Immanants of Combinatorial Matrices

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The arbitrary immanants of three matrices whose determinants are known to be generating functions for sets of combinatorial objects are examined. Combinatorial interpretations are given for the immanants of the Matrix-tree matrix, and a special case of the Jacobi-Trudi matrix. These allow us to deduce immediately the nonnegativity of the coefficients in the expansion of the immanants. A conjecture is made about the nonnegativity of coefficients of the expansion of the immanant of the Jacobi-Trudi matrix in the general case. This nonnegativity result is seen to fail for the Hankel matrix, and combinatorial reasons for this failure are given. All results can be translated into statements about the nonnegativity of Schur function expansions for the related symmetric functions.


1. INTRODUCTION

The matrix function that D. E. Littlewood [6] called the immanant has, as special cases at opposite extremes, the determinant and the permanent. The latter two have proved extremely useful for combinatorial purposes. However, in contrast, little is known about the immanant from a combinatorial point of view. In this paper we begin to fill this gap.

For positive integers \( \lambda_1, \ldots, \lambda_k \) such that \( \lambda_1 \geq \cdots \geq \lambda_k \geq 1 \) and \( \lambda_1 + \cdots + \lambda_k = n \), we say that \( \lambda = (\lambda_1, \ldots, \lambda_k) \) is a partition of \( n \) (and write \( \lambda \vdash n \)) with \( k \) parts. The latter is denoted by \( l(\lambda) = k \). We also write \( \lambda = (1^{i_1} 2^{i_2} \cdots n^{i_n}) \), where \( i_j \) of the parts of \( \lambda \) are equal to \( j \), for \( j = 1, \ldots, n \). The conjugate of \( \lambda \) is the partition \( \lambda' = (\lambda_1, \ldots, \lambda_m) \), where \( m = \lambda_1 \), and \( \lambda_i \) is the number of \( \lambda_j \)'s greater than or equal to \( i \), for \( i = 1, \ldots, m \). For \( \sigma \in S_n \), the symmetric group on \( N_n = \{1, \ldots, n\} \), let \( \chi^\lambda(\sigma) \) be the value at \( \sigma \) of the irreducible character, \( \chi^\lambda \), associated with the conjugacy class indexed by \( \lambda \).

**Definition 1.1.** The \( \lambda \)th immanant of the \( n \times n \) matrix \( A \), with \((i, j)\)-element \( a_{i,j} \), is defined by

\[
\text{Imm}_\lambda A = \sum_{\sigma \in S_n} \chi^\lambda(\sigma) \prod_{i=1}^n a_{\sigma(i),i}.
\]
For $\lambda = (1^n)$, $\chi^\lambda(\sigma) = \text{sgn}(\sigma)$, so $\text{Imm}_{(1^n)} A = \text{det} A$, the determinant of $A$, and for $\lambda = (n)$, $\chi^\lambda(\sigma) = 1$, so $\text{Imm}_{(n)} A = \text{per} A$, the permanent of $A$.

For a positive semidefinite matrix $A$, Schur [9] proved the dominance result $\text{Imm}_\lambda A \geq \chi^\lambda(1) \text{det} A$, where 1 is the identity permutation, and $\chi^\lambda(1)$ is a positive integer given by the degree or hook formula. Schur [9] actually proved the following stronger result.

**Lemma 1.2 (Schur's Dominance Result).** Let $\mathcal{H}$ be a subgroup of $\mathfrak{S}_n$ and

$$\text{Imm}_\mathcal{H} A = \sum_{\sigma \in \mathcal{H}} \chi^\lambda(\sigma) \prod_{i=1}^n a_{i,\alpha(i)}.$$

Then

$$\text{Imm}_\mathcal{H} A \geq \chi^\lambda(1) \text{det} A.$$

In this paper we examine the immanants of three matrices whose determinants are known to be the generating functions for sets of combinatorial objects. In each case the coefficients of each monomial in the expansion of the determinant is therefore nonnegative. The first of these is the matrix-tree determinant, which enumerates spanning trees, in Section 4. The second is the Jacobi-Trudi matrix, whose determinant enumerates tableaux, in Sections 5 and 6. The third is the Hankel matrix, whose determinant enumerates mutually nonintersecting lattice paths of a special type, in Section 7. In each of these cases, the immanant is a power series in many variables.

In the first of these instances, we give a combinatorial value to the coefficients that appear in the expansion of the arbitrary immanant, based on the combinatorial interpretation for the characters of $\mathfrak{S}_n$, which is discussed in Section 3. This allows us to deduce the fact that these coefficients are nonnegative. Furthermore, since the characters can also be interpreted in terms of an inner product for symmetric functions, discussed in Section 2, it follows that a certain symmetric function has a non-negative expansion in terms of Schur symmetric functions. Similar results hold for the border strip case of the second of these instances; these are given in Section 5.

However, in the general case of the Jacobi-Trudi matrix, we only make a conjecture. This is given in Section 6, and is expressed in terms of the nonnegativity of the Schur symmetric function expansion of the cycle indicator for a general class of objects in the group algebra of the symmetric group. We are hopeful that a better understanding of this situation will have deep combinatorial significance.

In the third instance, we show that these nonnegativity results fail, and we comment on the combinatorial difference between these situations.
2. CHARACTERS AND INNER PRODUCTS

Let $E(t; x_1, x_2, \ldots) = \prod_{j \geq 1} (1 + x_j t)$. Then the $k$th elementary symmetric function $e_k$ and the $k$th complete symmetric function $h_k$ in variables $x_1, x_2, \ldots$, are defined for each nonnegative integer $k$, by

$$E(t; x_1, x_2, \ldots) = \sum_{k \geq 0} e_k t^k = \left\{ \sum_{k \geq 0} h_k (-t)^k \right\}^{-1}.$$ 

The $k$th power sum symmetric function $p_k$ in the variables $x_1, x_2, \ldots$, is defined for each positive integer $k$, by

$$p_k = \sum_{j \geq 1} x_j^k.$$ 

For any partition $\lambda = (\lambda_1, \ldots, \lambda_m)$, let $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_m}$, $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_m}$, and $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_m}$. The Schur symmetric function $s_\lambda$ is given by

$$s_\lambda = \det[h_{\lambda_i - i + j}]_{m \times m} = \det[e_{\lambda_i - i + j}]_{\lambda_1 \times \lambda_i},$$

and the monomial symmetric function $m_\lambda$ by

$$m_\lambda = \sum_{\alpha} x_1^{\alpha_1} x_2^{\alpha_2} \cdots,$$

where the sum is over all nonnegative integers $x_1, x_2 \ldots$ in which the nonzero $x_i$'s form a distinct permutation of $(\lambda_1, \ldots, \lambda_m)$.

