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LINEAR ALGEBRA AND ITS APPLICATIONS

Digraph based determination of Jordan block size structure of singular matrix pencils

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Abstract

The generic Jordan block sizes corresponding to multiple characteristic roots at zero and at infinity of a singular matrix pencil will be determined graph-theoretically. An application of this technique to detect certain controllability properties of linear time-invariant differential algebraic equations is discussed. © 1998 Elsevier Science Inc. All rights reserved.

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1. Introduction

In this paper the authors study the correspondence between matrix pencils and directed graphs.

The determination of the Jordan block sizes associated with the eigenvalue zero of a matrix A has been of interest for years. Important results were presented by Brualdi [2], Hershkowitz [7,8] and Hershkowitz and Schneider [9]. The present authors make use of cycle families to graph-theoretically determine determinants, minors and determinantal divisors. In a forerunner paper the special case of regular matrix pencil was investigated [17]. In contrast with the regular case, we now take left and right Kronecker indices into consideration. The last part of this contribution deals with an application in control

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theory, namely impulse controllability. This problem has been mentioned in another recent paper [16]. Context and proofs are different.

Since 1960s, the state–space description $\dot{x} = Ax + Bu, y = Cx$ has been widely accepted by the control engineers' community. Generic multiplicities of poles and zeros of the transfer function $C(sI - A)^{-1}B$ were characterized by analysing the Rosenbrock matrix

$$\begin{pmatrix} sI - A & B \\ C & 0 \end{pmatrix}$$

graph-theoretically (see Andrei [1], Reinschke [13] and references cited there).

The state–space theory was generalized to differential algebraic equations of the form $E\dot{x} = Ax + Bu, y = Cx$, where the matrix E may be singular. Murota [11,12] and Reinschke [14,15] obtained various results using the matrix pencil

$$\begin{pmatrix} sE - A & B \\ C & 0 \end{pmatrix},$$

where Murota considered bipartite graphs instead of directed graphs.

In Section 2 of this paper, the reader is reminded of some topics from the matrix pencil theory. Structure matrices and directed graphs are introduced in Section 3. In Section 4 we determine the generic Jordan block sizes associated with characteristic roots at zero and at infinity of a possibly singular matrix pencil $\lambda E + \mu A$. The results derived there will be applied in Section 5 to analyse differential algebraic equations.

2. Matrix pencils

First, we recall some facts from the theory of matrix pencils [21,10,6,19]. Let $\lambda E + \mu A$ be a matrix pencil with $E, A \in \mathbb{R}^{m \times n}$. A matrix pencil is said to be *regular*, if $m = n$ and $\det(\lambda E + \mu A)$ is not the zero polynomial. Otherwise, the pencil is said to be *singular*. Each pencil can be transformed into *Kronecker canonical form*

$$P(\lambda E + \mu A)Q = \operatorname{diag}(\lambda E_{\mathrm{r}} + \mu A_{\mathrm{r}}, \lambda E_{\mathrm{s}} + \mu A_{\mathrm{s}}) \tag{1}$$

with regular matrices P and Q. The block diagonal matrix on the right-hand side consists of a *regular part* $\lambda E_{\mathrm{r}} + \mu A_{\mathrm{r}}$ and a *singular part* $\lambda E_{\mathrm{s}} + \mu A_{\mathrm{s}}$. Let us consider the regular part. A pair $(\lambda, \mu) \in \mathbb{C}^2 \setminus (0,0)$ is called a *characteristic root* if $\det(\lambda E + \mu A) = 0$. A characteristic root (λ, μ) is said to be a *characteristic root at zero* if $\lambda = 0$, a *characteristic root at infinity* if $\mu = 0$, and a *finite characteristic root* else. The regular part has the following structure

$$\lambda E_{\mathrm{r}} + \mu A_{\mathrm{r}} = \operatorname{diag}(\lambda I_{n_f} + \mu W, \lambda I_{n_0} + \mu N^0, \lambda N^\infty + \mu I_{n_\infty}). \tag{2}$$

The $n_f \times n_f$ matrix W is regular and the matrices N^0, N^∞ are $n_0 \times n_0$, $n_\infty \times n_\infty$ block diagonal matrices

$$N^0 = \mathrm{diag}(N_1^0, \ldots, N_{d_0}^0), \qquad N^\infty = \mathrm{diag}(N_1^\infty, \ldots, N_{d_\infty}^\infty) \tag{3}$$

consisting of nilpotent Jordan blocks. The characteristic roots at zero and at infinity are associated with the matrices N^0 and N^∞. We denote the sizes of the Jordan blocks by $s_1^0 \geqslant \cdots \geqslant s_{d_0}^0$ and $s_1^\infty \geqslant \cdots \geqslant s_{d_\infty}^\infty$ respectively. The *index* is defined by

$$\mathrm{ind}(\lambda E + \mu A) := \begin{cases} 0 & \text{if } n_\infty = 0, \\ s_1^\infty & \text{if } n_\infty > 0. \end{cases} \tag{4}$$

Obviously, the finite characteristic roots are given by the zeros of $\det(\lambda I_{n_f} + \mu W) = 0$, the characteristic roots at zero by the zeros of $\det(\lambda I_{n_0} + \mu N^0) = 0$, and the characteristic roots at infinity by the zeros of $\det(\lambda N^\infty + \mu I_{n_\infty})$.

