On the structure of positive solutions to an elliptic problem with a singular nonlinearity

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We consider the following boundary value problem

\[ \frac{\partial u}{\partial t} = \lambda \left[ u^{p} - u^{q} \right] \quad \text{in} \quad \Omega, \quad u = \kappa \quad \text{in} \quad \partial \Omega, \]

where \( p > q > 0 \) and \( \Omega \) is a bounded smooth domain in \( \mathbb{R}^{N} \) (\( N \geq 2 \)). It is shown that, for \( \kappa > 1 \), there is at least one solution \( u_{\lambda} \) of this problem satisfying \( u_{\lambda} > \kappa \) in \( \Omega \) for any \( \lambda > 0 \).

The profiles of \( u_{\lambda} \) for \( \lambda \to 0 \) and \( \lambda \to \infty \) are analyzed. For \( \kappa = 1 \), it is clear that \( u_{\lambda} \equiv 1 \) is a constant solution to the problem. We also present the branch of solutions satisfying \( 0 < u_{\lambda} < 1 \) in \( \Omega \) for the equation with \( \kappa = 1 \). We obtain more information on the structure of positive radial solutions of the problem when \( \Omega \) is an annulus.

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1. Introduction

We consider the structure of positive solutions to the following problem

\[ \frac{\partial u}{\partial t} = \lambda \left[ u^{p} - u^{q} \right] \quad \text{in} \quad \Omega, \quad u > \kappa \quad \text{in} \quad \Omega, \quad u = \kappa \quad \text{on} \quad \partial \Omega, \]

where \( \lambda > 0 \), \( \kappa \in [1, \infty) \), \( 0 < q < p < \infty \), \( \Omega \subset \mathbb{R}^{N} \) (\( N \geq 2 \)) is a bounded smooth domain.

By a solution \( u \) of (\( T^{\kappa}_{\lambda} \)) we mean that \( u \in C^{2}(\Omega) \cap C^{1}(\Omega) \) satisfies (\( T^{\kappa}_{\lambda} \)). The equation in (\( T^{\kappa}_{\lambda} \)) arises in the study of steady states of thin films. Equations of the type

\[ u_{t} = -\nabla \cdot \left( f(u) \nabla u \right) - \nabla \cdot \left( g(u) \nabla u \right) \]

have been used to model the dynamics of thin films of viscous liquids, where \( z = u(x, t) \) is the height of the air/liquid interface. The zero set \( \Sigma = \{ x \in \Omega : u(x, t) = 0 \} \) is the liquid/solid interface and is sometimes called set of ruptures. Ruptures play a very important role in the study of thin films. The coefficient \( f(u) \) reflects surface tension effects—a typical choice is \( f(u) = u^{3} \). The coefficient of the second-order term can reflect additional forces such as gravity \( g(u) = u^{2} \), van der Waals interactions \( g(u) = u^{m} - \gamma u^{l} \) with \( \gamma > 0 \), \( m < 0 \), \( l < 0 \) and \( |l| < |m| \). For background on (1.1), we refer to [1–3,25–28] and the references therein.

In general, let us assume \( f(u) = u^{3} \), \( g(u) = u^{m} - \kappa u^{l} \), where \( m, l \in \mathbb{R} \). Then if we consider the steady-state of (1.1), we see that \( u \) satisfying
is a steady state of (1.1), where $C = (C_1, C_2, \ldots, C_n)$ is a constant vector. Assuming $C = 0$, we see that
\[\Delta u + \frac{1}{m-2} u^{m-2} - \frac{g(x)}{m-2} u^{m-2} = 0 \quad \text{in } \Omega,\]
where $C$ is a constant. If we assume $C = 0$ and $v = (|x| + 2)^{1/(3-m)} u$, we see that $v$ satisfies
\[\Delta v = v^{m-2} - \frac{g(x)}{|x|+2} v^{m-2} \quad \text{in } \Omega,\]
which is the required form of the equation
\[\Delta v = v^{-p} - \tau v^{-q}, \quad 0 < q < p. \tag{1.2}\]
By a simple change: $u = \tau^{1/(p-q)} v$, we see that $u$ satisfies the equation:
\[\Delta u = \tau^{1/(p-q)} \left[ u^{-p} - u^{-q} \right] \tag{1.3}\]
which is the required form of $(T_2^\kappa)$.

The problem
\[-\Delta v = \frac{\lambda}{(1-v)^2} \quad \text{in } \Omega, \quad 0 < v < 1 \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial \Omega, \tag{1.4}\]
models a simple electrostatic Micro-Electromechanical System (MEMS) device consisting of a thin dielectric elastic membrane with boundary supported at 0 below a rigid plate located at $+1$. When a voltage—represented here by $\lambda$—is applied, the membrane deflects towards the ceiling plate and a snap-through may occur when it exceeds a certain critical value $\lambda^*$ (pull-in voltage). This creates a so-called “pull-in instability” which greatly affects the design of many devices (see [11] and [29,30] for a detailed discussion on MEMS devices). Note that two-dimensional domains are of real physical relevance.

In recent papers [8,11–14] and [24], the authors studied the problem
\[
\begin{aligned}
-\Delta u &= \lambda g(x) (1-v)^2 \quad \text{in } \Omega, \\
0 &< v < 1 \quad \text{in } \Omega, \\
v &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\tag{1.4}\]
where $g \in C(\Omega)$ is a nonnegative function. They gave a detailed study on the minimal solutions of the problem (1.4) with different forms of $g(x)$. Similar problems with singular nonlinearities to (1.4) have also been studied by the authors in [15,19,21–23] and the references therein.

In a recent paper [18], the authors studied the problem
\[
\begin{aligned}
-\Delta u &= \lambda \left[ u^{-p} - u^{-q} \right] \quad \text{in } \Omega, \\
0 &< u < \kappa \quad \text{in } \Omega, \\
u &= \kappa \quad \text{on } \partial \Omega,
\end{aligned}
\tag{T_{2}^{\kappa,1}}
\]
where $\lambda > 0$, $\kappa \in (0, 1)$, $0 < q < p < \infty$. They obtained that there exists $\lambda_* := \lambda_*(p,q,\kappa) > 0$ such that for $\lambda \in (0, \lambda_*)$, $(T_{2}^{\kappa,1})$ has a branch of maximal positive solutions $(\lambda, u_\lambda(x)) \in (0, \lambda_*) \times C^2(\Omega) \cap C^1(\Omega)$, which connects $(0, \kappa)$ and $(\lambda_*, u_\lambda(x))$. Moreover, for any $x \in \Omega$, $\lambda \mapsto u_\lambda(x)$ is decreasing. Meanwhile, in another recent paper [17], the authors showed that for some lower dimensional balls $\Omega$ and $0 < \kappa < 1$ the branch of positive radial solutions of $(T_{2}^{\kappa,1})$ has infinitely many turning points and the minima of the solutions on the branch go to 0 eventually.

In the present paper, we first study the problem $(T_{2}^{\kappa})$. Our main results of this paper are the following theorems.

**Theorem 1.1.** For any fixed $\kappa > 1$, there is a branch of solutions $(\lambda, u_{\lambda}^\kappa)$ in $(0, \infty) \times C^2(\Omega) \cap C^1(\Omega)$ of $(T_{2}^{\kappa,1})$, which connects $(0, \kappa)$ and $(\infty, \infty)$. For any $x \in \Omega$, the mapping $\lambda \mapsto u_{\lambda}^\kappa(x)$ is increasing. Moreover, the function: $\lambda \mapsto u_{\lambda}^\kappa$ from $(0, \infty)$ to $C^2(\Omega) \cap C^1(\Omega)$ is continuous.

For $\kappa = 1$, it is clear that $u_0 = 1$ is a trivial solution to the problem
\[
\Delta u = \lambda \left[ u^{-p} - u^{-q} \right] \quad \text{in } \Omega, \quad u|_{\partial \Omega} = 1 \tag{1.5}\]
for all $\lambda > 0$. Except this solution, we can obtain another solution to (1.5). We have the following theorem.

**Theorem 1.2.** Let $p > 1$, $0 < q < p$ and $\lambda^* = \sigma_1/(p-q)$, where $\sigma_1$ is the first eigenvalue of $-\Delta$ in $\Omega$ with 0 boundary value. Then there is an unlimited branch $\Gamma := \{(\lambda, u_{\lambda}): 0 < u_{\lambda} < 1 \text{ satisfies (1.5)} \}$ starting from $(\lambda^*, 1)$. 
By an unlimited branch, we mean a solution branch along which the solutions approach a singular state, i.e., there exists a sequence \( \{(\lambda_n, u_{\lambda_n})\} \) such that \( \lambda_n \to \lambda^* > 0 \) and \( \min_{\Omega} u_{\lambda_n} \to 0 \) as \( n \to \infty \).

