldots, q-1,$$

are $q$ linearly independent eigenvectors of $C$ with the eigenvalues 0 and $\left(\frac{j}{3}\right)$, respectively. Therefore the nullity of $C$ is $3q/p$ when $p>3$ and is $q/p$ when $p=3$, which yields the number of linearly independent eigenvectors if $c:=\sum_{b\in\mathbb{F}} Z(\mathbf{c},b,0)=0$. When $c\neq 0$, we note that

$$\mathbf{v}=\begin{pmatrix} 0 \\ ca^3 \\ ca^6 \\ \vdots \\ ca^{3(q-1)} \end{pmatrix}$$

is a solution of (9.6). Together with the homogeneous ones, we see that the 1-eigenspace of $T_{\mathfrak{P}}$ has dimension $q+3q/p$ if $p>3$ and $q+q/p$ if $p=3$.

Next we determine the eigenvectors with eigenvalue $-2P+3$. Such eigenvectors are double cusp forms by Proposition 13, so $Z(\mathbf{c},r,3)=0$ for all $r\in\mathbb{F}$. Thus the equations from (8.5) with $j=0,1,2$ can be simplified as

$$Z(\mathbf{c},r,0)=-\frac{3}{2}\sum_{b\neq r}\frac{Z(\mathbf{c},b,1)}{b-r}+\frac{3(P-1)}{2}\sum_{b\neq r}\frac{Z(\mathbf{c},b,2)}{(b-r)^2},$$

$$0=(P+1)\sum_{b\in\mathbb{F}}(b-r)Z(\mathbf{c},b,0)+(P+3)\sum_{\mathbb{F}}Z(\mathbf{c},b,1)-(P-3)\sum_{b\neq r}\frac{Z(\mathbf{c},b,2)}{b-r},$$

and

$$0=\sum_{b\in\mathbb{F}}(b-r)^2Z(\mathbf{c},b,0)+3\sum_{b\in\mathbb{F}}(b-r)Z(\mathbf{c},b,1)+3\sum_{b\in\mathbb{F}}Z(\mathbf{c},b,2)$$

for all $r\in\mathbb{F}$. Substituting the first relation into the second and the third, and simplifying the resulting expressions by using Proposition 16, we arrive at

$$\sum_{b\in\mathbb{F}}Z(\mathbf{c},b,1)=2\sum_{b\neq r}\frac{Z(\mathbf{c},b,2)}{r-b} \tag{9.7}$$

and

$$\sum_{b\in\mathbb{F}}Z(\mathbf{c},b,2)=0 \tag{9.8}$$

for all $r\in\mathbb{F}$. Write $c=\sum_{b\in\mathbb{F}}Z(\mathbf{c},b,1)$. Solve the system (9.7) using the same method as (9.1). When $c=0$, we get homogeneous solutions

$$\begin{pmatrix} Z(\mathbf{c},0,2) \\ Z(\mathbf{c},a,2) \\ Z(\mathbf{c},a^2,2) \\ \vdots \\ Z(\mathbf{c},a^{q-1},2) \end{pmatrix}=\begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 \\ 1 \\ a^j \\ \vdots \\ a^{(q-2)j} \end{pmatrix}, \quad j=1,\dots,q-1;$$

when $c \neq 0$, we get a nonhomogeneous solution

$$\begin{pmatrix} Z(\mathbf{c},0,2) \\ Z(\mathbf{c},a,2) \\ Z(\mathbf{c},a^2,2) \\ \vdots \\ Z(\mathbf{c},a^{q-1},2) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 \\ ca \\ ca^2 \\ \vdots \\ ca^{q-1} \end{pmatrix}.$$

Note that all solutions satisfy Eq. (9.8). Thus the $(-2P+3)$-eigenspace of $T_{\mathfrak{P}}$ has dimension $q + q/p$. Combined with the dimension of 1-eigenspace, we conclude that $T_{\mathfrak{P}}$ is not diagonalizable on $S_{5,m}(\Gamma(T))$ since the space has dimension $4q$. We summarize the above discussion in