The singular part in Eq. (1) has a generalized block diagonal form

$$\lambda E_s + \mu A_s = \mathrm{diag}(L_{\varepsilon_1}, \ldots, L_{\varepsilon_p}, \ldots, L_{\eta_1}^{\mathrm{T}}, \ldots, L_{\eta_q}^{\mathrm{T}}). \tag{5}$$

The $(\sum_{i=1}^p \varepsilon_i + \sum_{j=1}^q \eta_j + q) \times (\sum_{i=1}^p \varepsilon_i + \sum_{j=1}^q \eta_j + p)$ matrix pencil $\lambda E_s + \mu A_s$ is formed by $k \times (k+1)$ blocks L_k with

$$L_k = \left.\begin{pmatrix} \lambda & \mu & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & \lambda & \mu \end{pmatrix}\right\} k. \tag{6}$$

The integers $0 \leqslant \varepsilon_1 \leqslant \cdots \leqslant \varepsilon_p$ and $0 \leqslant \eta_1 \leqslant \cdots \leqslant \eta_q$ are called *right* and *left Kronecker indices*, respectively. In case of a rectangular pencil, one can obtain a square pencil by inserting zero rows or columns. Furthermore, we have $m = n$ if and only if $p = q$.

3. Structure matrices and digraphs

In this section we consider matrices whose entries are either fixed at zero or indeterminate values. Denoting the indeterminate entries of a matrix M by "×" and the zero entries by "0", one obtains a (*Boolean*) *structure matrix* $[M]$. Fixing all the indeterminate entries of $[M]$ at some particular values we obtain an *admissible realization*, for short, $M \in [M]$. The matrices M' and M'' are said to be structurally equivalent if $M' \in [M]$ and $M'' \in [M]$.

Consider a structure matrix $[M]$ with k non-zero entries. The set of admissible realizations $M \in [M]$ is isomorphic to the vector space $\mathbb{R}^k$. We say "a property holds *generically* for $[M]$" or, equivalently, "a property holds for *almost all* $M \in [M]$" if the property under consideration is met for all $M \in [M]$ belonging to an open and dense subset of $\mathbb{R}^k$. For example, the *generic rank* of a structure matrix is given by $\operatorname{rank}[M] := \max_{M\in[M]} \operatorname{rank} M$ (cf. [11,13,3]). Let $[\lambda E + \mu A]$ denote a pencil of $n \times n$ structure matrices $[E]$ and $[A]$. The generic rank of a pencil $[\lambda E + \mu A]$ is defined by $\operatorname{rank}\, [\lambda E + \mu A] := \max_{(E,A)\in[E,A]} \max_{\lambda,\mu\in\mathbb{C}} \operatorname{rank}\, (\lambda E + \mu A)$.

We consider an associated *digraph* (*directed graph*) $G([\lambda E + \mu A])$ with n vertices enumerated $1,\dots,n$, and E-edges and A-edges leading from vertex j to vertex i if $[e_{ij}] \neq 0$ or $[a_{ij}] \neq 0$, respectively. A *path* is a sequence of edges such that the initial vertex of the succeeding edge is the final vertex of the proceeding edge, where each vertex is incident to at most two edges. A path is said to be a *cycle* if the initial vertex of the first edge is the final vertex of the last edge. A *self-cycle* is a cycle consisting of exactly one edge. A set of vertex disjoint cycles is called a *cycle family*. Its length is given by the number of all the edges involved. A cycle family of length n is called a *spanning cycle family* (*scf*). An $n \times n$ structure matrix $[M]$ is generically regular ($\operatorname{rank}[M] = n$) if and only if there exists an scf within the associated digraph $G([M])$.

Let $G^k(\cdot)$ denote the set of digraphs resulting from $G^0(\cdot) := G(\cdot)$ by supplementing k additional edges. Furthermore, we define

$$\varrho := \min\{k\colon\ \exists \text{ scf within } G^k([\lambda E + \mu A])\},$$
$$\varrho^{[E]} := \min\{k\colon\ \exists \text{ scf within } G^k([\lambda E])\},$$
$$\varrho^{[A]} := \min\{k\colon\ \exists \text{ scf within } G^k([\mu A])\},$$

The integers $\theta_k^{[E]}$ and $\theta_k^{[A]}$ denote the minimal numbers of E-edges or A-edges, respectively, contained in an scf of $G^k([\lambda E + \mu A])$ involving k additional edges. Obviously, $\theta_k^{[E]}$ and $\theta_k^{[A]}$ are defined for $k \geqslant \varrho$ only.