We can easily know that for \( \lambda > \lambda^* \) there is no solution of
\[
\begin{align*}
\Delta u &= \lambda [u^{-p} - u^{-q}] \quad \text{in} \quad \Omega, \\
0 < u < 1 &\quad \text{in} \quad \Omega, \\
u &= 1 &\quad \text{on} \quad \partial \Omega.
\end{align*}
\]

Note that there is a difference between the cases of \( \kappa \in (0, 1) \) and \( \kappa = 1 \). For the case \( \kappa \in (0, 1) \), we know from [18] that the maximal solutions \( \overline{u}_\kappa \) of \((T^{1,1}_\kappa)\) exist for \( \lambda \in (0, \lambda^*) \). But we can only show that \( \min_{\Omega} \overline{u}_\kappa > 0 \) for some lower dimensional domains \( \Omega \). For \( \kappa = 1 \), the maximal solution of \((T^{1,1}_1)\) is \( \overline{u}_1 \equiv 1 \) for \( \lambda \in (0, \lambda^*) \) (where \( \lambda^* = \sigma_1/(p-q) \)), we have that \( \min_{\Omega} \overline{u}_1 > 0 \) for any dimensional domain \( \Omega \).

By a simple change: \( v = \kappa - u \), we obtain the following problem from \((T^{1,1}_\kappa)\):
\[
\begin{align*}
-\Delta v &= \lambda [\kappa - v]^{-p} - [\kappa - v]^{-q} \quad \text{in} \quad \Omega, \\
v &< 0 \quad \text{in} \quad \Omega, \\
v &= 0 \quad \text{on} \quad \partial \Omega.
\end{align*}
\]

We show that the problem \((S^{\kappa}_\lambda)\) has a branch \((\lambda, v_\lambda)\) connecting \((0, 0)\) and \((\infty, -\infty)\) and for any \( x \in \Omega \), the mapping \( \lambda \to v_\lambda(x) \) is decreasing.

Note that we can also consider \((T^{1,1}_\kappa)\) for \( \kappa > 1 \). But for such cases, if we make the change \( v = \kappa - u \), we easily see that \( v \equiv 0 \) is not a subsolution to the problem
\[
\begin{align*}
-\Delta v &= \lambda [(\kappa - v)^{-p} - (\kappa - v)^{-q}] \quad \text{in} \quad \Omega, \\
0 < v &< \kappa \quad \text{in} \quad \Omega, \\
v &= 0 \quad \text{on} \quad \partial \Omega.
\end{align*}
\]

The existence of solutions of this problem is still unclear.

The organization of the paper is as follows: In Section 2, we establish the existence and uniqueness of maximal solution \( \overline{v}_\kappa \) of \((S^{\kappa}_\lambda)\) for \( \kappa > 1 \). In Section 3, we study the profiles of the maximal solution \( \overline{v}_\kappa \) as \( \lambda \to 0 \) and \( \lambda \to \infty \). In Section 4, we study the structure of positive solutions of \((S^{1,1}_\lambda)\) and finally, we present the structure of positive radial solutions of \((S^{\kappa,1}_\lambda)\) for \( 0 < \kappa < 1 \) and \( \Omega \) being an annulus.

2. Existence and uniqueness of maximal solution to \((S^{\kappa}_\lambda)\)

For any \( \kappa > 1 \), in this section we study the existence and uniqueness of maximal solutions to \((S^{\kappa}_\lambda)\).

A solution \( \overline{v}_\kappa \) is said to be a maximal solution of \((S^{\kappa}_\lambda)\), if for any solution \( v \) of \((S^{\kappa}_\lambda)\) we have \( \overline{v}_\kappa \geq v \) in \( \Omega \). Note that a maximal solution to \((S^{\kappa}_\lambda)\) corresponds to a minimal positive solution of \((T^{1,1}_\lambda)\).

We first obtain the following theorem.

**Theorem 2.1.** For any \( \kappa > 1 \) and any \( \lambda > 0 \), there exists at least one solution to the problem \((S^{\kappa}_\lambda)\).

**Proof.** Define \( e(s) = (\kappa - s)^{-p} - (\kappa - s)^{-q} \). Note that the function \( e(s) \) depends on \( \kappa \). Then \( e(\kappa - 1) = 0 \) and \( \kappa - 1 \geq 0 \), Moreover, \( e(s) < 0 \) for \( s \in (-\infty, \kappa - 1) \) and \( e(s) > 0 \) for \( s \in (\kappa - 1, \kappa) \), \( \lim_{s \to -\infty} e(s) = 0 \), \( \lim_{s \to \kappa -} e(s) = \infty \) and \( e(s) \) has a unique minimum point \( s_0 = \kappa - (q/p)^{-1/(p-q)} \), \( e(s_0) = (q/p)^{1/(p-q)} - (q/p)^{1/(p-q)} < 0 \).

Now we construct a function \( g \in C^1((\infty, \infty)) \) with \( g(s) \leq 0 \) for \( s \in (-\infty, \infty) \) and \( g(s) < e(s) \) such that \( |g(s)| \) is bounded, \( g(s) \equiv 0 \) for \( s \in (\alpha, \infty) \), where \( \alpha \in (\kappa - 1, \kappa) \). Since \( |g| \) is bounded, if we define \( J(w) : H^1_0(\Omega) \to \mathbb{R} \) by
\[
J(w) = \frac{1}{2} \int_{\Omega} |\nabla w|^2 \, dx - \int_{\Omega} G(w) \, dx,
\]
where \( G(s) = \int_{s_0}^s g(\xi) \, d\xi \), \( J \) is sequentially weakly lower semi-continuous and coercive on \( H^1_0(\Omega) \) and so \( J \) possesses a global minimizer, which we denote by \( w_\lambda \). From the regularity of the Laplacian, \( w_\lambda \in C^2(\Omega) \), and \( w_\lambda \) satisfies the problem
\[
-\Delta w_\lambda = \lambda g(w_\lambda) \quad \text{in} \quad \Omega, \quad w_\lambda = 0 \quad \text{on} \quad \partial \Omega.
\]
(2.1)

Since \( g(0) < 0 \) and \( g(s) \leq 0 \) for \( s \in (-\infty, \infty) \), the maximum principle implies that \( w_\lambda < 0 \) in \( \Omega \). The fact that \( g(s) \leq e(s) \) for \( s \in (-\infty, \infty) \) implies that \( w_\lambda \) is a subsolution to the problem
\[
-\Delta v = \lambda e(v) \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial \Omega.
\]
(2.2)

It is clear that \( v \equiv \kappa - 1 \) is a supersolution to (2.2). Moreover, there exists \( M > 0 \) such that \( |e'(s)| \leq M \) for \( s \in (-\infty, \kappa - 1) \). Thus, \( h(s) := e(s) + Ms \) is a nondecreasing function for \( s \in (-\infty, \kappa - 1) \). □

The proof of Theorem 2.1 can be obtained from the following results.
**Lemma 2.2.** For any $\kappa > 1$ and any $\lambda > 0$, (2.2) has unique maximal negative solution $\overline{v}_{\lambda}^k < 0$ in $\Omega$.

**Proof.** For each fixed $\kappa > 1$, we define $u^0 = \kappa - 1$. For each $\lambda > 0$, we obtain a sequence $\{u^k\}_{k=0}^\infty \subset C^{2,\alpha}(\Omega)$ for some $0 < \alpha < 1$ by solving the problems

$$-\Delta u^k + \lambda Mu^k = \lambda h(u^{k-1}) \text{ in } \Omega, \quad u^k = 0 \text{ on } \partial \Omega.$$  

We see that

$$-\Delta u^1 + \lambda Mu^1 = \lambda h(u^0) \leq -\Delta u^0 + \lambda Mu^0 \text{ in } \Omega, \quad u^1 = 0 \text{ on } \partial \Omega.$$  

The maximum principle implies that

$$u^1 < \kappa - 1 \text{ in } \Omega. \quad (2.3)$$

Moreover, (2.3) and the monotonicity of $h(s)$ imply that

$$-\Delta u^2 + \lambda Mu^2 = \lambda h(u^1) < \lambda h(u^0) = -\Delta u^1 + \lambda Mu^1, \quad (u^2 - u^1)|_{\partial \Omega} = 0.$$  

The maximum principle again implies that

$$u^2 < u^1 \text{ in } \Omega. \quad (2.4)$$

Similar argument implies that $u^k < u^{k-1}$ for $k = 3, 4, \ldots$. On the other hand, since $w_\lambda < \kappa - 1$ in $\Omega$, the monotonicity of $h(s)$ implies that

$$-\Delta w_\lambda + \lambda Mw_\lambda < \lambda h(w_\lambda) \leq \lambda h(u^0) = -\Delta u^1 + \lambda Mu^1 \text{ in } \Omega.$$  

The maximum principle implies that

$$w_\lambda < u^1 \text{ in } \Omega. \quad (2.5)$$

The similar argument implies that

$$u^k \geq w_\lambda \text{ in } \Omega \text{ for all } k = 2, 3, \ldots. \quad (2.6)$$

Therefore, $\overline{v}_{\lambda}^k := \lim_{k \to \infty} u^k$ is a solution of (2.2). We easily see that $\overline{v}_{\lambda}^k \in C^2(\Omega) \cap C^1(\overline{\Omega})$. Moreover, $w_\lambda \leq \overline{v}_{\lambda}^k \leq \kappa - 1$ in $\Omega$. Since $e(\overline{v}_{\lambda}^k) \leq 0$, we see from the maximum principle that $\overline{v}_{\lambda}^k < 0$ in $\Omega$.

