Numerical analysis and computing of a non-arbitrage liquidity model with observable parameters for derivatives

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ARTICLE INFO	ABSTRACT
	This paper deals with the numerical analysis and computing of a nonlinear model of option pricing appearing in illiquid markets with observable parameters for derivatives. A consistent monotone finite difference scheme is proposed and a stability condition on the stepsize discretizations is given. © 2010 Elsevier Ltd. All rights reserved.

1. Introduction

Market liquidity has become currently an issue of very high concern in financial risk management. The Black–Scholes (B–S) model is only acceptable in idealized financial markets where one assumes that the market in the underlying asset is perfectly elastic so that trades do not affect prices in equilibrium. An updated summary of models, methods and techniques related to illiquid option pricing problems may be found in [1–4]. In this paper we deal with the non-arbitrage liquidity model of Backstein and Howison [5,6], which presents the suitable property of observability for parameters of derivatives, i.e., directly estimable from order book data,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \lambda \sigma^2 S^3 \left(\frac{\partial^2 V}{\partial S^2}\right)^2 + \frac{1}{2}\lambda^2 \mu^2 \sigma^2 S^4 \left(\frac{\partial^2 V}{\partial S^2}\right)^3 + rS\frac{\partial V}{\partial S} - rV = 0$$

$$V(S, T) = f(S) = \max\{S - K, 0\} \quad 0 < S < \infty, \quad 0 \le t < T$$

(1.1)

where $\lambda > 0$ models the market depth, which represents the elasticity of the stock price to the quantity traded. Parameter μ has the meaning of the slippage measure that transforms the average transaction price into the next published price, [5]. When $\lambda = \mu = 0$, model (1.1) becomes the (B–S) model. Here σ is the constant volatility, r is the interest rate, T is the maturity, K is the strike price and $V(S, t)$ is the option price depending on the underlying asset S and the time t. $\frac{\partial^2 V}{\partial S^2}$ is the Gamma of the option.

In Section 2 a suitable transformation is introduced allowing the consideration of the original problem as a nonlinear diffusion one. We choose the bounded numerical domain and introduce a numerical scheme construction for the transformed option price as well as for the transformed Gamma because of its leading influence in the numerics of the problem. Properties of the numerical solution are studied in Section 3. Finally Section 4 includes stability, consistency and illustrative examples. If $z = (z_1 z_2 \dots z_p)^\top$ is a vector in $\mathbb{R}^{p \times 1}$, its $\| \ \|_\infty$ is denoted by $\|z\|_\infty = \max\{|z_i|; 1 \le i \le p\}$.

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2. Problem transformation and numerical scheme construction

For the sake of convenience in the study of the numerical analysis of the problem (1.1) it is going to be transformed into a nonlinear diffusion problem. Let us consider the substitution defined by

$$X = e^{r(T-t)}S, \qquad \tau = \frac{\sigma^2}{2}(T-t), \qquad U = e^{r(T-t)}V. \tag{2.1}$$

Then problem (1.1) takes the form

$$U_\tau(X, \tau) - X^2 \Psi(X, U_{XX}(X, \tau))U_{XX}(X, \tau) = 0, \qquad U(X, 0) = f(X), \tag{2.2}$$

where $0 < X < \infty, 0 < \tau \leq \frac{\sigma^2 T}{2}$, and

$$\Psi(X, U_{XX}) = 1 + 2\lambda X U_{XX} + \lambda^2 \mu^2 X^2 U_{XX}^2 \tag{2.3}$$

involves the nonlinearity of the problem.

Using centered finite differences for the second order spatial partial derivative U_{XX} and forward finite difference for U_τ one gets, for $0 \leq n \leq \ell$,

$$U_{XX}(X_j, \tau^n) = \Delta_j^n + O(h^2), \qquad \Delta_j^n = \Delta_j^n(u) = \frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{h^2}, \quad 1 \leq j \leq N-1, \tag{2.4}$$

$$U_\tau(X_j, \tau^n) = \frac{u_j^{n+1} - u_j^n}{k} + O(k), \quad 0 \leq j \leq N, \tag{2.5}$$