4. Determination of the Jordan block size structure

In this section we apply the concepts introduced in the previous sections.

Lemma 4.1. *Let $[\lambda E + \mu A]$ be a pencil of $n \times n$ structure matrices. The numbers $d_{[0]}$ and $d_{[\infty]}$ of Jordan blocks corresponding to the characteristic roots at zero and at infinity, respectively, may be obtained for almost all $(E,A) \in [E,A]$ from the set of digraphs $G^k([\lambda E + \mu A])$ as follows:*

$$d_{[0]} = \varrho^{[A]} - \varrho, \quad d_{[\infty]} = \varrho^{[E]} - \varrho. \tag{7}$$

Furthermore, $\varrho = p = q$ holds generically.

Proof. The integers $\varrho, \varrho^{[E]}$ and $\varrho^{[A]}$ are the minimal numbers of additional matrix entries to be supplemented such that $[\lambda E + \mu A]$, $[E]$ and $[A]$, respectively, become generically regular. Consequently, $\varrho, \varrho^{[E]}$ and $\varrho^{[A]}$ can be interpreted as generic rank deficiencies of $[\lambda E + \mu A]$, $[E]$ and $[A]$, respectively. For almost all admissible realizations $(E,A) \in [E,A]$ there hold the equations rank

$E = \text{rank}[E]$, rank $A = \text{rank}[A]$, and $\max_{\lambda,\mu\in\mathbb{C}}$ rank $(\lambda E + \mu A) = \text{rank}\,[\lambda E + \mu A]$. Using the Kronecker canonical form, one obtains for almost all $(E, A) \in [E, A]$:

$$\begin{aligned}
\varrho &= n - \text{rank}[\lambda E + \mu A] = n - \max_{\lambda,\mu\in\mathbb{C}} \text{rank}(\lambda E + \mu A) \\
&= n - ((n_0 + n_f + n_\infty) + \text{rank}(\lambda E_s + \mu A_s)), \\
\varrho^{[E]} &= n - \text{rank}[E] = n - \text{rank}\, E = n - ((n_0 + n_f + n_\infty - d_\infty) \\
&\quad + \text{rank}\, E_s), \\
\varrho^{[A]} &= n - \text{rank}\,[A] = n - \text{rank}\, A = n - ((n_0 + n_f + n_\infty - d_0) \\
&\quad + \text{rank}\, A_s).
\end{aligned}$$

Because of Eqs. (5) and (6) we have rank $(\lambda E_s + \mu A_s) =$ rank $E_s =$ rank A_s for all $(\lambda, \mu) \in \mathbb{C}^2 \setminus \{(0,0)\}$. Hence,

$$\begin{aligned}
\varrho &= \varrho^{[E]} - d_{[\infty]}, \\
\varrho &= \varrho^{[A]} - d_{[0]}.
\end{aligned}$$

Furthermore, $\varrho = n - ((n_0 + n_f + n_\infty) + \sum_{i=1}^{p} \varepsilon_i + \sum_{j=1}^{q} \eta_j) = p = q$. □

Example 4.1. Consider the given structure matrices

$$[E] = \begin{pmatrix} 0 & 0 & \times & 0 \\ 0 & 0 & 0 & 0 \\ \times & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad [A] = \begin{pmatrix} \times & \times & 0 & 0 \\ 0 & 0 & \times & 0 \\ 0 & \times & 0 & \times \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with the digraph $G([\lambda E + \mu A])$ depicted in Fig. 1(a). The E-edges have been drawn as bold lines, and we will draw additional edges as dotted lines. Let us denote an A-edge from j to i by $j \to i$, an E-edge by $j \Rightarrow i$ and an additional edge by $j \Rrightarrow i$. At least $k = 1$ additional edge is needed to obtain an scf within the digraphs $G^k([\lambda E + \mu A])$ or $G^k([\mu A])$, for example $1 \to 1, 2 \to 3 \to 2, 4 \Rrightarrow 4$, see Fig. 1(b). Hence, $\varrho = \varrho^{[A]} = 1$.

The digraph $G([\lambda E])$ must be supplemented by at least two additional edges to obtain an scf, e.g. $1 \Rightarrow 3 \Rightarrow 1, 2 \Rrightarrow 4 \Rrightarrow 2$ (see Fig. 1(c)). Therefore, $\varrho^{[E]} = 2$. Referring to Lemma 4.1, we have no Jordan block corresponding to a characteristic root at zero and exactly one Jordan block corresponding to a characteristic root at infinity.

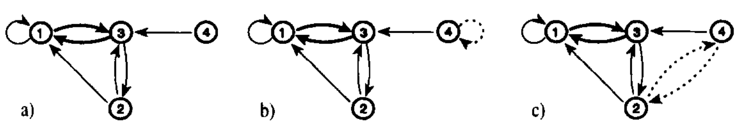

Fig. 1. Digraphs to Example 4.1.