To prove the maximality of $\overline{v}_{\lambda}^k$, we assume that $v_{\lambda}^k$ is a solution of $(S_\lambda^k)$. We can easily see that $v_{\lambda}^k \leq u^k$ for all $k = 0, 1, 2, \ldots$. Thus, $v_{\lambda}^k \leq \overline{v}_{\lambda}^k$ in $\Omega$. \qed

**Theorem 2.3.** If $\kappa \geq (q/p)^{-1/(p-q)}$, $(S_\lambda^k)$ has a unique solution for any $\lambda > 0$, i.e., the maximal solution $\overline{v}_{\lambda}^k$.

**Proof.** For any fixed $\kappa \geq (q/p)^{-1/(p-q)}$ and any fixed $\lambda > 0$, suppose $(S_\lambda^k)$ has another solution $v_{\lambda}^k$, which is different from $\overline{v}_{\lambda}^k$. Then, setting $w_{\lambda}^k = \overline{v}_{\lambda}^k - v_{\lambda}^k$, we see that $w_{\lambda}^k \geq 0$ in $\Omega$. Moreover, $w_{\lambda}^k$ satisfies the problem

$$-\Delta w_{\lambda}^k = \lambda e'(\xi_{\lambda}^k)w_{\lambda}^k \text{ in } \Omega, \quad w_{\lambda}^k = 0 \text{ on } \partial \Omega,$$

where $\xi_{\lambda}^k \in (v_{\lambda}^k, \overline{v}_{\lambda}^k)$. Noticing $e'(s) \leq 0$ for $s \leq 0$ and $\kappa \geq (q/p)^{-1/(p-q)}$, we see that $e'(\xi_{\lambda}^k) \leq 0$. Therefore, $\Delta w_{\lambda}^k \geq 0$. The maximum principle implies that $w_{\lambda}^k < 0$ in $\Omega$. This contradicts our assumption. \qed

**Theorem 2.4.** For any fixed $\kappa > 1$ and each $x \in \Omega$, the function $\lambda \mapsto \overline{v}_{\lambda}^k(x)$ is non-increasing on $(0, \infty)$. Moreover, the map $\lambda \mapsto \overline{v}_{\lambda}^k$ from $(0, \infty)$ to $C^2(\Omega) \cap C^1(\overline{\Omega})$ is continuous.

**Proof.** For $0 < \lambda_1 < \lambda_2$, since $\overline{v}_{\lambda_2}^k \leq \kappa - 1$, $e(\overline{v}_{\lambda_2}^k) \leq 0$ and $\overline{v}_{\lambda_2}^k$ satisfies

$$-\Delta \overline{v}_{\lambda_2}^k = \lambda_2 e(\overline{v}_{\lambda_2}^k) \leq \lambda_1 e(\overline{v}_{\lambda_2}^k) \text{ in } \Omega,$$

we see that $\overline{v}_{\lambda_2}^k$ is a subsolution to the problem

$$-\Delta v = \lambda_1 e(v) \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega. \quad (2.7)$$

Noticing that $w_{\lambda_2} \leq \overline{v}_{\lambda_2}^k < \kappa - 1$ in $\Omega$, there exists a solution $v_{\lambda_1}^k$ of (2.7) between $\overline{v}_{\lambda_2}^k$ and $\kappa - 1$. The maximality of $v_{\lambda_1}^k$ implies that $v_{\lambda_1}^k \leq \overline{v}_{\lambda_2}^k$, and hence $\overline{v}_{\lambda_2}^k \leq \overline{v}_{\lambda_1}^k$ in $\Omega$. We can argue as in Section 2.1 of Buffoni, Dancer and Toland [4,5] and [6] in the space $R \times C^2(\Omega) \cap C^1(\overline{\Omega})$ to obtain that the map $\lambda \mapsto \overline{v}_{\lambda}^k$ from $(0, \infty)$ to $C^2(\Omega) \cap C^1(\overline{\Omega})$ is continuous. \qed
3. Profiles of $v^k_{\lambda}$

In this section we will study the profiles of the solutions $v^k_{\lambda}$ as $\lambda \to 0$ and $\lambda \to \infty$.

**Theorem 3.1.** For any fixed $\kappa > 1$, $v^k_{\lambda} \to 0$ in $C^0(\Omega)$ as $\lambda \to 0^+$.

**Proof.** Since $|e(s)| \leq |e(s_0)| = (q/p)^{\theta/(p-q)} - (q/p)^{\theta/(p-q)}$ for $s \leq 0$, we see that
\[
\lambda |e(v^k_{\lambda})| \leq \lambda |e(s_0)| \to 0 \quad \text{as} \quad \lambda \to 0^+.
\]
The regularity of $-\Delta$ implies that $v^k_{\lambda} \to 0$ in $C^0(\Omega)$ as $\lambda \to 0^+$. \qed

Define
\[
\tau^k_{\lambda} = \min_{\Omega} v^k_{\lambda}.
\]
We see from Theorem 2.4 that $\tau^k_{\lambda} < 0$ is a decreasing function of $\lambda$.

**Lemma 3.2.** For any fixed $\kappa > 1$, $\tau^k_{\lambda} \to -\infty$ as $\lambda \to \infty$.

**Proof.** On the contrary, there exists $0 \geq \tau^k > -\infty$ such that $\tau^k \leq v^k_{\lambda} < 0$ in $\Omega$ and all $\lambda > 0$. Let $\phi$ with $\|\phi\|_{L^\infty(\Omega)} = 1$ be the first eigenfunction of the eigenvalue problem
\[
-\Delta \phi = \sigma \phi \quad \text{in} \quad \Omega, \quad \phi = 0 \quad \text{on} \quad \partial \Omega.
\]
We see that $\phi > 0$ in $\Omega$. Multiplying $\phi$ on both the sides of the equation of $v^k_{\lambda}$ and integrating it on $\Omega$, we see that
\[
\sigma \int_{\Omega} v^k_{\lambda}(x)\phi(x) \, dx = \lambda \int_{\Omega} e(v^k_{\lambda}(x))\phi(x) \, dx.
\]
We derive a contradiction from (3.1) as $\lambda \to \infty$ since $|e(v^k_{\lambda}(x))| \geq |e^0| > 0$, where $e^0 = \min_{e \in [\tau^k, 0]} |e(s)|$. \qed

**Theorem 3.3.** For any fixed $\kappa > 1$ and $x \in \Omega$, $v^k_{\lambda}(x) \to -\infty$ as $\lambda \to \infty$.

**Proof.** For any $x_0 \in \Omega$, we choose an $R > 0$ such that $B_R(x_0) \subseteq \Omega$, where $B_R(x_0)$ is the ball with center at $x_0$ and radius of $R$. We consider the problem
\[
-\Delta \nu = \lambda e(\nu) \quad \text{in} \quad B_R(x_0), \quad \nu = 0 \quad \text{on} \quad \partial B_R(x_0).
\]
Since $v^k_{\lambda}(B_R(x_0))$ is a subsolution to (3.2) and $\kappa - 1$ is a supersolution to (3.2), we see that (3.2) possesses a maximal solution $\nu_{\lambda}$ in the order interval $(v^k_{\lambda}, \kappa - 1)$ of $C^0(B_R(x_0))$. The maximum principle implies that $\nu_{\lambda} < 0$ in $B_R(x_0)$. By the result of [16], we see that $\nu^k_{\lambda}(x) = \nu_{\lambda}(r)$ with $r = |x - x_0|$ and the minimum of $\nu^k_{\lambda}$ attains at $x_0$. Arguments similar to those in the proof of Lemma 3.2 imply $\nu^k_{\lambda}(x_0) \to -\infty$ as $\lambda \to \infty$. Therefore, the fact that $\nu^k_{\lambda}(x_0) \leq v^k_{\lambda}(x_0)$ implies $v^k_{\lambda}(x_0) \to -\infty$ as $\lambda \to \infty$. This completes the proof. \qed

Let $x^k_{\lambda} \in \Omega$ such that $v^k_{\lambda}(x^k_{\lambda}) = \tau^k_{\lambda}$.