The sets $\{e_\lambda\}$, $\{h_\lambda\}$, $\{p_\lambda\}$, $\{s_\lambda\}$, and $\{m_\lambda\}$ each form a basis (see [7]) for the ring of symmetric functions in $x_1, x_2, \ldots$ over $Q$. For this ring we define a scalar product by

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda, \mu},$$

or equivalently,

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda, \mu}$$

for partitions $\lambda, \mu$. For $\sigma \in S_n$ with $i_j$ cycles of length $j$ in its (disjoint) cycle decomposition, $j = 1, 2, \ldots$, we define the cycle-type of $\sigma$ to be the partition $\tau(\sigma) = (1^{i_1} 2^{i_2} \ldots)$. The cycle indicator of $\sigma$ is the symmetric function $p_{\tau(\sigma)}$. (In Polya Theory, it is usual to use the term cycle indicator to describe $p_{\tau(\sigma)}$ written explicitly in terms of a finite number of the underlying variables $x_i$.)

In terms of the scalar product and the cycle indicator, the characters of the symmetric group are given by

$$\chi^\sigma(\sigma) = \langle s_\lambda, p_{\tau(\sigma)} \rangle.$$
This leads immediately to an expression for the arbitrary immanant in terms of the scalar product.

**PROPOSITION 2.1.** For $A = [a_{i,j}]_{n \times n}$ and $\lambda \vdash n$,

$$\text{Imm}_\lambda A = \left< s_\lambda, \sum_{\sigma \in S_n} p_{\tau(\sigma)} \prod_{i=1}^n a_{i,\sigma(i)} \right>.$$  \hfill (3)

An alternative way of writing Proposition 2.1 is

$$\sum_{\sigma \in S_n} p_{\tau(\sigma)} \prod_{i=1}^n a_{i,\sigma(i)} = \sum_{\lambda \vdash n} s_\lambda \text{Imm}_\lambda A,$$  \hfill (4)

from (2). Thus the symmetric function on the left-hand side of (4) is a generating function for various immanants of $A$ using the Schur function basis.

For a positive semidefinite matrix $A$, note that Schur’s dominance result immediately shows that $\sum_{\sigma \in \mathcal{N}} p_{\tau(\sigma)} \prod_{i=1}^n a_{i,\sigma(i)}$ has a nonnegative Schur function expansion, since, in this case, $\text{Imm}_\lambda A \geq \chi^\lambda(1) \det A \geq 0$. Of course, this can be extended to the immanant functions $\text{Imm}_\lambda^H A$ immediately, by replacing the range of summation, $S_n$, by the subgroup $H$. Thus, for instance, we obtain an immediate proof of the following well known result [9].

**THEOREM 2.2.** For any subgroup $H$ of $S_n$, $\sum_{\sigma \in H} p_{\tau(\sigma)}$ has a nonnegative Schur function expansion.

**Proof.** If $J = [1]_{n \times n}$, then $J$ is positive semidefinite, and

$$\sum_{\sigma \in H} p_{\tau(\sigma)} = \sum_{\sigma \in H} p_{\tau(\sigma)} \prod_{i=1}^n [J]_{i,\sigma(i)} = \sum_{\lambda \vdash n} s_\lambda \text{Imm}_\lambda^H J.$$  \hfill (5)

The result follows from Schur’s dominance result for $\text{Imm}_\lambda^H$.

Theorem 2.2 is usually proved by noting that $\sum_{\sigma \in H} p_{\tau(\sigma)}$ is the character of a representation [10], so the coefficient of the Schur function $s_\lambda$ in its expansion equals the multiplicity in the representation of the irreducible representation with character $s_\lambda$, and hence is nonnegative.

3. **Characters and Standard Tableaux**

The diagram of a partition $\nu = (\nu_1, ..., \nu_m)$ is an array of squares of equal size arranged in rows and columns, with $\nu_i$ squares in the $i$th row, for $i = 1, ..., m$, such that each row begins in the first column. Adjacent squares
in the same row or the same column are edge-connected. If \( \mu = (\mu_1, ..., \mu_m) \) is a partition with \( v_i \geq \mu_i \) for \( i = 1, ..., m \), then the skew diagram of \( v/\mu \) is the diagram of \( v \) with the squares of the diagram of \( \mu \) removed from the upper left-hand corner. If \( a, b, c \) are diagrams of \( v/\mu, v, \mu \), respectively, then we write \( a = b - c \). A (skew) diagram is a border strip if it contains no \( 2 \times 2 \) subsquare of squares, and if its squares are edge-connected.


**Theorem 3.1.** Let \( \lambda \vdash n \) and let \( \sigma \in \mathfrak{S}_n \) with \( \mu = (\mu_1, ..., \mu_m) \) any fixed permutation of the parts of \( \tau(\sigma) \vdash n \). Then

\[
\chi^\lambda(\sigma) = \sum_d (-1)^{ht(d)},
\]

where the summation is over all sequences \( d = (d^{(1)}, ..., d^{(m)}) \) of diagrams such that \( d^{(m)} \) has shape \( \lambda \), and \( d^{(1)}, d^{(2)} - d^{(1)}, ..., d^{(m)} - d^{(m-1)} \) are border strips with \( \mu_1, ..., \mu_m \) squares, respectively. Moreover

\[
ht(d) = \# \text{rows}(d^{(1)}) + \# \text{rows}(d^{(2)} - d^{(1)}) + \cdots + \# \text{rows}(d^{(m)} - d^{(m-1)}) - m.
\]

For the combinatorial expressions for the immanants that we develop in this paper, it is convenient to reexpress this rule in terms of standard tableaux.

A tableau of shape \( v/\mu \) is obtained by placing a positive integer in each square of the diagram of \( v/\mu \), such that the sequence of integers in each row is nondecreasing from left to right, and the sequence of integers in each column is strictly increasing from top to bottom. A standard tableau is a tableau containing one of each of \( 1, 2, ..., n \), where \( n \) is the number of squares to be filled. A canonical subtableau of a standard tableau is a set of consecutive integers whose diagram is a border strip, and such that for all \( i < j \), all the integers in row \( i \) are smaller than all the integers in row \( j \). For an arbitrary vector of positive integers \( a = (a_1, ..., a_m) \), an \( a \)-canonical subtableau of a standard tableau is a canonical subtableau consisting entirely of the elements of \( \mathcal{N}_{a_1} + \cdots + a_i - \mathcal{N}_{a_1} + \cdots + a_{i-1} \) for some \( i = 1, ..., m \). Since a canonical subtableau is a border strip filled in a unique way, we have the following restatement of the Murnaghan–Nakayama rule.