Theorem 4.1. *The generic Jordan block sizes* $s^0_{[1]},\dots,s^0_{[d_{[0]}]}$ *and* $s^\infty_{[1]},\dots,s^\infty_{[d_{[\infty]}]}$ *may be obtained from the set of digraphs* $G^k([\lambda E+\mu A])$ *as follows*:

$$\begin{aligned} s^0_{[1]} &= \theta^{[E]}_{\varrho} - \theta^{[E]}_{\varrho+1}, & s^\infty_{[1]} &= \theta^{[A]}_{\varrho} - \theta^{[A]}_{\varrho+1} \\ &\vdots & &\vdots \\ s^0_{[d_{[0]}]} &= \theta^{[E]}_{\varrho^{[A]}-1} - \theta^{[E]}_{\varrho^{[A]}}, & s^\infty_{[d_{[\infty]}]} &= \theta^{[A]}_{\varrho^{[E]}-1} - \theta^{[A]}_{\varrho^{[E]}}. \end{aligned}$$

The generic index is $\operatorname{ind}([\lambda E+\mu A]) = \theta^{[A]}_{\varrho} - \theta^{[A]}_{\varrho+1}$.

Proof. Let $\lambda E+\mu A$ be a pencil of $n\times n$ matrices $(E,A)\in[E,A]$, and $0\leqslant k\leqslant n$. The determinant Δ of an $(n-k)\times(n-k)$ submatrix pencil resulting from $\lambda E+\mu A$ by deletion of the rows $i_1,\dots,i_k$ and the columns $j_1,\dots j_k$ is either the zero polynomial or a homogeneous polynomial of degree $n-k$ within the variables λ and μ. This minor of order $n-k$ can be determined as follows

$$\begin{aligned} \Delta &= (-1)^{i_1+\cdots+i_k-j_1+\cdots+j_k}\det\begin{pmatrix} \lambda E+\mu A & e_{i_1} & \dots & e_{i_k} \\ e^{\mathrm T}_{j_1} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ e^{\mathrm T}_{j_k} & 0 & \dots & 0 \end{pmatrix} \\ &= \sum_{\nu=k}^{n} p_\nu \lambda^{n-\nu}\mu^{\nu-k}, \end{aligned} \tag{8}$$

where e_i is a column vector whose ith entry is one and the remaining $n-1$ entries are zero. The coefficients $p_\nu(k\leqslant\nu\leqslant n)$ may be obtained from the (weighted) digraph $G(\lambda E+\mu A)$ (see [15] and references cited there). For this purpose we supplement $G(\lambda E+\mu A)$ by k additional edges leading from j_1 to $i_1,\dots,j_k$ to i_k, respectively, each with weight 1. Each coefficient p_ν of Δ corresponds to the scf whose edge set consists of all the k supplementary edges, $n-\nu$ E-edges, and $\nu-k$ A-edges (cf. [17]). This implies that for almost all $(E,A)\in[E,A]$ the zero multiplicity (z.m.) of Δ with respect to λ is the smallest number of E-edges within such an scf.

The determinantal divisors $D_{n-k}(\lambda,\mu)$ of $\lambda E+\mu A$ are defined as the greatest common divisors (gcd) of all minors of order $n-k$ of the pencil $\lambda E+\mu A$. Eq. (8) yields

$$D_{n-k}(\lambda,\mu) = \gcd_{\substack{1\leqslant i_1<\cdots<i_k\leqslant n\\ 1\leqslant j_1<\cdots<j_k\leqslant n}} \det\begin{pmatrix} \lambda E+\mu A & e_{i_1} & \dots & e_{i_k} \\ e^{\mathrm T}_{j_1} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ e^{\mathrm T}_{j_k} & 0 & \dots & 0 \end{pmatrix}.$$

To obtain the z.m. of $D_{n-k}(\lambda,\mu)$ one has to find the smallest ν with $p_\nu \neq 0$ for all minors of order $n-k$. Hence, for almost all $(E,A) \in [E,A]$ the z.m. of $D_{n-k}(\lambda,\mu)$ with respect to λ is the smallest number of E-edges involved in an scf of the set $G^k([\lambda E + \mu A])$, i.e., $\theta_k^{[E]} =$ z.m. $_\lambda(D_{n-k}(\lambda,\mu))$ holds generically. Because of Lemma 4.1 we have $D_{n-k}(\lambda,\mu) \equiv 0$ for $k < \varrho$ and z.m. $(D_{n-k}(\lambda,\mu)) = 0$ for $k \geqslant \varrho^{[A]}$. The z.m. of the determinantal divisors with respect to λ determine the sizes of the Jordan blocks corresponding to the characteristic roots at zero (see [10,6])

$$\begin{array}{rcccccccc} D_{n-k}(\lambda,\mu) & = & s_1^0 & + & s_2^0 & + & \cdots & + & s_{d_0}^0 \\ D_{n-k-1}(\lambda,\mu) & = & & & s_2^0 & + & \cdots & + & s_{d_0}^0 \\ & \vdots & & & & & & & \vdots \\ D_{n-\varrho^{[A]}}(\lambda,\mu) & = & & & & & & & s_{d_0}^0 \end{array}$$

for $\varrho = n - \max_{\lambda,\mu\in\mathbb{C}} \operatorname{rank}(\lambda E + \mu A)$ and $\varrho^{[A]} = n - \operatorname{rank} A$ (which holds generically because of Lemma 4.1). This implies $s_{[1]}^0 = \theta_\varrho^{[E]} - \theta_{\varrho+1}^{[E]}, \ldots, s_{[d_{[0]}]}^0 = \theta_{\varrho^{[A]}-1}^{[E]} - \theta_{\varrho^{[A]}}^{[E]}$.