**Theorem 3.4.** For any fixed $\kappa > 1$,

\[
\lim_{\lambda \to \infty} \lambda^{1/2}d(x^k_{\lambda}, \partial \Omega) = \omega.
\]

**Proof.** In the following, we omit $\kappa$ of $x^k_{\lambda}$, $v^k_{\lambda}$ and $\tau^k_{\lambda}$. Setting $w_{\lambda}(x) : = v_{\lambda}/\tau_{\lambda}$, we see that $w_{\lambda} \geq 0$ in $\Omega$. Suppose that there is a subsequence $\{\lambda_n\}$ with $\lambda_n \to \infty$ as $n \to \infty$ and $\lambda_n^{1/2}d(x_{\lambda_n}, \partial \Omega) \leq \eta$, $0 < \eta < \infty$ for all $n$ sufficiently large. In the following, we denote $\{x_n\} = \{x_{\lambda_n}\}$, $\{\tau_n\} = \{\tau_{\lambda_n}\}$. Let $\tilde{x}_n$ be the point of $\partial \Omega$ closest to $x_n$. Suppose $\tilde{x}_n \to \tilde{x} \in \partial \Omega$. Choose coordinates such that $\tau_{\lambda}(\partial \Omega) = \{x \in \mathbb{R}^N : x_1 = 0\}$ and $\eta_1 = \epsilon_1 = (1, 0, 0, \ldots, 0)$. Making the transformations:
\[
y^n = \lambda_n^{1/2}(x - \tilde{x}_n), \quad \tilde{w}_n(y^n) = w_{\lambda_n}(x),
\]
$\tilde{w}_n$ satisfies the problem
\[
-\Delta \tilde{w}_n = \frac{e(\tilde{w}_n)}{\tau_n} \quad \text{in} \quad \Omega_{\lambda_n}, \quad \tilde{w}_n = 0 \quad \text{on} \quad \partial \Omega_{\lambda_n}.
\]
where

\[ \Omega_n = \{ y^n := \lambda_{n}^{1/2}(x - \bar{x}_n): x \in \Omega \}. \]

Note that, in the new coordinates, \( \tilde{\omega}_n(Z_n) = 1 \), where \( Z_n = \lambda_{n}^{1/2}(x_n - \bar{x}_n) \) is at distance at most \( Z \) from 0. By a boundary blow-up argument as in [7,20], we see that \( \tilde{\omega}_n \to \tilde{w} \) in \( C^1_{\text{loc}}(T_1) \) as \( n \to \infty \) (we can choose subsequences if necessary) and \( \tilde{w} \) satisfies the problem

\[
-\Delta w = 0 \quad \text{in } T_1, \quad w = 0 \quad \text{on } \partial T_1,
\]

\( \tilde{w} \geq 0 \) in \( T_1 \) and \( \tilde{w} \) is nontrivial because \( \tilde{\omega}_n(Z_n) = 1 \) and \( d(0, Z_n) \leq Z \). Moreover, \( 0 \leq \tilde{w} \leq 1 \) in \( T_1 \).

We show that \( \tilde{w} \) does not exist. The proof is divided into three steps.

**Step 1.** We first find a solution \( \hat{u}(x_1) := 1 + x_1 \) which satisfies \( \hat{u}''(x_1) = 0 \) and \( \hat{u}(x_1) > 0 \) for \( x_1 > 0 \) and \( \lim_{x_1 \to \infty} \hat{u}(x_1) = \infty \).

**Step 2.** If (3.3) has a nontrivial bounded nonnegative solution \( w \) and \( x_1 > 0 \), then \( w \) can be chosen so that \( T(x_1) := \sup_{y \in \mathbb{R}^{n-1}} w(x_1, y) \) is achieved.

Obviously, there exists \( y_n \in \mathbb{R}^{n-1} \) such that \( w(x_1, y_n) \to T(x_1) \) as \( n \to \infty \). Let \( \tilde{\omega}_n(x_1, y) = w(x_1, y_n - y) \). It is easy to see that \( \tilde{\omega}_n \) is a solution of (3.3) and that

\[ \tilde{\omega}_n(x_1, 0) \to T(x_1) = \sup_{y \in \mathbb{R}^{n-1}} \tilde{\omega}_n(x_1, y) \quad \text{as } n \to \infty. \]

We now use an argument similar to that in our blow-up constructions to choose a subsequence of \( \tilde{\omega}_n \) converging on compact subsets of \( T_1 \) to a nonnegative bounded solution \( \mathcal{W} \) of (3.3). Moreover, \( \mathcal{W}(x_1, 0) = T(x_1) \) by our choice of \( \tilde{\omega}_n \). Since it is easy to see that

\[ \sup_{y \in \mathbb{R}^{n-1}} \mathcal{W}(x_1, y) \leq \sup_{y \in \mathbb{R}^{n-1}} \tilde{\omega}_n(x_1, y) = T(x_1), \]

we see that \( \sup_{y \in \mathbb{R}^{n-1}} \mathcal{W}(x_1, y) = \mathcal{W}(x_1, 0) \). This proves Step 2. Note that our argument shows that \( \sup\{ \mathcal{W}(x_1, y): y \in \mathbb{R}^{n-1} \} \leq T(x_1) \) for all \( x_1 \geq 0 \). This will be useful later.

**Step 3.** We show that \( \tilde{w} \) does not exist. If \( \tilde{w} \) exists, using the notation of Step 2, we consider \( r(x) = \tilde{w}(x)/\hat{u}(x_1) \), where \( \hat{u}(x_1) \) is the function defined in Step 1. Applying standard elliptic estimates on balls of radius 1/2 and half balls with centers at points where \( x_1 = 0 \) and of radius 1, we see that \( \nabla \tilde{w} \) is bounded on \( T_1 \). Thus \( \tilde{w} \) is uniformly continuous on \( T_1 \) and hence \( T(x_1) := \sup_{y \in \mathbb{R}^{n-1}} \tilde{w}(x_1, y) \) is continuous. By Step 1 and the boundedness of \( \tilde{w} \), it follows that \( \lim_{x_1 \to \infty} T(x_1)/\hat{u}(x_1) = 0 \). Thus, since \( T(0) = 0 \), we can find \( 0 < \tilde{x}_1 < x_1 \) (\( \tilde{x}_1 \) is a large number) such that

\[ \sup\{ \tilde{T}(x_1)/\hat{u}(x_1): 0 \leq x_1 \leq \tilde{x}_1 \} = \tilde{T}(\tilde{x}_1)/\hat{u}(\tilde{x}_1). \]

By Step 2, \( \tilde{w} \) can be chosen so that \( \tilde{w}(\tilde{x}_1, y) \) achieves its maximum on \( \mathbb{R}^{n-1} \) at 0. (Our construction of the new \( \tilde{w} \) may decrease \( T(x_1) \) for \( x_1 \neq \tilde{x}_1 \) but the maximum will still be attained at \( \tilde{x}_1 \).) By our construction, \( r(x) \) achieves its maximum on \( \{(x_1, y): 0 \leq x_1 \leq \tilde{x}_1, y \in \mathbb{R}^{n-1} \} \) at the interior point \( (\tilde{x}_1, 0) \). However, since \( \tilde{u} \) satisfies \( \tilde{u}''(x_1) = 0 \), a simple calculation shows that \( r \) satisfies an elliptic equation

\[ \frac{\partial^2 r}{\partial x_1^2} + 2 \frac{\partial r}{\partial x_1} + \Delta_{n-1} r = 0, \]

where \( \Delta_{n-1} \) denotes the Laplacian in the \( y \) variables. Hence, by applying the maximum principle on compact sets, we see that \( r(x_1, y) \) is constant if \( 0 \leq x_1 \leq \tilde{x}_1, y \in \mathbb{R}^{n-1} \). This is impossible since \( r = 0 \) when \( x_1 = 0 \). The proof of this theorem is completed. \( \square \)

4. The case of \( \kappa = 1 \)

In this section, we consider the case \( \kappa = 1 \) and \( p > 1 \) with \( 0 < q < p \). For this special case, we can obtain solutions for the problem

\[
\begin{align*}
\Delta u &= \lambda [u^{-p} - u^{-q}] \quad \text{in } \Omega, \\
0 &< u < 1 \quad \text{in } \Omega, \\
u &= 1 \quad \text{on } \partial \Omega.
\end{align*}
\]

Setting \( v = 1 - u \), we obtain from \( (T_{\kappa}^{1,1}) \) the problem

\[
\begin{align*}
-\Delta v &= \lambda [(1 - v)^{-p} - (1 - v)^{-q}] \quad \text{in } \Omega, \\
0 &< v < 1 \quad \text{in } \Omega, \\
v &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]
It is clear that \( v_\lambda \equiv 0 \) is a solution of the equation in \((S_{\lambda}^{1,1})\) for all \( \lambda \in (0, \infty) \). In this section, we will obtain nontrivial solutions of \((S_{\lambda}^{1,1})\).