**Corollary 3.2.** Let \( \lambda \vdash n \) and let \( \sigma \in \mathfrak{S}_n \) with \( \mu \) any fixed permutation of the parts of \( \tau(\sigma) \vdash n \). Then

\[
\chi^\lambda(\sigma) = \sum_t (-1)^{ht(d(t))},
\]
where the sum is over all standard tableaux of shape $\lambda$ that can be partitioned into $\mu$-canonical subtableaux. The sum of the numbers of rows in the $\mu$-canonical subtableaux, minus the number of parts of $\mu$, equals $h_{\mu}(t)$.

For example, if $\lambda = (2, 2, 1)$ and $\mu = (2, 2, 1)$, there are 3 standard tableaux of shape $\lambda$ that can be partitioned into $\mu$-canonical subtableaux. They are displayed in Fig. 1, with the $\mu$-canonical subtableaux circled, and labelled $t_1$, $t_2$, and $t_3$. But $h_{\mu}(t_1) = 0$, $h_{\mu}(t_2) = 1$, and $h_{\mu}(t_3) = 2$, so in this case the character has value $1 - 1 + 1 = 1$.

We shall use this modified rule to give combinatorial expressions for coefficients in the expansion of immanants by deducing sign-reversing involutions to cancel the contribution of all standard tableaux with an odd value of $h_{\mu}$, thus assuring nonnegativity.

4. IMMANANTS AND THE MATRIX-TREE THEOREM

A tree is a connected, acyclic graph. An in-directed tree rooted at a vertex $v$ is a directed graph formed from a tree containing the vertex $v$, by directing all edges of the tree towards $v$. If

$$T_n = \begin{bmatrix} \delta_{ij} \left( \sum_{i=1}^{n+1} a_{i,j} \right) - a_{i,j} \end{bmatrix}_{n \times n}$$

then the matrix-tree theorem states that $\det T_n$ is the generating function for in-directed trees on $\mathcal{N}_{n+1}$ rooted at $n+1$, with $a_{ij}$ marking the occurrences of the directed edge $(i, j)$. Thus every coefficient in the expansion of $\det T_n$ is nonnegative.

In this section we give a combinatorial interpretation for the expansion of $\mathrm{Imm}_2 T_n$ for arbitrary $\lambda \vdash n$. This involves functional digraphs and standard tableaux. As a corollary, we deduce that every coefficient in this expansion is nonnegative.

Consider an arbitrary function $f: \mathcal{N}_n \to \mathcal{N}_{n+1}$. The functional digraph of $f$ is a directed graph on the vertex set $\mathcal{N}_{n+1}$, with an edge directed from
i to f(i) for each $i \in \mathcal{N}_{n+1}$. These functional digraphs can be characterised as follows:

1. the connected component containing vertex $n+1$ is an in-directed tree rooted at $n+1$;
2. each other connected component, if any, consists of a nonempty collection of in-directed trees, whose root vertices are joined in a directed cycle.

For example, for $n = 8$, the function $g$ with $g(1) = 8$, $g(2) = 8$, $g(3) = 3$, $g(4) = 9$, $g(5) = 1$, $g(6) = 2$, $g(7) = 8$, $g(8) = 6$ has the functional digraph displayed in Fig. 2.

If the functional digraph of $f$ has $i_j$ directed cycles of length $j$, for $j \geq 1$, the cycle-type of $f$ is the partition $\tau(f) = (1^{i_1} 2^{i_2} \ldots)$. Thus, for $g$ given above, $\tau(g) = (3, 1)^4$. Note that the cycle-type of functional digraphs with a single component is the empty partition. A comprehensive account of the enumeration of functional digraphs with respect to various characteristics is given by Goulden and Jackson [5].

Now consider $\mathcal{F}_n$, the set of all functions $f : \mathcal{N}_n \to \mathcal{N}_{n+1}$ with no fixed points. Thus the cycle-types of such functions do not contain 1's. This set of functions is important because each monomial that arises in the expansion of $\text{Imm}_\lambda T_n$ is of the form $\prod_{i=1}^n a_{i, f(i)}$ for some $f \in \mathcal{F}_n$. Thus we write

$$\text{Imm}_\lambda T_n = \sum_{f \in \mathcal{F}_n} b_\lambda(f) \prod_{i=1}^n a_{i, f(i)}.$$ 

Since immanants are multilinear, we are able to write $b_\lambda(f)$ as an immanant.

**Lemma 4.1.** For $f \in \mathcal{F}_n$ with $\tau(f) = \mu = (\mu_1, \mu_2, \ldots)$, let $F$ be an $n \times n$ matrix whose $(i, j)$-element is 1 if $j = f(i) \leq n$, and 0 otherwise. Then

1. $b_\lambda(f) = \text{Imm}_\lambda (I - F)$,
2. $b_\lambda(f) = \left( s_\lambda, p^{n-|\mu|} \prod_{j \geq 1} \left( p_j^{\mu_j} + (-1)^{\mu_j} p_{n-j} \right) \right).$

![Fig. 2. The functional digraph of g.](image-url)
Proof. (1) The indeterminate $a_{i,j}$ appears only in row $i$, linearly, for all $i$ and $j$, in $T_n$. The coefficient of $a_{i,j}$ is 1 in position $(i,i)$ of $T_n$, and $-1$ in position $(i,j)$, and the result follows immediately.

(2) Every nonzero term in the expansion of the immanant in (1) corresponds to a permutation whose cycles are either cycles of $f$, or are cycles of length 1. Thus the cycle indicator for this set of permutations is

$$p_1^n - 1 \prod_{j = 1}^{n} (p_1^{\mu_j} + p_{\mu_j}).$$

But in the immanant, each $\mu_j$-cycle contributes $(-1)^{\mu_j}$, from the entries of $-F$, and the result follows.

We now use the inner product representation given by Lemma 4.1.2 to derive a combinatorial value for $b_{\lambda}(f)$ in terms of standard tableaux.