Similarly, the integers $\theta_k^{[A]}$ can be interpreted as the generic z.m. of $D_{n-k}(\lambda,\mu)$ with respect to μ. One obtains $s_{[1]}^\infty = \theta_\varrho^{[A]} - \theta_{\varrho+1}^{[A]} = \operatorname{ind}([\lambda E + \mu A])$ etc. for $\varrho < \varrho^{[E]}$, i.e., for $d_{[0]} > 0$. In case of $d_{[0]} = 0$ $(\varrho = \varrho^{[E]})$ we have $\theta_k^{[A]} = 0$ for all $k \geqslant \varrho$, which implies $\operatorname{ind}([\lambda E + \mu A]) = 0$. This completes the proof. □

Example 4.2. For the matrix pencil of Example 4.1 we have only to determine the size of the Jordan block at infinity. Within the set $G^1([\lambda E + \mu A])$ the minimal number of A-edges contained in an scf is $\theta_1^{[A]} = 2$ (see Fig. 1(b) and consider the scf $1 \Rightarrow 3 \to 2 \to 1, 4 \Rrightarrow 4$). As mentioned in Example 4.1, there is an scf without A-edges with the set $G^2([\lambda E + \mu A])$, i.e., $\theta_2^{[A]} = 0$. Hence, the size of the only Jordan block corresponding to a characteristic root at infinity is $s_{[1]}^\infty = 2 = \operatorname{ind}([\lambda E + \mu A])$. A symbolical decomposition of the example matrix pencil into the Kronecker canonical form yields actually a 2×2 Jordan block at infinity and the remaining singular part with $p = q = 1$:

$$\begin{aligned} P(\lambda E + \mu A)Q &= \operatorname{diag}(\lambda E_r + \mu A_r, \lambda E_s + \mu A_s) \\ &= \operatorname{diag}(\lambda N_1^\infty + \mu I_2, L_{\varepsilon_1}, L_{\eta_1}^{\mathrm{T}}) \\ &= \operatorname{diag}(\lambda N_1^\infty + \mu I_2, L_1, L_0^{\mathrm{T}}) = \begin{pmatrix} \mu & \lambda & & \\ 0 & \mu & & \\ & & \lambda & \mu \\ & & 0 & 0 \end{pmatrix}. \end{aligned}$$

5. Applications in control theory

Many real-world systems can be modelled by *differential algebraic equations* (*DAE*). We confine ourselves to linear time-invariant DAE of the form

$$E\dot{x}(t) = Ax(t) + Bu(t); \quad E, A \in \mathbb{R}^{n\times n}; \quad B \in \mathbb{R}^{n\times m}, \tag{9}$$

or, equivalently, of the Laplace-transformed form

$$(sE - A)X(s) = Ex(0) + BU(s). \tag{10}$$

We set $(\lambda, \mu) := (s, -1)$ to use the notations common in the control engineers' community. The following pencils derived from Eq. (9) play important roles:

(i) Consider $sE - A$. Eqs. (9) and (10) have an unique solution if and only if $sE - A$ is regular. In the following we assume $sE - A$ to be regular. The generic regularity of $[sE - A]$ is equivalent to the existence of an scf within the digraph $G([sE - A])$, i.e., $\varrho = 0$ [15], Cor. to Th. 1.

(ii) The controllability properties of Eq. (9) can be characterized by the pencil $(sE - A, -B)$. Adding m zero rows one gets an $(n + m) \times (n + m)$ matrix pencil

$$(s\tilde{E} - \tilde{A}) := \begin{pmatrix} sE - A & -B \\ 0_{m\times n} & 0_{m\times m} \end{pmatrix} \tag{11}$$

One needs m additional edges to obtain an scf within the digraph associated with Eq. (11), for example self-cycles at the vertices $n + 1, \ldots, n + m$, i.e., $\varrho = m$.