**Proposition 21.** *Suppose $\mathbb{F}$ has cardinality $q \geqslant 4$ and characteristic $p > 2$. For a maximal degree one ideal $\mathfrak{P} \neq (T)$, $1$ and $-2P+3$ are the two distinct eigenvalues of the Hecke operator $T_{\mathfrak{P}}$ on $S_{5,m}(\Gamma(T))$. The 1-eigenspace has dimension $q + 3q/p$ if $p > 3$ and dimension $q + q/p$ if $p = 3$. The $(-2P+3)$-eigenspace has dimension $q + q/p$. Further, $T_{\mathfrak{P}}$ is not diagonalizable on $S_{5,m}(\Gamma(T))$.*

Now we turn to the case when $p = 2$. In this case, we have only one eigenvalue, namely, 1. Then (8.5) for $j = 0, 1, 2$ become

$$\begin{aligned} 0 &= \bigl(P^2 + P + 1\bigr) \sum_{b \in \mathbb{F}} Z(\mathbf{c}, b, 0) + \sum_{b \neq r} \frac{Z(\mathbf{c}, b, 1)}{b - r} + (P-1) \sum_{b \neq r} \frac{Z(\mathbf{c}, b, 2)}{(b-r)^2} \\ &\quad + (P-1)^2 \sum_{b \neq r} \frac{Z(\mathbf{c}, b, 3)}{(b-r)^3}, \\ 0 &= \sum_{b \in \mathbb{F}} (b-r) Z(\mathbf{c}, b, 0) + \sum_{b \in \mathbb{F}} Z(\mathbf{c}, b, 1) + \sum_{b \neq r} \frac{Z(\mathbf{c}, b, 2)}{b - r} + \sum_{b \neq r} \frac{Z(\mathbf{c}, b, 3)}{(b-r)^2}, \end{aligned}$$

and

$$0 = \sum_{b \in \mathbb{F}} (b-r)^2 Z(\mathbf{c}, b, 0) + \sum_{b \in \mathbb{F}} (b-r) Z(\mathbf{c}, b, 1) + \sum_{b \in \mathbb{F}} Z(\mathbf{c}, b, 2) + \sum_{b \neq r} \frac{Z(\mathbf{c}, b, 3)}{(b-r)}$$

for all $r \in \mathbb{F}$. Observe that we can represent the above system as a homogeneous matrix equation $M\mathbf{x} = \mathbf{0}$, where $M$ is a $3q \times 4q$ matrix. Moreover, it is clear that rank $M > 1$. Thus the eigenspace is has dimension less than $4q$, so that the Hecke operator $T_{\mathfrak{P}}$ is not diagonalizable. Therefore we have shown

**Proposition 22.** *Suppose $\mathbb{F}$ has cardinality $q \geqslant 4$ and characteristic $p = 2$. For a maximal degree one ideal $\mathfrak{P} \neq (T)$, $1$ is the only eigenvalue of the Hecke operator $T_{\mathfrak{P}}$ on $S_{5,m}(\Gamma(T))$. The eigenspace of $T_{\mathfrak{P}}$ has dimension less than $4q$, the dimension of $S_{5,m}(\Gamma(T))$. Hence $T_{\mathfrak{P}}$ is not diagonalizable on $S_{5,m}(\Gamma(T))$.*

As for the action of $T_{\mathfrak{P}}$ on $S^2_{5,m}(\Gamma(T))$, by the same computation as before, we see that for $q$ odd, the 1-eigenspace is $(q-1)$-dimensional and the $(3-2P)$-eigenspace has dimension

$q + q/p$ so that the total dimension is less than $3q - 1$, the dimension of $S^2_{5,m}(\Gamma(T))$; for $q$ even, the matrix $M$ is $3q \times 3q$ with rank at least two, thus the eigenspace is at most $(3q - 2)$-dimensional. Hence in both cases, $T_{\mathfrak{P}}$ on $S^2_{5,m}(\Gamma(T))$ is not diagonalizable.

**Remark.** For Drinfeld cusp forms, what happens in case (iii) is representative of the general weights. For example, when the weight $k = 6$, we have three distinct eigenvalues $1, 4 - 3P$ and $6 - 6P + P^2$ if $p \neq 3$ and two distinct eigenvalues 1 and $P^2$ if $p = 3$. The computations for $Z(\mathbf{c}, b, u)$ are similar.

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