Consider a standard tableau $t$ on $\mathcal{M}_n$, and a partition $\mu \vdash k \leq n$. Let $\varepsilon_{\mu}(t)$ be the number of $\mu$-canonical subtableaux of $t$ for which the number of columns is even. Similarly, let $\alpha_{\mu}(t)$ be the number of $\mu$-canonical subtableaux of $t$ for which the number of columns is odd. For example, if $\mu = (4, 3, 2, 2)$ and $t$ is given by Fig. 3 then the $\mu$-canonical subtableaux of $t$ are given in Fig. 4 so $\varepsilon_{\mu}(t) = 1$ and $\alpha_{\mu}(t) = 1$.

We are now ready to give a combinatorial expression for $b_{\lambda}(f)$.

**Theorem 4.2.** For any $f \in \mathcal{F}_n$ with $\mu = \tau(f)$, and any $\lambda \vdash n$,

$$b_{\lambda}(f) = \sum_{\iota} 2^{\varepsilon_{\mu}(t)},$$

where 8 and 10 11

Fig. 4. The $\mu$-canonical subtableaux of $t$. 

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**FIG. 3.** The tableau $t$.

**FIG. 4.** The $\mu$-canonical subtableaux of $t$. 

where the summation is over all standard tableaux \( t \) of shape \( \lambda \), such that \( o_{\mu}(t) = 0 \).

**Proof.** From Lemma 4.1.2, if \( \mu = (\mu_1, \ldots, \mu_m) \), then

\[
b_\lambda(f) = \left\langle s_\lambda, \sum_v (-1)^{N(v)} p_v \right\rangle,
\]

where the summation is over all vectors \( v \) obtained from \( (\mu_1, \ldots, \mu_m, 1^{n-|\mu|}) \) by replacing some subset of the \( \mu_i \)'s by \( \mu_i, 1 \)'s (hence, there are \( 2^m \) such choices of \( v \)) and \( N(v) \) is the sum of all the parts of \( v \) that are not equal to 1. Thus, from Corollary 3.2 and the inner product expression for the characters,

\[
b_\lambda(f) = \sum_v (-1)^{N(v)} \sum_t (-1)^{ht(t)}
\]

\[
= \sum_v \sum_t (-1)^{N(v) + ht(t)},
\]

where the inner sum is over all \( v \) obtained as above, such that \( t \) can be partitioned into \( \nu \)-canonical subtableaux. There are \( 2^k \) such \( v \), where \( k \) is the number of \( \mu \)-canonical subtableaux of \( t \). But for a border strip, \( \# \text{cells} = \# \text{rows} + \# \text{columns} - 1 \), so \( (-1)^{N(v) + ht(t)} = (-1)^{\text{col}(t)} \), where \( \text{col}(t) \) is the sum of the numbers of columns in the \( \nu \)-canonical subtableaux of \( t \) with more than one cell. Now if \( t \) has any \( \mu \)-canonical subtableaux with an odd number of columns, then for every \( v \) containing the corresponding part of \( \mu \), there is a corresponding \( v' \) with this part replaced by 1's, and \( \text{col}(t) \) and \( \text{col}_{\nu}(t) \) have opposite parity. Thus the inner sum is 0 unless \( o_{\mu}(t) = 0 \). When \( o_{\mu}(t) = 0 \), \( \text{col}(t) \) is even for all \( v \), so the inner sum equals \( 2^k \), and the result follows since \( k = \varepsilon_{\mu}(t) \).

We can deduce the Matrix-tree theorem [1] from Theorem 4.2, as follows.

**Corollary 4.3 (Matrix-Tree Theorem).** \( \det T_n \) is the generating function for in-directed trees rooted at \( n + 1 \).

**Proof.** The determinant corresponds to the choice of \( \lambda = (1^n) \) in Theorem 4.2. There is a single standard tableau \( t \) of shape \( (1^n) \), namely, the elements of \( N_n \) arranged in a column, in increasing order.

But, for \( f \in \mathcal{F} \) with \( \tau(f) = \mu \), if \( o_{\mu}(t) = 0 \) then since \( o_{\mu}(t) = l(\mu) \) for this \( t, \mu \) must be the empty partition. Thus \( b_{(1^n)}(f) = 1 \) if \( \mu \) is the empty partition and 0 otherwise. Now if \( \tau(f) \) is empty, \( f \) is an in-directed tree rooted at \( n + 1 \), and this gives the result.

The specialisation of Theorem 4.2 to the permanent yields a compact result, as follows.
COROLLARY 4.4. For an arbitrary $f \in \mathcal{F}_n$, with cycle-type $\mu$, the coefficient of $\prod_{i=1}^n a_{i, f(i)}$ in $\text{per} \, T_n$ is

$$0 \quad \text{if } \mu \text{ has any odd parts},$$

$$2^{\ell(\mu)} \quad \text{if } \mu \text{ has no odd parts}.$$  

Proof. The permanent corresponds to the choice of $\lambda = (n)$ in Theorem 4.2. There is a single standard tableau of shape $(n)$, namely the elements of $\mathcal{N}_n$ arranged in a row, in increasing order. Thus

$$o(\mu) = \text{number of odd parts in } \mu,$$

$$e(\mu) = \text{number of even parts in } \mu$$

and the result follows. 

For the arbitrary immanant, Theorem 4.2 can be interpreted to yield a number of nonnegativity results.

COROLLARY 4.5. (1) For all $\lambda \vdash n$, $\text{Imm}_\lambda T_n$ has a nonnegative expansion in monomials in the $a_{i,j}$.

(2) For all $\lambda \vdash n$, and $f \in \mathcal{F}_n$, $\text{Imm}_\lambda (I - F) \geq 0$.

(3) For all $\mu = (\mu_1, \mu_2, \ldots) \vdash k \leq n$

$$p_{\mu}^{n-|\mu|} \prod_{j \geq 1} (p_{\mu_j}^n + (-1)^{\mu_j} p_{\mu_j})$$

has a nonnegative Schur function expansion.

Proof. This follows immediately from Lemma 4.1 and Theorem 4.2.

Corollary 4.5.2 and the Matrix-tree theorem establish a Schur dominance result for the matrix $I - F$, which is not symmetric, so it does not follow from Schur's result.

Note that Corollary 4.5.3 follows by multiplication from the fact that $p_{\mu}^n + (-1)^m p_m$ has a nonnegative Schur function expansion for all $m$. This is a consequence of the fact that if $\lambda \vdash m$ then $\langle p_m^n, s_\lambda \rangle \geq 1$, but $|\langle p_m, s_\lambda \rangle| \leq 1$.