Now, the reader is reminded of some facts from control theory [18,22,20,4,5]. The system (9) can be transformed into the canonical form

$$\begin{pmatrix} I_{n_f} & 0 & 0 \\ 0 & I_{n_0} & 0 \\ 0 & 0 & N^\infty \end{pmatrix} \dot{z} = \begin{pmatrix} W & 0 & 0 \\ 0 & N^0 & 0 \\ 0 & 0 & I_{n_\infty} \end{pmatrix} z + \begin{pmatrix} B^f \\ B^0 \\ B^\infty \end{pmatrix}, \tag{12}$$

where $z(t) := Q^{-1}x(t)$, comp. Eqs. (1) and (2). Setting $J := \operatorname{diag}(W, N^0)$ and $B^1 := \begin{pmatrix} B^f \\ B^0 \end{pmatrix}$ we obtain two subsystems:

$$\dot{z}_1(t) = Jz_1(t) + B^1u(t), \tag{13}$$

$$N^\infty \dot{z}_2(t) = z_2(t) + B^\infty u(t). \tag{14}$$

The subsystem (13) associated with the finite characteristic roots and the characteristic roots at zero of $sE - A$ is called *slow subsystem* because the responses z_1 when B^1u is a unit step are continuous functions. The so-called *fast subsystem* (14) is associated with the characteristic roots at infinity. Here, the responses z_2 when $B^\infty u$ is a unit step are discontinuous functions. The subspace defined by the image im $N^\infty \subseteq \mathbb{R}^{n_\infty}$ is called the *impulse subspace*.

Lemma 5.1. *Let $[sE - A]$ be a pencil of $n \times n$ structure matrices such that the digraph $G([sE - A])$ contains an scf. Then the following hold for almost $(E, A) \in [E, A]$:*

$$\textit{Dimension of fast subsystem: } \theta_0^{[A]}, \tag{15}$$

$$\textit{Dimension of slow subsystem: } n - \theta_0^{[A]}, \tag{16}$$

$$\textit{Dimension of impulse subspace: } \theta_0^{[A]} - \varrho^{[E]}. \tag{17}$$

Proof. The existence of an scf with $G([sE - A])$ implies $\varrho = 0$, and $p = q = 0$ for almost all $(E, A) \in [E, A]$. From the degree $\deg_s \det(sE - A) = n_f + n_0 = n - n_\infty$ (see Eq. (2)) the statements Eqs. (15) and (16) follow immediately, comp. [13], p. 232, [14], Th. 2. The matrix $N^\infty \in \mathbb{R}^{n_\infty \times n_\infty}$ consists of d_∞ Jordan blocks N_i^∞, therefore $\dim \operatorname{im} N^\infty = n_\infty - d_\infty$, i.e., the image of N^∞ is $(n_\infty - d_\infty)$-dimensional. For almost all $(E, A) \in [E, A]$ we have $n_\infty = \theta^{[A]}$ and $d_\infty = \varrho^{[E]}$ (Lemma 4.1). Hence, $\dim \operatorname{im} N^\infty = \theta^{[A]} - \varrho^{[E]}$ holds generically. □

Example 5.1. The DAE system

$$\underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \times & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \times & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \times \\ 0 & \times & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{[E]} \dot{x}$$

$$= \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & \times & 0 & 0 & 0 \\ 0 & \times & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \times & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \times & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \times \\ 0 & 0 & \times & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \times & 0 & 0 \\ \times & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{[A]} x + \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \times & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \times \end{pmatrix}}_{[B]} u$$

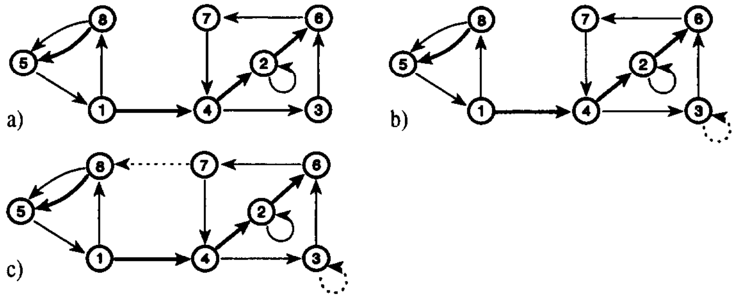

Fig. 2. Digraphs of the sets $G^0([sE-A]), G^1([sE-A]), G^2([sE-A])$.

is associated with the digraph $G([sE-A])$ depicted in Fig. 2(a). The minimal number of A-edges contained in an scf is $\theta_0^{[A]} = 7$ $(1 \rightarrow 8 \Rightarrow 5 \rightarrow 1, 2 \rightarrow 2, 4 \rightarrow 3 \rightarrow 6 \rightarrow 7 \rightarrow 4)$. Supplementing the digraph with a self-cycle at vertex 3, we obtain $\theta_1^{[A]} = 4$ $(1 \rightarrow 8 \Rightarrow 5 \rightarrow 1, 2 \Rightarrow 6 \rightarrow 7 \rightarrow 4 \Rightarrow 2, 3 \Rrightarrow 3)$, see Fig. 2(b). With a further additional edge from 7 to 8, the minimal number of A-edges involved in an scf of $G^2([sE-A])$ can be reduced to $\theta_2^{[A]} = 2$ (Fig. 2(c)). Introducing additional edges from 6 to 7 and from 5 to 1, one gets $\theta_3^{[A]} = 1$ and $\theta_4^{[A]} = 0$. This implies $d_{[\infty]} = 4, s_{[1]}^{\infty} = 3, s_{[2]}^{\infty} = 2$ and $s_{[3]}^{\infty} = s_{[4]}^{\infty} = 1$. Therefore, the generic dimension of the fast subsystem is 7, of the slow subsystem is 1, and we have a 3-dimensional impulse subspace.