where $X_j = jh, \tau^n = nk, h = \Delta X, k = \Delta\tau, Nh = b, k\ell = \tau$ and b, which defines the right spatial boundary of the numerical domain $\Omega = [0, b] \times [0, \tau], 0 < \tau \leq \frac{\sigma^2 T}{2}$, is chosen like in [3]. Hence the numerical scheme for the approximation $u_j^n \approx U(X_j, \tau^n)$ takes the form

$$u_j^{n+1} = \left(1 - \frac{2k}{h^2}\beta_j^n\right)u_j^n + \frac{k}{h^2}\beta_j^n\left(u_{j-1}^n + u_{j+1}^n\right); \quad 1 \leq j \leq N-1, 0 \leq n \leq \ell-1, \tag{2.6}$$

$$u_j^0 = f(X_j), \quad 0 \leq j \leq N, \tag{2.7}$$

where

$$\beta_j^n = X_j^2 \Psi_j^n, \qquad \Psi_j^n = 1 + 2\lambda X_j \Delta_j^n + \lambda^2 \mu^2 X_j^2 (\Delta_j^n)^2. \tag{2.8}$$

Since the value u_j^{n+1} is expressed in terms of u_{j-1}^n, u_j^n and u_{j+1}^n, we need to know the boundary values u_0^n and u_N^n. These values are obtained by imposing (2.6) at $j = 0, j = N$ and using linear extrapolation by assigning to the external artificial values u_{-1}^n and u_{N+1}^n as follows:

$$u_{-1}^n = 2u_0^n - u_1^n, \qquad u_{N+1}^n = 2u_N^n - u_{N-1}^n. \tag{2.9}$$

Thus the numerical values at the numerical boundaries of the domain turn out

$$u_0^{n+1} = u_0^n = f(0); \qquad u_N^{n+1} = u_N^n = \cdots = u_N^0 = f(b). \tag{2.10}$$

For the sake of convenience to show the positiveness of coefficients of scheme (2.6) it is convenient to study the evolution of the numerical transformed *gamma* $\{\Delta_j^n\}$.

Lemma 1. *With the previous notation, the numerical transformed gamma* Δ_j^n *satisfies the scheme*

$$\left.\begin{array}{l} \Delta_j^{n+1} = \left(1 - \frac{2k}{h^2}\beta_j^n\right)\Delta_j^n + \frac{k}{h^2}\beta_{j-1}^n\Delta_{j-1}^n + \frac{k}{h^2}\beta_{j+1}^n\Delta_{j+1}^n, \quad 1 \leq j \leq N-1 \\ \Delta_0^n = \Delta_N^0 = 0; \quad 0 \leq n \leq \ell. \end{array}\right\} \tag{2.11}$$

Proof. Let u^n and Δ^n be the vectors in $\mathbb{R}^{N+1}$ defined by

$$u^n = [u_0^n\ u_1^n\ \ldots\ u_N^n]^\top, \qquad \Delta^n = [\Delta_0^n\ \Delta_1^n\ \ldots\ \Delta_N^n]^\top. \tag{2.12}$$

From (2.4) and (2.9) one gets for $j = 0$ and $j = N$, that

$$\Delta_0^n = \Delta_N^0 = 0; \quad 0 \leq n \leq \ell. \tag{2.13}$$

From (2.4), (2.12) and (2.13) it follows that

$$\Delta^n = \frac{1}{h^2}Au^n, \tag{2.14}$$

where

$$
A = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \cdots & 1 & -2 & 1 \\ 0 & \cdots & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{(N+1)\times(N+1)}. \tag{2.15}
$$

This scheme (2.6), (2.10) can be written in matrix form

$$
u^{n+1} = \left(I + \frac{k}{h^2} B(n) A \right) u^n, \qquad u^0 = [f(0), \qquad f(X_1), \ldots, f(b)]^\top \tag{2.16}
$$

where $0 \le n \le \ell - 1$ and

$$
B(n) = \operatorname{diag}\left(\beta_0^n, \beta_1^n, \ldots, \beta_N^n \right). \tag{2.17}
$$

Since $\Delta^{n+1} = \frac{1}{h^2} A u^{n+1}$, from (2.14) and (2.16) one gets

$$
\Delta^{n+1} = \left(I + \frac{2k}{h^2} A B(n) \right) \Delta^n. \tag{2.18}
$$

Writing Δ^{n+1} in a componentwise form one gets (2.11).   □

3. Properties of the numerical solution

We begin this section by showing that coefficients of scheme (2.6) for a vanilla call option problem are positive under appropriate relationship between stepsize discretization h and k.