5. IMMANANTS AND BORDER STRIP SKEW SCHUR FUNCTIONS

Consider partitions $\nu = (\nu_1, \ldots, \nu_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$, where $\nu_i \geq \mu_i \geq 0$ for $i = 1, \ldots, n$. We define the skew Schur symmetric function $s_{\nu/\mu}$ in $x_1, x_2, \ldots$, by $\langle s_\lambda, s_{\nu/\mu} \rangle = \langle s_\lambda s_\mu, s_\nu \rangle$ for all partitions $\lambda$. If

$$S_{\nu/\mu} = [h_{\nu_i - \mu_j + j - i}]_{n \times n}$$
then the Jacobi–Trudi identity [7] states that \( \det S_{v/\mu} = s_{v/\mu} \). Thus (see [7]) \( \det S_{v/\mu} \) is the generating function for tableaux of shape \( v/\mu \), with \( x_i \) marking the occurrence of \( i \), for \( i \geq 1 \), and therefore every monomial in \( x_1, x_2, \ldots \) has a nonnegative coefficient in its expansion.

In this section we give a combinatorial interpretation of the expansion of \( \text{Imm}_\lambda S_{v/\mu} \) for arbitrary \( \lambda \vdash n \) where \( v/\mu \) is a border strip (or union of border strips). This interpretation is in terms of standard tableaux. As a corollary, we deduce that every coefficient in this expansion is nonnegative. We also consider arbitrary \( v/\mu \) in Section 6.

In our examination of \( \text{Imm}_\lambda S_{v/\mu} \), it is convenient to consider the lattice path interpretation of the terms in its expansion given by Gessel and Viennot [4] in their combinatorial proof of the Jacobi–Trudi identity. A lattice path is a sequence of vertices, located at integer lattice points, connected by edges. The edges are called steps, and in this case we permit only two types of steps: vertical steps, which increase the \( y \)-coordinate of a vertex by 1, and horizontal steps, which increase the \( x \)-coordinate of a vertex by 1.

Now consider generating functions for such paths in which \( x_i \) marks each horizontal step at \( y \)-coordinate \( i \), for each \( i = 1, 2, \ldots \). Then, clearly, \( h_k \) is the generating function for the set of paths beginning at \((t, 1)\) and ending at \((t + k, \infty)\), for any integer \( t \). Thus, for any permutation \( \sigma \) of \( \mathcal{N}_n \),

\[
\prod_{i=1}^{n} h_{v_i - \mu_\sigma(i) + \sigma(i) - i}
\]

is the generating function for \( n \)-tuples of lattice paths, in which the \( i \)th such path begins at \((\mu_\sigma(i) - \sigma(i), 1)\) and ends at \((v_i - i, \infty)\), for \( i = 1, \ldots, n \). This leads immediately to the following combinatorial interpretation of the arbitrary immanant of \( S_{v/\mu} \).

**Proposition 5.1.** \( \text{Imm}_\lambda S_{v/\mu} \) is the generating function for \( n \)-tuples of lattice paths, from some permutation \( \sigma \) of the points \( P_1 = (\mu_1 - 1, 1) \), \( P_2 = (\mu_2 - 2, 1) \), \ldots, \( P_n = (\mu_n - n, 1) \) to the points \( Q_1 = (v_1 - 1, \infty) \), \( Q_2 = (v_2 - 2, \infty) \), \ldots, \( Q_n = (v_n - n, \infty) \), weighted by \( \chi^\lambda(\sigma) \).

Gessel and Viennot [4] evaluated \( \det S_{v/\mu} \), for which \( \chi^\lambda(\sigma) = \text{sgn}(\sigma) \), by finding a sign-reversing involution for the \( n \)-tuples with at least one pairwise intersection of paths. This proves that the contribution of such \( n \)-tuples is 0, so \( \det S_{v/\mu} \) is the generating function for \( n \)-tuples of mutually nonintersecting paths, which are easily shown to be equivalent to skew column-strict plane partitions of shape \( v/\mu \).

For each \( n \)-tuple \( \mathbf{r} \) of paths in Proposition 5.1, let \( \mathcal{M}(\mathbf{r}) \) be the multiset consisting of all steps in all \( n \) of the paths, where we retain the vertex at which a step starts, and regard horizontal (or vertical) steps starting at
different vertices \((i, j)\) as different. Also, let \(\sigma(\mathbf{r})\) be the permutation of the starting points given in Proposition 5.1. Then we can write

\[
\text{Imm}_{\lambda} S_{v/\mu} = \sum_{\alpha} x_1^{a_1} x_2^{a_2} \cdots \sum_{\mathbf{r}, M(\mathbf{r}) = \alpha} \chi^\lambda(\sigma(\mathbf{r})),
\]

where the outer sum is over all multisets \(\alpha\) of steps, \(a_i\) equals the number of horizontal steps in \(\alpha\) starting at \((i, j)\), for all integers \(j\), for \(i = 1, 2, \ldots\), and the inner sum is over all \(n\)-tuples \(\mathbf{r}\) of paths given in Proposition 5.1.

Let this inner sum be denoted by \(c_\lambda(\lambda, v, \mu)\). In the remainder of this section we demonstrate that this sum is nonnegative when \(v/\mu\) is a border strip (or union of border strips), and hence that \(\text{Imm}_{\lambda} S_{v/\mu}\) has a nonnegative expansion in monomials in \(x_1, x_2, \ldots\) for arbitrary \(\lambda\).

Note that to determine the \(n\)-tuples of paths \(\mathbf{r}\) such that \(M(\mathbf{r}) = \alpha\) for any given \(\alpha\), it is easiest to draw the steps in \(\alpha\) to form a directed multigraph (horizontal steps directed to the right, and vertical steps upwards), which we call the diagram of \(\alpha\). Then the ways of covering the edges of this graph with \(n\) directed paths (in which no distinction is made between the edges constituting a multiple edge) are precisely the choices of \(\mathbf{r}\) such that \(M(\mathbf{r}) = \alpha\).