Let \( \sigma_1, \phi_1 > 0 \) be the first eigenvalue and the first eigenfunction of the problem

\[
-\Delta \phi = \sigma_1 \phi \quad \text{in} \quad \Omega, \quad \phi|_{\partial \Omega} = 0.
\]

Defining \( \lambda^* = \sigma_1/(p - q) \), we easily see the fact: If \((S_{\lambda}^{1,1})\) has a solution pair \((\lambda, v_\lambda)\), then \( \lambda \leq \lambda^* \). Let \( v_\lambda \) be a solution of \((S_{\lambda}^{1,1})\). Then multiplying \( \phi_1 \) on both sides of \((S_{\lambda}^{1,1})\) and integrating it on \( \Omega \), we see that

\[
\sigma_1 \int_{\Omega} v_\lambda \phi_1 \, dx = \lambda \int_{\Omega} |(1-v_\lambda)^{-p} - (1-v_\lambda)^{-q}| \phi_1 \, dx \geq \lambda(p - q) \int_{\Omega} v_\lambda \phi_1 \, dx.
\]

The last inequality is obtained from the fact that the function \( f(s) := (1-s)^{-p} - (1-s)^{-q} \geq (p - q)s \) for \( s \in [0, 1) \).

Our Theorem 1.2 can be obtained from the following theorem.

**Theorem 4.1.** There exists an unlimited solution branch

\[
\Gamma^* := \{ (\lambda, v_\lambda) : 0 < v_\lambda < 1 \text{ satisfies } (S_{\lambda}^{1,1}) \}
\]

starting from \((\lambda^*, 0)\).

**Proof.** Arguments similar to those in \([8]\) and \([4]\) imply that there exists a solution branch which starts from \((\lambda^*, 0)\). Let \( D \) denote the component of \( \{(\lambda, v) \in (0, \lambda^*) \times C(\Omega) : -\Delta v = \lambda [(1-v)^{-p} - (1-v)^{-q}], \quad 0 < v < 1 \quad \text{in} \Omega, \quad v = 0 \quad \text{on} \partial \Omega \} \) containing its closure \((\lambda^*, 0)\). Note that we can talk about the component since we know from arguments similar to those in \([8]\) that it is a simple curve near the end point. It is known from Theorem 2.2 of \([4]\) that there exists an analytic curve \( \lambda = \lambda(s), u = u(s) \) for \( s \geq 0 \) such that \( \max_{\partial \Omega} u(s) \to 1 \) as \( s \to \infty \), \( (\lambda(s), u(s)) \in D \) for \( s \geq 0 \). Note that we allow the curve \((\lambda(s), u(s))\) to have isolated intersections and that for each \( s > 0 \), \( u(s) \in C^2_0(\Omega) \). If we now use the usual trick of finding a minimal continuum in \( \{(\lambda(s), u(s)) : s \geq 0 \} \) joining \((\lambda(0), u(0))\) to “infinity”, we obtain a curve with no self intersections but it is only piecewise analytic and continuous. It is easy to see that the minimal continuum is an unlimited solution branch. This completes the proof. \( \square \)

**Remark 4.2.** It follows from \([18]\) that for \( \kappa \in (0, 1) \), there exists \( \lambda_*(p, q, \kappa) \in (0, \infty) \) such that for \( \lambda \in (0, \lambda_*(p, q, \kappa)) \), the problem

\[
\begin{cases}
-\Delta v = \lambda [((\kappa - v)^{-p} - (\kappa - v)^{-q}) & \text{in} \quad \Omega,
0 < v < \kappa & \text{in} \quad \Omega,
 v = 0 & \text{on} \quad \partial \Omega
\end{cases}
\]

has a unique minimal positive solution \( v_{\kappa}^x \). Moreover, we see from \([18]\) that the mapping \( \kappa \mapsto \lambda_*(p, q, \kappa) \) is increasing and for any \( x \in \Omega \), the mapping \( \kappa \mapsto v_{\kappa}^x(x) \) is decreasing. Therefore, \( \lim_{\kappa \to 1^-} \lambda_*(p, q, \kappa) = \lambda_\infty \), where \( \lambda_\infty \) is defined in Theorem 4.1 and \( \lim_{\kappa \to 1^-} v_{\kappa}^x(x) = 0 \) uniformly for \( \lambda \in (0, \lambda_\infty) \) and \( x \in \Omega \).

**Remark 4.3.** We still do not know whether the problem \((S_{\lambda}^{\kappa})\) has a solution \( v_{\kappa}^x \) for \( \kappa \in (0, 1) \) and \( \lambda \in (0, \infty) \) or not. We suspect that such solution does not exist. This means that for \( \kappa \in (0, 1) \), the problem

\[
\Delta u = \lambda [u^{-p} - u^{-q}] \quad \text{in} \quad \Omega, \quad u = \kappa \quad \text{on} \quad \partial \Omega
\]

does not have solution \( u_{\kappa}^x \) satisfying \( u_{\kappa}^x > \kappa \) in \( \Omega \); for \( \kappa > 1 \), (4.1) does not have solution \( u_{\kappa}^x \) satisfying \( 0 < u_{\kappa}^x < \kappa \) in \( \Omega \).

5. Positive radial solutions of \((T_{\lambda}^{\kappa})\) and \((T_{\lambda}^{\kappa,1})\) when \( \Omega \) is an annulus

In this section we study positive radial solutions of \((T_{\lambda}^{\kappa})\) and \((T_{\lambda}^{\kappa,1})\) \( \Omega = \{ x \in \mathbb{R}^N : 0 < r < \| x \| < b \} \) being an annulus in \( \mathbb{R}^N \) \( (N \geq 2) \). In this case, we can obtain more information on the structure of positive solutions of \((T_{\lambda}^{\kappa})\) and \((T_{\lambda}^{\kappa,1})\). Since we are interested in positive radial solutions of \((T_{\lambda}^{\kappa})\) and \((T_{\lambda}^{\kappa,1})\), we write the main equations in \((T_{\lambda}^{\kappa})\) and \((T_{\lambda}^{\kappa,1})\) in the form

\[
u''(r) + \frac{N-1}{r} \nu'(r) = \lambda [u^{-p} - u^{-q}] \quad \text{in} \quad (a, b), \quad \nu(a) = \nu(b) = \kappa.
\]

In the following, we assume that \( 1 < q < p < \infty \).

**Theorem 5.1.** For any fixed \( \kappa > 1 \), there is a continuous branch of positive radial solutions \((\lambda, u_{\kappa}^x)\) in \((0, \infty) \times C^2(a, b) \cap C^1([a, b]) \) of (5.1), which connects \((0, \kappa)\) and \((\infty, \infty)\). Moreover, for any \( r \in (a, b) \), the mapping \( \lambda \mapsto u_{\kappa}^x(r) \) is increasing.
Theorem 5.2. For any fixed $κ ∈ (0, 1)$, there exists a $0 < λ_∗(p, q, κ) < ∞$ such that the problem (5.1) has no positive radial solution for $λ > λ_∗(p, q, κ)$, one and only one radial solution for $λ = λ_∗(p, q, κ)$ and exactly two radial solutions for $0 < λ < λ_∗(p, q, κ)$.

Theorem 5.3. For $κ = 1$, problem (5.1) has no positive radial solution for $λ > σ_1/(p − q)$, and exactly two radial solutions for $0 < λ < σ_1/(p − q)$, one of them is the constant solution $u_1 = 1$, where $σ_1 > 0$ is the first eigenvalue of the problem

$$−Δφ = σφ \quad \text{in } Ω, \quad φ|_{Ω^c} = 0.$$  

We only need to show Theorem 5.2, the other two theorems can be obtained by arguments similar to those in the proofs of Theorems 1.1 and 5.2. Indeed, since the solution $y^*_2$ obtained in Theorem 1.1 is the minimal solution larger than the boundary value $κ$, then it is radially symmetric when $Ω$ is the annulus. Notice that Theorem 5.1 holds for $0 < q < p < ∞$.

Proof of Theorem 5.2. We consider the initial value problem

$$\ddot{u}(r) + \frac{N−1}{r} \dot{u}(r) = λ[u^{p−2}(r) − u^{q−2}(r)], \quad u(0) = κ, \quad u′(0) = −β \in (−∞, 0)$$  

and denote the solution of (5.2) by $u(r) = u(r, κ, β)$.