Lemma 2. *Let Δ_j^n be the numerical transformed gamma and $S^n = \sum_{j=0}^{N} \Delta_j^n$, $0 \le n \le \ell$, $Nh = b$. Let λ be the market depth parameter, μ the slippage parameter and b the numerical domain boundary parameter. Assuming that stepsizes h and k satisfy the condition*

$$
k \le \frac{h^4}{2b^2(\lambda^2 \mu^2 b^2 + 2\lambda\mu h + h^2)}, \tag{3.1}
$$

then

(i) *Sequence solution $\{\Delta_j^n\}$ of (2.11) is nonnegative and $\{S^n\}$ is non-increasing.*
(ii) *Coefficients of (2.6) and (2.11) satisfy*

$$
\beta_j^n \ge 0; \qquad 1 - \frac{2k}{h^2} \beta_j^n \ge 0, \quad 0 \le j \le N, 0 \le n \le \ell - 1.
$$

Proof. Part (i) is proved using the induction principle over index n and (ii) will be a direct consequence of part (i). First of all, note that from (2.11) one gets

$$
\sum_{j=0}^{N} \Delta_j^{n+1} = \sum_{j=1}^{N-1} \Delta_j^{n+1} = \sum_{j=1}^{N-1} \Delta_j^n - \frac{2k}{h^2} \sum_{j=1}^{N-1} \beta_j^n \Delta_j^n + \frac{k}{h^2} \sum_{j=0}^{N-2} \beta_j^n \Delta_j^n + \frac{k}{h^2} \sum_{j=2}^{N} \beta_j^n \Delta_j^n
$$

$$
= \sum_{j=1}^{N-1} \Delta_j^n - \frac{k}{h^2} \left(\beta_1^n \Delta_1^n + \beta_{N-1}^n \Delta_{N-1}^n \right), \quad 0 \le n \le \ell - 1. \tag{3.2}
$$

If $n = 0$, from the transformed payoff function $f(x) = \max\{X - K, 0\}$ and (2.4) it follows that

$$
\Delta_{j_0-1}^0 = \frac{\theta}{h}, \qquad \Delta_{j_0}^0 = \frac{1-\theta}{h} \quad \text{and} \quad \Delta_j^0 = 0 \quad \text{otherwise}, \tag{3.3}
$$

where j_0 is chosen so that

$$
h(j_0 - 1) < K \le h j_0; \qquad K = h(j_0 - \theta), \quad 0 \le \theta < 1. \tag{3.4}
$$

From (3.3), (3.4) and (2.8), (3.1) one gets $\Delta_j^0 \ge 0$, $\beta_j^0 \ge 0$, and $1 - \frac{2k}{h^2} \beta_j^0 \ge 0$, for $0 \le j \le N$. Taking into account (3.2) for $n = 0$ and (2.13), (3.3) it follows that $S^1 \le S^0 = \frac{1}{h}$. Furthermore from (2.11), $\Delta_j^1 \ge 0$, $0 \le j \le N$.

Assume the induction hypothesis

$$\Delta_j^n \geq 0 \quad \text{for } 0 \leq j \leq N \quad \text{and} \quad S^n \leq S^{n-1} \leq \cdots \leq S^0 = \frac{1}{h}. \tag{3.5}$$

From (2.8) one gets $\beta_j^n \geq 0$ and from (3.1), (3.5) it follows that

$$
\begin{aligned}
1 - \frac{2k}{h^2}\beta_j^n &= 1 - \frac{2k}{h^2}X_j^2\left(1 + 2\lambda X_j\Delta_j^n + \lambda^2\mu^2 X_j^2(\Delta_j^n)^2\right) \\
&\geq 1 - \frac{2k}{h^2}b^2\left(1 + 2\frac{\lambda b}{h} + \frac{\lambda^2\mu^2 b^2}{h^2}\right) \geq 0.
\end{aligned}
\tag{3.6}
$$

From (3.2) and (3.5) one gets $S^{n+1} \leq S^n$ and from (2.11) and (3.5), (3.6) it follows that $\Delta_j^{n+1} \geq 0, 0 \leq j \leq N, 0 \leq n \leq \ell - 1$. Thus the result is established. $\square$

The next result shows the nice properties of positiveness and monotonicity of the numerical solution.