If \(v/\mu\) is a border strip then \(P_i\) and \(Q_{i+1}\) have the same \(x\)-coordinate for \(i = 1, \ldots, n-1\). This means that each multiple edge must join a pair of vertices with the same one of these \(n-1\) \(x\)-coordinate values, and can consist only of two edges. Thus all allowable diagrams have \(k\) components, for some \(k \geq 1\), in which each component has the diagram given in Fig. 5, for some \(l \geq 1\), where the vertical undirected edges represent vertical undirected paths, the angled edges represent directed paths between vertices with different \(x\)-coordinates, and the \(l-1\) internal vertices represent pairs of vertical undirected paths. Note that these simplifications make no difference in the values of \(\sigma(\mathbf{r})\) for \(\mathbf{r}\) with \(M(\mathbf{r}) = \alpha\) for such a diagram, and that in Fig. 5 the directed path ending at \(Q_i\) can begin at \(P_{\sigma(i)}\) only for \(\sigma(i) \geq i-1\). This

![Diagram](image_url)

**Fig. 5.** The diagram for components when \(v/\mu\) is a border strip.
immanants of combinatorial matrices

Immediately yields, for border strip \( v/\mu \) and \( \alpha \) whose diagrams have a single component,

\[
c_{\alpha}(\lambda, v, \mu) = \text{Imm}_x U_n,
\]

where \( U_n \) is an \( n \times n \) matrix with \( [U_n]_{i,j} = \begin{cases} 1, & \text{for } j \geq i - 1, \\ 0, & \text{otherwise}. \end{cases} \)

Furthermore, note that the cycles of a permutation \( \sigma \) with \( \sigma(i) \geq i - 1 \) are \((\beta_1, \beta_1 - 1, \ldots, 2, 1), (\beta_1 + \beta_2, \ldots, \beta_1 + 1), \ldots, (\beta_1 + \beta_2 + \cdots + \beta_i, \ldots, \beta_1 + \beta_2 + \cdots + \beta_{i-1} + 1)\) for any \( \beta_1, \beta_2, \ldots, \beta_i \geq 1 \) with \( \beta_1 + \beta_2 + \cdots + \beta_i = l \), and any \( i \geq 1 \). Thus the cycle indicator for this set of permutations is

\[
\sum_{i \geq 1} \sum_{\beta_1, \beta_2, \ldots, \beta_i \geq 1} p_{\beta_1} p_{\beta_2} \cdots p_{\beta_i} = [t^l](1 - p_1 t - p_2 t^2 - \cdots)^{-1},
\]

so, for border strip \( v/\mu \) and \( \alpha \) whose diagram has a single component

\[
c_{\alpha}(\lambda, v, \mu) = [t^n] \langle s_\lambda, (1 - p_1 t - p_2 t^2 - \cdots)^{-1} \rangle.
\]

The situation in which the diagram of \( \alpha \) has \( k \) components follows straightforwardly.

**Lemma 5.2.** For border strip \( v/\mu \) and \( \alpha \) whose diagram has \( k \) components, containing

\[
\{P_1, P_2, \ldots, P_{l_1}\}, \{P_{l_1+1}, \ldots, P_{l_1+l_2}\}, \ldots, \{P_{l_1+\cdots+l_{k-1}+1}, \ldots, P_{l_1+\cdots+l_k}\},
\]

where \( l_1, l_2, \ldots, l_k \geq 1 \) and \( l_1 + l_2 + \cdots + l_k = n \), then

1. \( c_{\alpha}(\lambda, v, \mu) = \text{Imm}_x (U_{l_1} \oplus \cdots \oplus U_{l_k}) \)
2. \( c_{\alpha}(\lambda, v, \mu) = \langle s_\lambda, \prod_{i=1}^k [t^{l_i}](1 - p_1 t - p_2 t^2 - \cdots)^{-1} \rangle. \)

We now use the inner product representation given by Lemma 5.2.2 to derive a combinatorial value for \( c_{\alpha}(\lambda, v, \mu) \), in terms of standard tableaux.

Consider a standard tableau \( t \) on \( \mathcal{N}_n^\alpha \), and \( l_1, l_2, \ldots, l_k \geq 1 \) with \( l_1 + \cdots + l_k = n \). Then \( w_{l_1, \ldots, l_k}(t) \) is the number of ways of covering \( t \) with canonical subtableaux of one row each, such that:

1. the subtableaux consist entirely of elements chosen from exactly one of

\[
L_1 = \mathcal{N}_{l_1}^\alpha, \ L_2 = \mathcal{N}_{l_1+l_2}^\alpha - \mathcal{N}_{l_1}^\alpha, \ldots, \ L_k = \mathcal{N}_{l_1+\cdots+l_k}^\alpha - \mathcal{N}_{l_1+\cdots+l_{k-1}}^\alpha,
\]
(2) no pair of the subtableaux together form a canonical subtableau (on two rows) containing only elements chosen from exactly one of \(L_1, L_2, \ldots, L_k\).

**THEOREM 5.3.** For border strip \(v/\mu\) and \(x\) whose diagram has \(k\) components, containing \(\{P_1, \ldots, P_{l_1}\}, \{P_{l_1+1}, \ldots, P_{l_2}\}, \ldots, \{P_{l_1+\cdots+l_{k-1}+1}, \ldots, P_{l_k}\}\), where \(l_1, l_2, \ldots, l_k \geq 1\) and \(l_1 + l_2 + \cdots + l_k = n\), then

\[
c_x(\lambda, v, \mu) = \sum_t w_{l_1, \ldots, l_k}(t),
\]

where the sum is over all standard tableaux \(t\) of shape \(\lambda\).

**Proof.** Suppose there are \(k = 1\) components. Then, from Lemma 5.2.2,

\[
c_x(\lambda, v, \mu) = \langle s_\lambda, [t^n](1 - p_1 t - p_2 t^2 - \cdots)^{-1}\rangle
\]

from Corollary 3.2, where the summation over \(\beta\) is for all \(i \geq 1\) and \(\beta = (\beta_1, \ldots, \beta_i)\), such that \(\beta_1, \ldots, \beta_i \geq 1\) and \(\beta_1 + \cdots + \beta_i = n\).

Now, for fixed standard tableaux \(t\), consider the inner summation over \(\beta\). For a given \(\beta\), let \(j \in \mathcal{N}_n\) be the smallest element such that either

1. \(j\) lies in a \(\beta\)-canonical subtableau of \(t\) with more than one row, or
2. \(j\) lies in a \(\beta\)-canonical subtableau of \(t\) with one row, which, together with another \(\beta\)-canonical subtableau, forms another canonical subtableau.

In case (1), if the canonical subtableau with more than one row has \(\beta_m\) elements, with \(a\) elements in its first row, then let \(\psi(\beta)\) be obtained from \(\beta\) by replacing \(\beta_m\) by the pair \(a, \beta_m - a\). Thus \(\text{ht}_\beta(t) = \text{ht}_{\psi(\beta)}(t) + 1\).