Let be $E, A, \tilde{E}, \tilde{A}$ and B as defined in Eqs. (11) and (12), and denote the number of Jordan blocks associated with the characteristic roots at infinity of $s\tilde{E}-\tilde{A}$ by $\tilde{d}_{\infty}$. In general we have rank $\tilde{E} = \operatorname{rank} E$ and $\tilde{d}_{\infty} = \max_{s\in\mathbb{C}} \operatorname{rank}(s\tilde{E}-\tilde{A}) - \operatorname{rank}\tilde{E} \leqslant \max_{s\in\mathbb{C}} \operatorname{rank}(\lambda E + \mu A) - \operatorname{rank} E = d_{\infty}$ (see Proof of Lemma 4.1). Under the assumption $\det(sE-A) \not\equiv 0$, for any matrix $B \in \mathbb{R}^{n\times m}$ the pencils $sE-A$ and $s\tilde{E}-\tilde{A}$ (Eq. (11)) have the same number of Jordan blocks associated with the characteristic roots at infinity, and we will use the notation d_{∞}. We denote the Jordan block sizes of $s\tilde{E}-\tilde{A}$ by $\tilde{s}_1^{\infty} \geqslant \cdots \geqslant \tilde{s}_{d_{\infty}}^{\infty}$, and we define $\langle M|\bar{B}\rangle := \operatorname{im}\bar{B} + \operatorname{im} M\bar{B} + \operatorname{im} M^2\bar{B} + \cdots + \operatorname{im} M^{k-1}\bar{B}$, where $M \in \mathbb{R}^{k\times k}, \bar{B} \in \mathbb{R}^{k\times m}$. The subspace $\langle N^{\infty}|N^{\infty}B^{\infty}\rangle \subseteq \operatorname{im} N^{\infty}$ is called *impulse controllable subspace* of Eq. (14) and can be interpreted as the set of points of $\mathbb{R}^{n_{\infty}}$ reachable by impulse solutions of Eq. (14) induced by non-impulsive excitations u (see [20,4] for details).

Theorem 5.1. *The generic dimension of the impulse controllable subspace of Eqs. (9) and (14) can be obtained from the digraphs* $G([sE-A])$ *and* $G([s\tilde{E}-\tilde{A}])$ *as follows*

$$\dim\langle N^{\infty}|N^{\infty}B^{\infty}\rangle = \theta_0^{[A]} - \theta_m^{[\tilde{A}]}.$$

Proof. Let us start with the Jordan block sizes associated with the characteristic roots at infinity of the pencils $(sE - A)$ and $(sE - A, -B)$. For almost all $(E, A, B) \in [E, A, B]$ the generic block sizes are equal to the numerical block sizes, i.e., $d_\infty = d_{[\infty]}$ and $s_i^\infty = s_{[i]}^\infty, \tilde{s}_i^\infty = \tilde{s}_{[i]}^\infty$ for $1 \leqslant i \leqslant d_\infty$. Theorem 4.1 implies:

$$\begin{aligned} \sum_{i=1}^{d_\infty} s_i^\infty &= \theta_0^{[A]} - \theta_{d_{[\infty]}}^{[A]} = \theta_0^{[A]}, \\ \sum_{i=1}^{d_\infty} \tilde{s}_i^\infty &= \theta_m^{[\tilde{A}]} - \theta_{d_{[\infty]}+m}^{[\tilde{A}]} = \theta_m^{[\tilde{A}]}. \end{aligned} \tag{18}$$

Consider such a triple $(E, A, B) \in [E, A, B]$ and the associated pencils $sE - A$ and $s\tilde{E} - \tilde{A}$ fulfilling Eq. (18). Using the representation (13) and (14) of (9) and (10) it has been shown that each Jordan block $\tilde{s}_i^\infty$ corresponds to an $(\tilde{s}_i^\infty - 1)$-dimensional subspace, $\mathscr{U}_i \subseteq \operatorname{im} N^\infty$ of the impulse subspace not contained in the impulse controllable subspace (see [18], Proof of Th. 4 and [20] Sec. IV. A), i.e., $\mathscr{U}_i \cap \langle N^\infty | N^\infty B^\infty \rangle = \{0\}$. This implies that the dimension of the impulse controllable subspace can be obtained as the difference of the Jordan block sizes of the pencils of $sE - A$ and $s\tilde{A} - \tilde{E}$ associated with the characteristic roots at infinity

$$d := \dim \langle N^\infty | N^\infty B^\infty \rangle = \sum_{i=1}^{d_\infty} s_i^\infty - \sum_{i=1}^{d_\infty} \tilde{s}_i^\infty.$$