Lemma 5.4. There exist $τ := τ(κ, β)$, $R := R(κ, β)$ satisfying $a < τ < R$ such that $R$ is the first point larger than $a$ with $u(R) = κ$ and $τ$ is the unique minimum point of $u$ in the interval $(a, R)$, i.e.,

$$u′(r) < 0 \quad \text{for } r ∈ (a, τ) \quad \text{and} \quad u′(r) > 0 \quad \text{for } r ∈ (τ, b).$$

Proof. We first show that $u(r) > 0$ for $r ∈ (a, ∞)$. On the contrary, there is $ξ ∈ (a, ∞)$ such that $u(ξ) = 0$. But we can derive a contradiction from the fact that the function

$$F(r) := \frac{1}{2}(u′(r))^2 + \frac{λ}{p−1}u^{1−p}(r) − \frac{λ}{q−1}u^{1−q}(r)$$

is decreasing for $r ∈ (a, ξ)$. Note that $F(ξ) = ∞$. Suppose that there is $0 < η < ∞$ such that $\lim_{r→η−} u(r) = +∞$. We see that $u′(r) → ∞$ as $r → η−$. This also contradicts the fact that the function $F(r)$ is decreasing in $(a, η)$. Therefore, $0 < u(r) < ∞$ for $r ∈ (a, ∞)$. Suppose that $u(r)$ is decreasing in $(a, ∞)$, we see that there exists $0 < A < 1$ such that $\lim_{r→∞} u(r) = A$ and $\lim_{r→∞} u′(r) = 0$. If we define $g(s) := s^{p−2} − s^{q−2}$, we see that $g(A) = 0$. This is impossible. Thus, there is a minimum point $τ := τ(κ, β) > a$ of $u(r)$ such that $u′(τ) = 0$. There are two cases of $u(r)$ for $r > τ$: (i) $u′(r) > 0$ for $r > τ$ and $\lim_{r→∞} u(r) = B$. (ii) There is a maximum point $τ < y < ∞$ of $u$. If (i) occurs, we see that $\lim_{r→∞} u(r) = 0$ and $g(B) = 0$. This implies that $B = 1$. Thus, there is $R := R(κ, β) > τ$ such that $u(R) = κ$. If (ii) occurs, we see that $u′(γ) > 1$. Indeed, suppose $u′(γ) < 1$, we see that $u′(γ) = g(u(γ)) > 0$. This contradicts the fact that $γ$ is a maximum point. Thus, the $R$ such that $u(R) = κ$ and $R > τ$ also exists. This completes the proof. □

For $N ≥ 3$, in terms of variables $s = r^{2−N}$ and $w(s) = κ − u(r)$, problem (5.1) can be rewritten as

$$w''(s) + λρ(s)f(w(s)) = 0 \quad \text{in } (s_1, s_2), \quad w(s_0) = w(s_1), \quad (5.3)$$

where $ρ(s) = (N−2−2s_k−k = (2N−2)/(N−2), s_0 = b^{2−N}, s_1 = a^{2−N}$ and $f(t) = (κ − t)^{−p} − (κ − t)^{−q}$. For $N = 2$, in terms of variables $s = −\ln r$, $w(s) = κ − u(r)$, problem (5.1) can be rewritten as

$$w''(s) + λe^{−2s}f(w(s)) = 0 \quad \text{in } (s_0, s_1), \quad w(s_0) = w(s_1), \quad (5.4)$$

where $s_0 = −\ln b$ and $s_1 = −\ln a$.

In the following we only consider the case of $N ≥ 3$. The case of $N = 2$ can be treated similarly. The main idea to treat the case of $N = 2$ is similar to that of [10].

We now consider the initial value problem

$$w''(s) + λρ(s)f(w(s)) = 0, \quad w(s_0) = 0, \quad w′(s_0) = θ > 0.$$  

Let $w(s) = w(s, θ, κ, λ)$ be the solution of (5.5). It follows from Lemma 5.4 that there are $S := S(θ, κ, λ) = \min(S > s_0, w(S) = 0)$ and $τ := τ(κ, λ) ∈ (s_0, S)$ such that $w′(s) > 0$ for $s ∈ (s_0, τ)$, $w′(s) < 0$ for $s ∈ (τ, S)$ and $w′(τ) = 0$. Note the $τ$ here is different from that in Lemma 5.4. We see that

$$w(s) = θ(s − s_0) + λ \int_{s_0}^{s} (η − s)ρ(η)f(w(η))dη.$$
Lemma 5.5.

\[ \lim_{\theta \to 0^+} S(\theta, \kappa, \lambda) = \lim_{\theta \to 0^+} \tau(\theta, \kappa, \lambda) = s_0. \]

**Proof.** Suppose otherwise, there exist a \( \lambda > 0 \), \( \epsilon > 0 \) and a sequence \( \{\theta_n\} \) with \( \theta_n \to 0 \) as \( n \to \infty \) such that \( S_n \equiv S(\theta_n, \kappa, \lambda) \geq s_0 + \epsilon \). Therefore, \( w_n(s) \equiv w(s, \theta_n, \kappa, \lambda) > 0 \) for \( s \in (s_0, s_0 + \epsilon) \) and all \( n \). A simple calculation shows \( 0 < w_n(s) < \kappa \) for \( s \in (s_0, s_0 + \epsilon) \) provided \( \epsilon \) sufficiently small. By the standard theory of ordinary differential equation, we see that \( w_n(s) \to w_0(s) \) as \( n \to \infty \) (we can choose a subsequence if necessary) for \( s \in (s_0, s_0 + \epsilon) \) and \( w_0(s) \) is the solution of the initial value problem

\[ w''(s) + \lambda \rho(s) f(w(s)) = 0, \quad w(s_0) = 0, \quad w'(s_0) = 0. \]

Since

\[ w_0(s) = \lambda \int_{s_0}^{s} (\eta - s) \rho(\eta) f(w_0(\eta)) d\eta, \]

we see that \( w_0(s) < 0 \) for \( s \in (s_0, s_0 + \epsilon) \). But this contradicts the fact that \( w_n(s) > 0 \) for \( s \in (s_0, s_0 + \epsilon) \) and all \( n \). The proof of \( \lim_{\theta \to 0^+} \tau(\theta, \kappa, \lambda) = 0 \) is trivial. \( \Box \)

Lemma 5.6.

\[ \lim_{\theta \to +\infty} S(\theta, \kappa, \lambda) = \lim_{\theta \to +\infty} \tau(\theta, \kappa, \lambda) = s_0. \]

**Proof.** Suppose \( \lim_{\theta \to +\infty} \tau(\theta, \kappa, \lambda) \neq s_0 \). Then there exist a \( \tau_0 > s_0 \) and a sequence \( \{\theta_n\} \) with \( \theta_n \to +\infty \) as \( n \to \infty \) with \( w_n(s) \equiv w(s, \theta_n, \kappa, \lambda) > 0 \) and \( w_n(s) > 0 \) for \( s \in (s_0, \tau_0) \). Let \( \tau = s_0 + \frac{\kappa - s_0}{2} \). We claim

\[ \lim_{n \to \infty} \sup_{\tau} w_n(\tau) = \kappa. \]

Otherwise, there exists \( \epsilon > 0 \) such that \( 0 < w_n(\tau) \leq \kappa - \epsilon \). It follows that

\[ w_n(\tau) = \theta_n(\tau - s_0) + \lambda \int_{s_0}^{\tau} (s - \tau) \rho(s) f(w_n(s)) \, ds \]

\[ \geq \theta_n(\tau - s_0) + \lambda \int_{s_0}^{\tau} (s - \tau) \rho(s) [e^{-p} - e^{-q}] \, ds \]

which is impossible since \( \theta_n \to \infty \). Hence choosing a subsequence if necessary, we may assume

\[ \lim_{n \to \infty} w_n(\tau) = \kappa. \]

Note that \( w_n \) satisfies

\[ w''(s) + \lambda \rho(s) \frac{f(w_n(s))}{w_n(s)} w(s) = 0 \quad \text{in} \ (\tau, \tau_0). \]

Let

\[ M_n = \inf \left\{ \frac{f(w_n)}{w_n} : s \in (\tau, \tau_0) \right\}. \]

Then \( \lim_{n \to \infty} M_n = \infty \). Note that \( \lambda \rho(s) \geq \lambda \rho(\tau_0) \) in \( (\tau, \tau_0) \). Let \( v_n \) solves

\[ v''(s) + \lambda \rho(\tau_0) M_n v = 0 \quad \text{in} \ (\tau, \tau_0). \]

It follows that \( v_n \) has at least two zeros in \( (\tau, \tau_0) \) when \( n \) is sufficiently large. By Sturm comparison principle, \( w_n \) has at least one zero in \( (\tau, \tau_0) \). But this is impossible. Hence

\[ \lim_{\theta \to +\infty} \tau(\theta, \kappa, \lambda) = s_0. \]

Finally we show \( \lim_{\theta \to +\infty} S(\theta, \kappa, \lambda) = s_0 \). Otherwise, there exist a point \( \hat{s} > s_0 \) and a sequence \( \{\theta_n\} \) with \( \theta_n \to \infty \) as \( n \to \infty \) with

\[ \lim_{\theta \to +\infty} \tau(\theta, \kappa, \lambda) = \hat{s}. \]
\( w_n(s) > 0 \) and \( w_n' \leq 0 \) in \((\tau_n, \hat{s})\).

where \( w_n \equiv w(s, \theta_n, \kappa, \lambda) \) and \( \tau_n = \tau(\theta_n, \kappa, \lambda) \). Let \( \hat{s} = s_0 + \frac{1 - s_0}{2} \). In view of previous lemma that \( \lim_{n \to \infty} \tau(\theta, \kappa, \lambda) = s_0 \), we may assume \( \hat{s} > \tau_n \) for all \( n \). We claim

\[
\lim_{n \to \infty} \sup w_n(\hat{s}) < \kappa. \tag{5.6}
\]

Otherwise, by Sturm comparison principle, \( w_n \) has zeros in \((\tau_n, \hat{s})\) when \( n \) is sufficiently large which is impossible since \( \tau_n \to s_0 \) as \( n \to \infty \).