In case (2), if the subtableau with one row has \(a\) elements and the other subtableau has \(b\) elements, then let \(\psi(\beta)\) be obtained from \(\beta\) by replacing the parts \(a, b\) (which must appear consecutively in \(\beta\), by construction) by \(a + b\). Thus \(\text{ht}_\beta(t) = \text{ht}_{\psi(\beta)}(t) - 1\).

Now in both (1) and (2), \((-1)^{\text{ht}(t)} + (-1)^{\text{ht}_{\psi(\beta)}(t)} = 0\), so the contributions of \(\beta\) and \(\psi(\beta)\) cancel in the inner sum. Moreover, for fixed \(t\), \(\psi\) is an involution for all \(\beta\) satisfying either (1) or (2), so the total contribution of all such \(\beta\) is 0. For all other \(\beta\), \(\text{ht}_\beta(t) = 0\), and there are \(w_n(t)\) such \(\beta\). Thus the result is true for \(k = 1\).

The result follows straightforwardly from this for all \(k\). \(\blacksquare\)

We have the following immediate corollaries for the determinant and permanent of \(S_{v/\mu}\).
COROLLARY 5.4. Under the conditions of Theorem 5.3,

1. \( c_s(1^n), v, \mu) = \delta_{n,k} \)

2. \( c_s(n), v, \mu) = 2^n - k. \)

Proof. (1) The only such \( t \) is the one-column tableau with 1, 2, ..., \( n \) arranged in increasing order from the top. This tableau can be covered in the desired way by choosing the individual entries as the subtableaux. But if any \( l_i > 1 \) then any pair of consecutive entries in \( L_i \) will form a canonical tableau on two rows. Thus the value of \( w \) is 0 unless \( l_1 = l_2 = \cdots = l_k = 1 \), or, equivalently, \( n = k \), and the value is 1 when \( n = k \).

(2) The only such \( t \) is the tableau 1 2 \cdots n. Now there are \( 2^k - 1 \) ways of covering the elements of \( L_i \), one for each ordered composition of \( l_i \), and thus \( \prod_{i=1}^k 2^k - 1 = 2^k + \cdots + k - k = 2^n - k \) ways of covering \( t \). But none of these violates condition (2).

Of course, Corollary 5.4.1 is immediate in Gessel and Viennot's combinatorial procedure, since they showed that \( s_{\nu/\mu} \) counts mutually nonintersecting lattice paths only, which lead only to diagrams \( \alpha \) with \( n = k \).

For the arbitrary immanant, Theorem 5.3 can be interpreted to yield a number of nonnegativity results.

COROLLARY 5.5. (1) For border strips \( \nu/\mu \), \( \text{Imm}_\lambda s_{\nu/\mu} \) has a nonnegative expansion in monomials in \( x_1, x_2, \ldots \).

(2) For all \( \lambda \vdash n \) and \( l_1, \ldots, l_k \geq 1 \) with \( l_1 + \cdots + l_k = n \),

\[ \text{Imm}_\lambda(U_{l_1} \oplus \cdots \oplus U_{l_k}) \geq 0. \]

(3) The symmetric function \( (1 - p_1 - p_2 - \cdots)^{-1} \), has a nonnegative Schur function expansion.

Proof. This follows immediately from Lemma 5.2 and Theorem 5.3.

Gessel [3] has pointed out to us that Corollary 5.5.3 can be deduced directly by noting that

\[ \left(1 - \sum_{i \geq 1} p_i\right)^{-1} = \left(\sum_{j \geq 0} h_j\right)\left(1 - \sum_{j \geq 2} (j - 1)h_j\right)^{-1}, \]

since the right-hand side has a nonnegative expansion in the complete symmetric functions, and products of complete symmetric functions have nonnegative Schur function expansions through the Kostka numbers.

Note that Corollary 5.5.2 establishes a Schur dominance result for direct sums of \( U_i \)'s, which are not symmetric, and so does not follow from Schur's result.
In this section we consider the evaluation of $c_\alpha(\lambda, v, \mu)$ in the arbitrary case. As noted in Section 5, for given $\alpha$ this can be carried out by covering the diagram of $\alpha$ with $n$ directed paths.

Now consider the following reduction operations on the diagram: remove (by identifying its incident vertices) a (multiple) edge if it is the only (multiple) edge directed away from some vertex other than one of the $P_i$ or if it is the only (multiple) edge directed towards some vertex other than one of the $Q_j$. After applying this recursively as often as possible, draw the reduced diagram, as a planar graph, with the $P_i$'s and $Q_j$'s in their original fixed positions, so that all directed edges have a vertical component from bottom to top, and so that all vertices except the $P_i$'s and $Q_j$'s have different $y$-coordinates. Denote these vertices by $v_1, v_2, \ldots$ in increasing order of $y$-coordinate. Such a reduced diagram for $n = 5$ is given in Fig. 6.

Let $\mathcal{S}_\alpha$, for $\alpha \subseteq N_n$, be the sum, in the group algebra of $\mathcal{S}_n$, of all permutations of elements in $\alpha$, with the elements of $N_n - \alpha$ as fixed points. The multiset of values of $\sigma(r)$ for 5-tuples of directed paths covering the reduced diagram given in Fig. 6 is simply

$$\mathcal{S}_{\{1, 2\}} \mathcal{S}_{\{4, 5\}} \mathcal{S}_{\{2, 3, 4\}} \mathcal{S}_{\{1, 2, 3\}} \mathcal{S}_{\{4, 5\}}.$$

How do we determine the multiset? We begin the process of determining the 5-tuples of directed paths covering the diagram by starting five paths at the $P_i$'s and moving upwards. The first point of choice is encountered at $v_1$, where paths 4 and 5 can switch, and hence the rightmost factor $\mathcal{S}_{\{4, 5\}}$.

![Fig. 6. A reduced diagram.](image-url)
The second point of choice is $v_2$, where paths 1, 2, 3 can be permuted in any order, giving rise to the factor $\mathfrak{S}_{\{1,2,3\}}$.

In general, there is one factor $\mathfrak{S}_x$ for each of the vertices $v_1, v_2, \ldots$, and these factors are multiplied in the order that the corresponding $v_i$ are encountered as we move from bottom to top. Note that each set $x$ consists of consecutive integers only, since the reduced diagrams are embedded in the plane. We conjecture that

$$\sum_{\sigma} \chi^x(\sigma) \geq 0,$$

where the summation multiset is any product of such $\mathfrak{S}_x$'s. Another way of expressing this, using the inner product expression for the character, is as follows.