Theorem 1. *Let $\{u_j^n\}$ be the solution of scheme (2.6)–(2.8) for a vanilla call option problem (2.2) with $f(x) = \max\{X - K, 0\}$. Then under hypothesis (3.1) $u_j^n \geq 0$ and $\{u_j^n\}$ is nondecreasing for j, for each fixed n.*

Proof. The positivity of u_j^n is a direct consequence of the nonnegative payoff function (2.7) and part (ii) of Lemma 2. The proof of monotonicity is done using the induction principle. Note that for $n = 0$, the monotonicity comes out from the nondecreasing property of the payoff function. Assume that for a fixed n,

$$u_{j+1}^n - u_j^n \geq 0, \quad 0 \leq j \leq N - 1. \tag{3.7}$$

From (2.6), (3.7) and using that $\beta_j^n \geq 0$ under hypothesis (3.1) one gets

$$\frac{-k}{h^2}\beta_j^n\left(u_j^n - u_{j-1}^n\right) \leq u_j^{n+1} - u_j^n \leq \frac{k}{h^2}\beta_j^n\left(u_{j+1}^n - u_j^n\right), \quad 1 \leq j \leq N - 1. \tag{3.8}$$

Taking into account (2.10), in the boundary of the domain we have

$$u_0^{n+1} = u_0^n \quad \text{and} \quad u_N^{n+1} = u_N^n. \tag{3.9}$$

From (3.1), (3.6) and (3.8) and part (ii) of Lemma 2 for $0 \leq j \leq N - 1$, one gets

$$
\begin{aligned}
u_{j+1}^{n+1} - u_j^{n+1} &= \left(u_{j+1}^{n+1} - u_{j+1}^n\right) + \left(u_{j+1}^n - u_j^n\right) - \left(u_j^{n+1} - u_j^n\right) \\
&\geq \left(u_{j+1}^n - u_j^n\right)\left(1 - \frac{2k}{h^2}\max\{\beta_{j+1}^n, \beta_j^n\}\right) \geq 0.
\end{aligned}
\tag{3.10}
$$

Thus the result has been established. $\square$

4. Stability and consistency

For the sake of clarity in the presentation we introduce the concept of stability used here.

Definition 1. The numerical scheme (2.16) for the initial value problem (1.1) is said to be $\| \ \|_\infty$-*stable* in the fixed station sense in the domain $[0, b] \times [0, \frac{\sigma^2 T}{2}]$ if, given τ with $0 < \tau \leq \frac{\sigma^2 T}{2}$, for every partition with $k = \Delta\tau, h = \Delta X, \tau = \ell k$, and every N with $Nh = b$, one gets $\|u^n\|_\infty \leq C, 0 \leq n \leq \ell$, where $C > 0$ is independent of h, k and N.

Note that for a vanilla call option, from (2.10) and Theorem 1 if follows that

$$\|u^n\|_\infty = \max\{u_j^n; 0 \leq j \leq N\} = \max\{b - K, 0\}.$$

Hence we have established the following conditional stability result:

Theorem 2. *Under condition (3.1) the numerical scheme (2.16) for solving the vanilla call option transformed problem (1.1) is stable.*

The following example shows that if condition (3.1) is not satisfied, then the monotonicity and stability are not granted.