Since Eq. (18) is valid for almost all $(E, A, B) \in [E, A, B]$, the integer d is the generic dimension of the impulse controllable subspace, i.e., $d = \theta_0^{[A]} - \theta_m^{[\tilde{A}]}$ holds generically. □

Example 5.2. Consider the DAE system of Example 5.1 with $m = 2$. From $G([sE - A])$ we get $\theta_0^{[A]} = 7$. The digraph $G([s\tilde{E} - \tilde{A}])$ has been sketched in Fig. 3(a). Supplementing $G([s\tilde{E} - \tilde{A}])$ with two additional edges one obtains an scf with only 4 $\tilde{A}$-edges, $10 \to 8 \Rightarrow 5 \to 1 \Rightarrow 4 \Rightarrow 2 \Rightarrow 6 \to 7 \Rrightarrow 10, 9 \to 39$ (see Fig. 3(b)), i.e., $\theta_2^{[\tilde{A}]} = 4$. Therefore, almost all systems of the structure given by Example 5.1 have a 3-dimensional impulse controllable subspace.

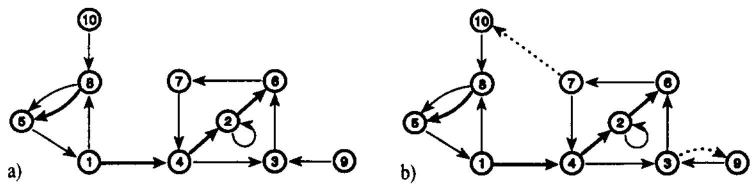

Fig. 3. Digraph $G([s\tilde{E} - A])$ and a modification.

Let $\tilde{\varepsilon}_1 \leqslant \cdots \leqslant \tilde{\varepsilon}_m$ and $\bar{\varepsilon}_1 \leqslant \cdots \leqslant \bar{\varepsilon}_m$ denote the right Kronecker indices of the pencils $s\tilde{E} - \tilde{A}$ and $(sN^\infty - I, -B^\infty)$, respectively. It has been shown (see [18] and [4], Th. 5) that

$$\dim\langle N^\infty | N^\infty B^\infty \rangle = \sum_{i=1}^{m} \bar{\varepsilon}_i \leqslant \sum_{i=1}^{m} \tilde{\varepsilon}_i.$$

So Theorem 5.1 can be used to determine $\sum_{i=1}^{m} \bar{\varepsilon}_i$ and to give a lower bound for $\sum_{i=1}^{m} \tilde{\varepsilon}_i$.

A DAE system is said to be *impulse controllable* if $\operatorname{im} N^\infty = \langle N^\infty | N^\infty B^\infty \rangle$.

Theorem 5.2. *A DAE system (9) is generically impulse controllable if and only if for the digraph $G([s\tilde{E} - \tilde{A}])$ there holds*

$$\theta_m^{[\tilde{A}]} - \theta_{m+1}^{[\tilde{A}]} \leqslant 1. \tag{19}$$

Proof. The difference (19) is the generic index $k := \operatorname{ind}([s\tilde{E} - \tilde{A}])$, see Th. 4.1, and the non-negative integer k is equal to the numerical index (4) for almost all $(E, A, B) \in [E, A, B]$. For such a "typical" realization (E, A, B) there are three possible cases:

$k = 0$: We have no Jordan block associated with the characteristic roots at infinity within both pencils, i.e., the impulse subspace is zero dimensional.

$k = 1$: As $s\tilde{E} - \tilde{A}$ has exactly d_∞ Jordan blocks associated with the characteristic roots at infinity and the sequence $(\tilde{s}_i^\infty)$ is non-increasing, we have $1 = \tilde{s}_1^\infty = \cdots = \tilde{s}_{d_\infty}^\infty$ and $\sum_{i=1}^{d_\infty} \tilde{s}_i^\infty = d_\infty$. Because of $\sum_{i=1}^{d_\infty} s_i^\infty = n_\infty$, the impulse controllable subspace is $(n_\infty - d_\infty)$-dimensional. This means, the impulse controllable subspace is the whole impulse subspace, i.e., the system is impulse controllable.

$k > 1$: In this case $\sum_{i=1}^{d_\infty} \tilde{s}_i^\infty > d_\infty$ and therefore $\dim\langle N^\infty | N^\infty B^\infty \rangle < n_\infty - d_\infty = \dim \operatorname{im} N^\infty$. The system is not impulse controllable. □

Example 5.3. Supplementing $G^2([s\tilde{E} - \tilde{A}])$ of Example 5.2 with a further additional edge, the number of $\tilde{A}$-edges involved in an scf of $G^3([s\tilde{E} - \tilde{A}])$ can be reduced form $\theta_2^{[A]} = 4$ to $\theta_3^{[A]} = 3$. Hence, almost all admissible realizations of the example system are impulse controllable.

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