**Conjecture 6.1.** The cycle indicator of

$$\prod_{i=1}^{k} \mathfrak{S}_{v_{b_i}^{-1}} v_{a_i},$$

where $1 < a_i < b_i \leq n$ for $1 = 1, \ldots, k$, has a nonnegative Schur function expansion.

This conjecture has been verified,\(^1\) for a large number of cases when $n \leq 6$. It has been proven in the case $a_i = i$, $b_i = i + 1$, for $i = 1, \ldots, k$ in Section 5, since the set of permutations for each component in Lemma 5.2 is precisely

$$\mathfrak{S}_{\{1,2\}} \mathfrak{S}_{\{2,3\}} \cdots \mathfrak{S}_{\{i-1,i\}}.$$

This conjecture, if true, would prove that $c_x(\lambda, v, \mu)$ is nonnegative, and hence that $\text{Imm}_x \mathfrak{S}_{v/\mu}$ has a nonnegative expansion in monomials in the $x$'s for the arbitrary case. The reason for this is as follows. The multiset of values of $\sigma$ for $n$-tuples of paths covering the diagram of $x$ and the reduced diagram of $x$ are not the same in all cases. This is because multiple-edges at a later stage of the reduction may correspond to distinct directed paths between a pair of vertices. Thus we must distinguish between these edges, which is not done in the reduction procedure. However, the effect of these multiedges would be to add a factor of $\mathfrak{S}_x$ where $x$ consists of consecutive integers corresponding to the edges in the multiple-edge, and such factors are built into the conjecture in all possible ways. Thus the conjecture accounts for all possible diagrams.

Note that the extension of Conjecture 6.1 to $\prod_{i=1}^{k} \mathfrak{S}_{a_i}$ for arbitrary

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\(^1\) This was obtained by a computer calculation using MAPLE.
subsets \( a_i \) of \( \mathcal{N}_n \), \( i = 1, \ldots, k \), is not true in general. For example, when \( n = 4 \), the cycle indicator of

\[
\mathcal{S}_{\{1,2\}} \mathcal{S}_{\{2,3\}} \mathcal{S}_{\{1,3\}} \mathcal{S}_{\{3,4\}}
\]

is

\[
r_1^4 + 5r_2^2 + 6r_3 + 3r_4,
\]

and the Schur function expansion of this is

\[
16s_4 + 4s_3 + 2s_2,
\]

which has a negative coefficient when expanded in terms of Schur functions. The combinatorial significance of such arbitrary subsets is that they could not arise from planar diagrams. Thus we believe that there may be some deep connection between the nonnegativity of these expansions, and the planarity of their corresponding networks.

Prompted by this conjecture, Stembridge [11] has been able to make similar nonnegativity conjectures for the expansion of a number of other classes of symmetric functions.

7. IMMANANTS AND THE HANKEL DETERMINANT

Consider the (Jacobi) continued fraction

\[
\frac{1}{1 - a_0 x - \frac{b_0 x^2}{1 - a_1 x - \frac{b_1 x^2}{1 - a_2 x - \frac{b_2 x^2}{1 - a_3 x - \cdots}}}}
\]

and suppose that the series expansion in powers of \( x \) for this continued fraction is \( \sum_{n \geq 0} J_n x^n \). If

\[
H_m = [J_{i+j-2}(m+1) \times (m+1)],
\]

then \( H_m \) is a Hankel matrix, and it is well known [5] that

\[
\det H_m = \prod_{j=0}^{m-1} b_j^{m-j}.
\]

Thus the Hankel determinant is a polynomial in \( a_0, a_1, \ldots; b_0, b_1, \ldots \) in which every monomial has a nonnegative coefficient.

Flajolet [2] (see also [5, Sect. 5.2]) has shown that \( J_n \) is the generating function for a certain set of lattice paths from \((0, 0)\) to \((n, 0)\). These paths
have three types of steps: *rises* increase the $x$- and $y$-coordinates each by 1; *levels* increase the $x$-coordinate by 1; *falls* increase the $x$-coordinate by 1 and decrease the $y$-coordinate by 1. Rises and levels which start at a vertex with $y$-coordinate equal to $i$ are marked by $b_i$ and $a_i$, respectively, for $i \geq 0$. Moreover, these paths must not reach a vertex whose $y$-coordinate is negative.

Viennot [12] has noted that the Hankel determinant $\det H_m$ is the generating function for $(m + 1)$-tuples of paths of the type described above, beginning at $(m, 0), (m - 1, 0), \ldots, (0, 0)$, and ending at some permutation of $(m, 0), (m + 1, 0), \ldots, (2m, 0)$, weighted by the sign of this permutation. He has exhibited a sign-reversing involution such that the $(m + 1)$-tuples containing at least one pairwise intersection of paths cancel in pairs, leaving a single $(m + 1)$-tuple of pairwise nonintersecting paths which contributes $\prod_{i=0}^{m-1} b_i^{m-i}$ to the generating function.

This combinatorial proof of the Hankel determinant evaluation differs from that of Gessel and Viennot's [4] combinatorial proof of the Jacobi–Trudi identity referred to in Section 5. In the Hankel case, paths can intersect without having a vertex in common (when a rise from $(i, j - 1)$ meets a fall from $(i, j)$). Viennot [12] calls such an intersection a *virtual crossing*, and his sign-reversing involution treats these separately.

In view of the results of earlier sections, it is reasonable to suppose that every immanant of $H_m$ has a nonnegative expansion. To check this, all immanants of $H_m$ were calculated for $m \leq 3$. All immanants of $H_0, H_1$, and $H_2$ have nonnegative expansions, as does $\text{Imm}_2 H_2$ for $\lambda = (3, 1)$ (as well, of course, as $\lambda = (1, 1, 1, 1), (4)$). However, this is not the case for $\lambda = (2, 1, 1), (2, 2)$.

For example, the expansion of $\text{Imm}_{(2,1,1)} H_3$ contains the terms $-2a_0^3a_1b_0^2b_1^2$, $-2a_0^3a_3b_0^2b_1^2$, and $-4a_0^2a_1a_3b_0^2b_1^2$, while the expansion of $\text{Imm}_{(2,2)} H_3$ contains 14 terms with negative coefficients.

If indeed our conjectured nonnegativity results in Section 6 are true for skew Schur functions, it seems that a general nonnegativity result for combinatorial matrices, if it exists, does not apply in the case of Hankel matrices because of the virtual crossings. However, the reason for this is not at all clear to us.

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2 This was obtained by using MAPLE.
REFERENCES