Note that

\[
\frac{d}{ds} \left( \int_{\tau_n}^{s} \lambda \rho(\eta) f(w_n(\eta)) d\eta \right) = \frac{d}{ds} \left( \int_{\tau_n}^{\hat{s}} \lambda \rho(\eta) f(w_n(\eta)) d\eta \right) + \int_{\tau_n}^{\hat{s}} \lambda \rho'(\eta) f(w_n(\eta)) d\eta.
\]

and if we define

\[
U_n(s) = \frac{1}{2} w_n'(s)^2 + \lambda \rho(s) F(w_n(s))
\]

where

\[
F(t) = \frac{1}{p - 1}(\kappa - t)^{1 - p} - \frac{1}{q - 1}(\kappa - t)^{1 - q}.
\]

It is clear that

\[
U_n'(s) = \lambda \rho'(s) F(w_n(s)) \tag{5.8}
\]

and so

\[
U_n(\hat{s}) = U_n(\tau_n) + \lambda \int_{\tau_n}^{\hat{s}} \rho'(\eta) F(w_n(\eta)) d\eta.
\]

That is,

\[
\frac{1}{2} w_n'(\hat{s})^2 = -\lambda \rho(\hat{s}) F(w_n(\hat{s})) + \lambda \rho(\tau_n) F(w_n(\tau_n)) + \lambda \int_{\tau_n}^{\hat{s}} \rho'(\eta) F(w_n(\eta)) d\eta.
\]

Since

\[
F(t) \leq f(t) \quad \text{for } t \in (0, 1), \tag{5.9}
\]

we have

\[
\frac{1}{2} w_n'(\hat{s})^2 - \lambda \int_{\tau_n}^{\hat{s}} \rho'(\eta) F(w_n(\eta)) d\eta \leq \frac{1}{2} w_n'(\hat{s})^2 + C \lambda \int_{\tau_n}^{\hat{s}} \rho(\eta) f(w_n(\eta)) d\eta,
\]

where \( C > 0 \). Now we need to show (5.9). To obtain (5.9), it is enough to show that for \( 1 < q < p \),

\[
\frac{1}{p - 1}s^{1 - p} - \frac{1}{q - 1}s^{1 - q} \leq s^{-p} - s^{-q} \quad \text{for } s \in (0, 1).
\]

Define

\[
e(s) = s^{-p} - s^{-q} - \frac{1}{p - 1}s^{1 - p} - \frac{1}{q - 1}s^{1 - q}.
\]

We see that \( e(1) > 0 \). Now we show that \( e(s) \leq 0 \) for \( s \in (0, 1) \). Indeed,

\[
e'(s) = -ps^{-(p+1)} + qs^{-(p+1)} + s^{-p} - s^{-q} = s^{-p}(-ps^{-1} + 1) + s^{-q}(qs^{-1} - 1)
\]

\[
\leq -(p - q)s^{-(p+1)} < 0.
\]

Then (5.9) holds. Hence, using (5.7), we obtain

\[
-\lambda \rho(\hat{s}) F(w_n(\hat{s})) + \lambda \rho(\tau_n) F(w_n(\tau_n)) \leq \frac{1}{2} w_n'(\hat{s})^2 + C|w_n'(\hat{s})|. \tag{5.10}
\]
Integrating (5.8) on \((s_0, \tau_n)\), we have
\[
\lambda \rho(\tau_n) F(w_n(\tau_n)) - \lambda \int_{s_0}^{\tau_n} \rho'(\eta) F(w_n(\eta)) \, d\eta = \frac{1}{2} \theta_n^2 + \lambda \rho(s_0) F(0).
\]
(5.11)

Since \(F'(s) \geq 0\) for \(s \in (0, \kappa)\), we see that
\[
F(w_n(s)) \leq F(w_n(\tau_n)) \quad \text{for } s \in (s_0, \tau_n).
\]

Thus,
\[
-\lambda \int_{s_0}^{\tau_n} \rho'(\eta) F(w_n(\eta)) \, d\eta \leq -\lambda \int_{s_0}^{\tau_n} \rho'(\eta) F(w_n(\tau_n)) \, d\eta = \lambda \rho(s_0) F(w_n(\tau_n)) - \lambda \rho(\tau_n) F(w_n(\tau_n)).
\]

This and (5.11) imply that
\[
\lambda \rho(s_0) F(w_n(\tau_n)) \geq \frac{1}{2} \theta_n^2 + \lambda \rho(\kappa) F(\kappa).
\]
(5.12)

Combining (5.10), (5.12) and (5.6), we see that
\[
w_n'(s) \to -\infty \quad \text{as } n \to \infty.
\]

Thus, we have
\[
w_n(\hat{s}) < \liminf_{n \to \infty} w_n(\tilde{s}) \to -\infty
\]
a contradiction to \(w_n(\hat{s}) > 0\). This completes the proof. \(\square\)

**Lemma 5.7.** Define \(\bar{S} = \bar{S} (\lambda) = \sup \{S(\theta, \kappa, \lambda); \theta > 0\}\). Then \(\bar{S}(\lambda)\) is strictly decreasing.

**Proof.** Let \(0 < \lambda_1 < \lambda_2 \) and \(w_2\) be a solution at \(\lambda_2\) on \((s_0, \bar{S}(\lambda_2))\). We see that \(w_2\) satisfies
\[
w'' + \lambda_1 \rho(s) f(w(s)) = 0, \quad w(s_0) = w(\bar{S}(\lambda_2)) = 0.
\]

Setting \(t = s - s_0\) and \(u(t) = w_2(s)\), we see that \(u(t)\) satisfies
\[
u''(t) + \lambda_1 \rho(t + s_0) f(u(t)) = 0, \quad u(0) = u(\bar{S}(\lambda_2) - s_0) = 0.
\]

Let \(v(\xi) = cu(t)\) with \(t = \xi/c\) where \(c\) is some constant greater but close to \(1\). It is easy to see that \(v(0) = 0\) and \(v(\bar{S}(\lambda_2) + \epsilon_1 - s_0) = 0\) for \(\epsilon_1 = (c - 1)\bar{S}(\lambda_2)\). We note that
\[
v''(\xi) + \lambda_1 \rho(\xi + s_0) f(v(\xi)) = \frac{1}{c} \left[ u''(t) + \lambda_1 c^{-(k-1)} \rho \left( t + \frac{s_0}{c} \right) f(cu(t)) \right]
\]
\[
= -\frac{1}{c} \left[ \lambda_2 \rho(t + s_0) f(u(t)) - \lambda_1 c^{-(k-1)} \rho \left( t + \frac{s_0}{c} \right) f(cu(t)) \right] \leq 0
\]
for \(c\) is sufficiently close to \(1\). Note that \(\bar{S}(\lambda_2) + \epsilon_1 - s_0 > \bar{S}(\lambda_2) + \epsilon - s_0\) for \(0 < \epsilon < (c - 1)\bar{S}(\lambda_2) - s_0\). Hence, \(v\) is a supersolution to the problem
\[
v''(\xi) + \lambda_1 \rho(\xi + s_0) f(v(\xi)) = 0, \quad v(0) = v(\bar{S}(\lambda_2) + \epsilon - s_0) = 0.
\]
(5.13)

Since \(v = 0\) is a subsolution to (5.13) (note that \(0 < \kappa < 1\)), we see that (5.13) has a positive solution between \(0\) and \(v\). This implies \(\bar{S}(\lambda_1) \geq \bar{S}(\lambda_2) + \epsilon\). Hence \(\bar{S}(\lambda)\) is strictly decreasing. This completes the proof. \(\square\)

**Lemma 5.8.**
\[
\lim_{\lambda \to +\infty} \bar{S}(\lambda) = +\infty, \quad \lim_{\lambda \to +\infty} \bar{S}(\lambda) = s_0.
\]

**Proof.** Suppose that \(\lim_{\lambda \to +\infty} \bar{S}(\lambda) \neq +\infty\). Then the monotonicity of \(\bar{S}(\lambda)\) implies that there exists a number \(S^* > s_0\) and sequences \(\{\lambda_n\}\) and \(\{\theta_n\}\) with \(\lambda_n \to 0^+\) and \(\lim_{n \to \infty} S(\lambda_n) = \lim_{n \to \infty} S(\theta_n, \kappa, \lambda_n) = S^*\). Lemmas 5.5 and 5.6 then imply that there does not exist any subsequence of \(\{\theta_n\}\) (still denoted by \(\{\theta_n\}\)) such that \(\theta_n \to \infty\) as \(n \to \infty\) or \(\theta_n \to 0\) as \(n \to \infty\). Therefore, there exist \(0 < \theta_1 < \theta_2 < \infty\) such that \(\theta_1 \leq \theta_n \leq \theta_2\). Let us write \(w_n(s) = w(s, \theta_n, \lambda_n)\). Then arguments similar to those in the proof of Lemma 5.4 imply that there exists \(\epsilon > 0\) such that \(w_n(s) \leq \kappa - \epsilon\) for all \(n\). Therefore,
\[0 = \omega_n(S^*) = \theta_n(S^* - s_0) + \lambda_n \int_{s_0}^{S^*} (\eta - S^*) \rho(\eta) f(\omega_n(\eta)) \, d\eta \]
\[\geq \theta_n(S^* - s_0) + \lambda_n \left[ e^{-p} - e^{-q} \right] \int_{s_0}^{S^*} (\eta - S^*) \rho(\eta) \, d\eta.\]