Example 1. Consider the vanilla call option problem (1.1) with $K = 50, T = 0.25$ years, $r = 6\%, \sigma = 40\%, b = 200$ for an illiquid market with parameters $\lambda = 10^{-6}$ and $\mu = 0.15$. Taking $h = 2, k = 5.9701 \cdot 10^{-5}$ the stability condition is broken and Fig. 1 shows the oscillations of the numerical solution
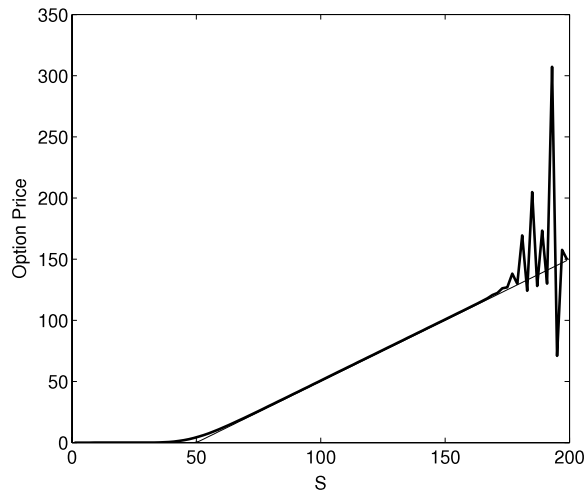
Fig. 1. Oscillations when the stability condition is broken.
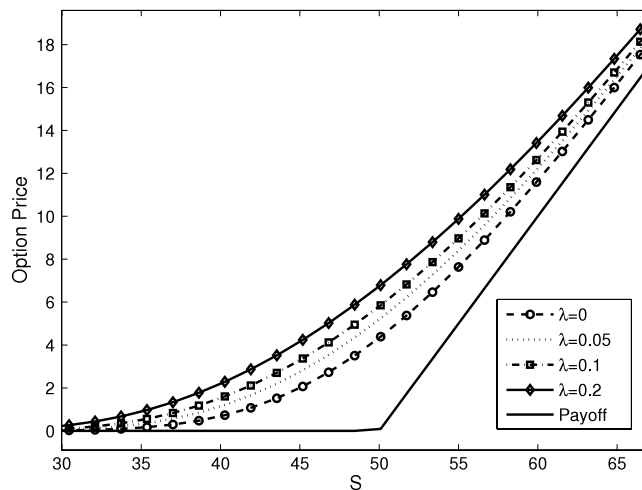

Fig. 2. Influence of the market depth parameter.

Consistency of a numerical scheme with respect to a partial differential equation means that the exact solution of the finite difference scheme approximates an exact solution of the PDE (see [7, p.100]). In order to prove the consistency of scheme (2.6) with (1.1), we need to show that the truncation error

$$T_j^n(U) = F(U_j^n) - L(U_j^n) \tag{4.1}$$

satisfies (see [7, p.100])

$$T_j^n(U) \to 0, \quad \text{as } h = \Delta X \to 0, k = \Delta \tau \to 0,$$

where U_j^n denotes the theoretical value of the solution of (1.1) at (X_j, τ^n), and

$$F(U_j^n) = \frac{U_j^{n+1} - U_j^n}{k} - \beta_j^n(U)\Delta_j^n(U). \tag{4.2}$$

Using Taylor's expansion about (X_j, τ^n) of the theoretical solution of the (1.1) and assuming the existence of up to order four continuous partial derivatives with respect to X and continuous second order partial derivatives with respect to τ, one gets

$$\Delta_j^n(U) = \frac{\partial^2 U}{\partial X^2}(X_j, \tau^n) + h^2 E_j^n(1); \qquad \frac{U_j^{n+1} - U_j^n}{k} = \frac{\partial U}{\partial \tau}(X_j, \tau^n) + k E_j^n(2), \tag{4.3}$$

$$E_j^n(1) = \frac{1}{12}\frac{\partial^4 U}{\partial X^4}(\eta, \tau^n), \qquad X_j - h < \eta < X_j + h, \tag{4.4}$$

$$E_j^n(2) = \frac{1}{2}\frac{\partial^2 U}{\partial \tau^2}(X_j, \tau), \qquad \tau^n < \tau < \tau^{n+1}. \tag{4.5}$$

Taking into account (2.8), (4.1)–(4.3) and (4.5), after some mathematical analysis one gets $T_j^n(U) = O(h^2) + O(k)$. Thus the scheme (2.6) is consistent of order 2 in h and order 1 in k with (1.1).

The next example shows the smooth variation of the numerical solution with the illiquidity market parameter λ.

Example 2. Consider the problem of Example 1 under the stability condition (3.1) with a fixed $\mu = 0.1501$ and several different values of the market depth parameter λ (Fig. 2).

Acknowledgements

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