Hence, \( \theta_n \to 0^+ \) or \( S^* = s_0 \). But these contradict the facts \( \lim_{\theta \to 0^+} S(\theta, \kappa, \lambda) = s_0 \) and \( S(\lambda) \) is strictly decreasing. Similarly we can show the second statement. This completes the proof. \( \square \)

Finally, for any given \( \lambda \) and \( \kappa \), we study the shape of \( S(\theta) \). Notice that \( S(\theta) \) is determined by the implicit equation
\[w(\omega_n(\theta), \theta) = 0. \tag{5.14}\]

Differentiating Eq. (5.14) with respect to \( \theta \) we get the following equation for the derivatives of \( S \):
\[w_S(S(\theta), \theta) S'(\theta) + w_\theta(S(\theta), \theta) = 0, \tag{5.15}\]
\[w_S(S(\theta), \theta) S'(\theta) + w_\theta(S(\theta), \theta) S''(\theta) + w_\theta S(S(\theta), \theta) = 0. \tag{5.16}\]

If we write \( h(s, \theta) = w_\phi(s, \theta), z(s, \theta) = w_\theta(s, \theta) \) and \( v(s, \theta) = w_\omega(s, \theta) \), we can rewrite (5.15) as
\[v(S(\theta), \theta) S'(\theta) + h(S(\theta), \theta) = 0. \tag{5.17}\]

Also notice that when \( S'(\theta) = 0 \), from Eq. (5.16) we have
\[v(S(\theta), \theta) S''(\theta) + z(S(\theta), \theta) = 0. \tag{5.18}\]

We have the following important lemma.

**Lemma 5.9.** For a given \( \lambda \), if \( S'(\theta) = 0 \), then \( S''(\theta) < 0 \).

**Proof.** Note that \( h(s, \theta) \) satisfies the following initial value problem:
\[\begin{align*}
&h'(s) + \lambda \rho(s) [p(k - w)^{-(p+1)} - q(k - w)^{-(q+1)}] h = 0, \\
&h(s_0, \theta) = 0, \quad h'(s_0, \theta) = 1.
\end{align*} \tag{5.19}\]

If \( S'(\theta) = 0 \), then Eq. (5.15) gives us \( h(S(\theta), \theta) = 0 \). We claim that \( h(s, \theta) > 0 \) for \( s \in (s_0, S(\theta)) \). Otherwise let \( h(\xi(\theta), \theta) = 0 \) and \( h > 0 \) on \( (s_0, \xi(\theta)) \) with \( s_0 < \xi(\theta) < S(\theta) \). Note that \( v \) satisfies the following
\[\begin{align*}
&v'' + \lambda \rho(s) [p(k - w)^{-(p+1)} - q(k - w)^{-(q+1)}] v + \lambda \rho'(s) [p(k - w)^{-p} - q(k - w)^{-q}] = 0, \\
v(s_0, \theta) = \theta, \quad v'(s_0, \theta) = -\lambda \rho(s_0) [p(k - w)^{-p} - q(k - w)^{-q}].
\end{align*} \tag{5.20}\]

Recall that \( v(\tau(\theta), \theta) = 0 \). If \( \xi(\theta) \geq \tau(\theta) \), then \( v < 0 \) on \( (\xi(\theta), S(\theta)) \). By Sturm comparison principle, \( v \) should have a zero on \( (\xi(\theta), S(\theta)) \) since \( h(\sigma(\theta), \theta) = 0 \). This is impossible. (Here we use arguments similar to those in the paragraph after (5.21).)

If \( \xi(\theta) < \tau(\theta) \), then \( v < 0 \) on \( (\tau(\theta), S(\theta)) \). Note that \( h(\tau(\theta), \theta) = 0 \). Otherwise, there is another zero point \( \tilde{\xi}(\theta) \in (\xi(\theta), \tau(\theta)) \) of \( h \). This implies that \( v \) has a zero point in \( (\xi(\theta), \tilde{\xi}(\theta)) \). This is impossible. Since \( 0 = v(\tau(\theta), \theta) > h(\tau(\theta), \theta) \), by Sturm comparison theorem, \( v > h \) on \( (\tau(\theta), S(\theta)) \), which is impossible since \( h \) has to cross over \( v \) and reaches zero at \( S(\theta) \). Next we claim \( z(S(\theta), \theta) < 0 \). Note that
\[\begin{align*}
z'' + \lambda \rho(s) [p(k - w)^{-(p+1)} - q(k - w)^{-(q+1)}] z + \lambda \rho(s) [p(k - w)^{-(p+2)} - q(k + 1)(k - w)^{-(q+2)}] h^2 = 0, \\
z(s_0, \theta) = 0, \quad z'(s_0, \theta) = 0.
\end{align*} \tag{5.21}\]

We claim that \( z \) is negative in some neighborhood of \( s_0 \). Otherwise there exists \( \epsilon_0 > 0 \) such that \( z \geq 0 \) in \( (s_0, s_0 + \epsilon_0) \). Observing Eq. (5.21), we have \( z'' < 0 \) in \( (s_0, s_0 + \epsilon_0) \). It follows that \( z' < 0 \) in \( (s_0, s_0 + \epsilon_0) \) since \( z'(s_0, \theta) = 0 \). This contradicts our assumption \( z \geq 0 \) in \( (s_0, s_0 + \epsilon_0) \).

Next we claim that \( z < 0 \) in \( (s_0, S(\theta)) \). Otherwise, let \( z(s, \theta) = 0 \) with \( z < 0 \) in \( (s_0, \hat{s}) \). Comparing Eq. (5.19) and Eq. (5.21), we see that \( h \) must have a zero in \( (s_0, \hat{s}) \) which contradicts our previous statement. We need to explain a little here. We see from (5.21) that
\[z'' + \lambda \rho(s) [p(k - w)^{-(p+1)} - q(k - w)^{-(q+1)}] z < 0 \quad \text{in} \ (s_0, \hat{s}).\]

Since \( z(s, \theta) = 0 \) and \( z < 0 \) in \( (s_0, \hat{s}) \), we see that \( z'(s) \geq 0 \). On the other hand, since \( h > 0 \) in \( (s_0, \hat{s}) \), multiplying \( h \) on both the sides of (5.21) and integrating by parts in \( (s_0, \hat{s}) \) we see from (5.19) that
This is impossible. Hence \( S''(\theta) < 0 \) and it follows from (5.18) that \( S''(\theta) < 0 \).

We are now in the position to prove Theorem 5.2. For any \( a > 0 \), by Lemma 5.8 that there exists \( \lambda_a \) such that \( S(\lambda_a) = a^{2-N} \) and by Lemmas 5.7 and 5.9 there is a unique \( \theta \) such that \( S(\theta, \lambda_a) \), \( \lambda_a \) is a solution of the nonlinear elliptic eigenvalue problem (5.16). Indeed, suppose that there are \( \theta_1 \) and \( \theta_2 \) such that \( S(\theta_1, \lambda_a) = S(\theta_2, \lambda_a) = a^{2-N} \). Then by standard ODE theory, we see that there exists \( \tilde{\theta} \in (\theta_1, \theta_2) \) such that \( S(\tilde{\theta}, \lambda_a) \), \( \lambda_a \) has a minimum at \( \tilde{\theta} = \tilde{\theta} \). This contradicts the result in Lemma 5.9. Thus, there exists a unique radial solution at \( \lambda = \lambda_a \). For \( \lambda < \lambda_a \), by Lemma 5.7, we see that \( \tilde{S}(\lambda) > \tilde{S}(\lambda_a) \). If \( \tilde{S}(\lambda) = S(\tilde{\theta}, \lambda) \), then \( S(\tilde{\theta}, \lambda) > \tilde{S}(\lambda) \) and \( S(\cdot, \lambda) \) attains its maximum at \( \tilde{\theta} \). Thus, we can find \( \theta_1 \) and \( \theta_2 \) such that \( S(\theta_1, \lambda) = S(\theta_2, \lambda) = \tilde{S}(\lambda) \) \( (= a^{2-N}) \). The problem has two radial solutions in this case. For \( \lambda > \lambda_a \), since \( \tilde{S}(\lambda) < a^{2-N} \), there is no radial solution.  

Acknowledgment

We would like to thank the referee for his/her valuable suggestions. The research of the first author is supported by grants of NSFC (10571022) and (10871060).

References