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Embedding uniformly convex spaces into spaces with very few operators ☆

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Abstract

We prove that every separable uniformly convex Banach space X embeds into a Banach space Z which has the property that all bounded linear operators on Z are compact perturbations of scalar multiples of the identity. More generally, the result holds for all separable reflexive Banach spaces of Szlenk index ω_0.

Keywords: Scalar plus compact; Bourgain–Delbaen spaces; Embedding Banach spaces; Very few operators

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1. Introduction

What is now known as the "scalar plus compact" problem asked if there existed an infinite-dimensional Banach space with the property that every bounded linear operator on the space is equal to a scalar times the identity plus a compact operator. This was listed by Lindenstrauss as Question 1 in his 1976 list of open problems in Banach space theory [17], though his problem was well known before then. Part of the reason for the interest in the "scalar plus compact" problem, is due to the fact that every compact operator has an invariant subspace [5]. Thus a Banach space with the property that every bounded linear operator on the space is equal to a scalar times the identity plus a compact operator, also has the property that every bounded operator has an invariant subspace.

Recently, the first and third author solved the "scalar plus compact problem" by creating an infinite-dimensional Banach space with the property that every bounded linear operator on the space is equal to a scalar times the identity plus a compact operator [2]. The space was constructed by modifying a procedure of Bourgain and Delbaen [6] which produces an $\mathcal{L}_\infty$ Banach space with dual isomorphic to ℓ_1. Recently as well, the second, fourth, and sixth named authors [10] modified the Bourgain–Delbaen procedure in a different manner to prove that if X is a Banach space with separable dual, then X embeds into an $\mathcal{L}_\infty$ Banach space with a shrinking basis with dual isomorphic to ℓ_1. The main goal of this paper is to combine the two modifications of the Bourgain–Delbaen procedure to prove the following theorem.

Theorem A. *Let X be a separable uniformly convex Banach space. Then X embeds in a Banach space Z, whose dual space is isomorphic to ℓ_1, and which has the property that all $T \in \mathcal{L}(Z)$, i.e. all bounded linear operators T on Z, are of the form $T = \lambda\,\mathrm{Id} + K$, where* Id *denotes the identity, λ is a scalar and K is a compact operator on Z.*

In particular, Theorem A shows that the subspace structure of a Banach space with the "scalar plus compact" property can be quite general and can contain unconditional basic sequences. This is in stark contrast to [2], where the constructed space was hereditarily indecomposable.

The space Z, constructed in the proof of Theorem A, will have some additional interesting properties. As in [2], we have that:

(i) Z is somewhat reflexive, i.e., every infinite-dimensional subspace of Z contains an infinite-dimensional reflexive subspace.
(ii) $\mathcal{L}(Z)$ is amenable as a Banach algebra.
(iii) $\mathcal{L}(Z)$ is separable.
(iv) Every $T \in \mathcal{L}(Z)$ admits a non-trivial invariant subspace.
(v) An operator $T : X \to X$ lifts to an operator $\overline{T} : Z \to Z$ such that $\overline{T}|_X = T$ if and only if T is equal to scalar times the identity operator on X plus a compact operator.

Many of these properties are significant in their own right, and merit further discussion. In particular, as the invariant subspace problem for Hilbert spaces is one of the most important open problems in operator theory, we note that Theorem A implies the related result that a separable Hilbert space (or more generally $L_p[0,1]$ for $1 < p < \infty$) embeds into a Banach space with separable dual such that every bounded operator has an invariant subspace.

In the 50's and 60's, Grothendiek and Lindenstrauss worked on determining the lifting properties of compact operators. One result of Lindenstrauss in particular [16] gives that because Z

is a isomorphic predual of ℓ_1, it has the injective property for compact operators. Thus, for every compact operator $K : X \to X$, there exists a compact operator $\overline{K} : Z \to Z$ such that $\overline{K}|_X = K$. Thus an operator $T : X \to X$ lifts to an operator $\overline{T} : Z \to Z$ such that $\overline{T}|_X = T$ if and only if T is equal to scalar times the identity operator on X plus a compact operator.

In his 1972 memoir [14], B.E. Johnson set up the theory of cohomology of Banach algebras, and introduced the notion of an amenable Banach algebra. A Banach algebra A is called *amenable* if every bounded derivation D from A to a dual Banach A-bimodule X^* is inner. That is, if a bounded linear operator $D : A \to X^*$ satisfies $D(ab) = a \cdot (Db) + (Da) \cdot b$, for each $a, b \in A$, then there exists $x^* \in X^*$ such that $Da = a \cdot x^* - x^* \cdot a$. The name amenable was appropriately chosen for this property, as the group algebra of a locally compact group is amenable as a Banach algebra if and only if the group is amenable [14]. Johnson posed the question of whether the algebra $\mathcal{L}(X)$ can ever be amenable for an infinite-dimensional Banach space X. Whether $\mathcal{L}(X)$ is amenable remains an important open problem for a number of concrete Banach spaces, including ℓ_p for $p \neq 1, 2$ [8]. It is shown in [13] that the algebra of compact operators $K(X)$ is amenable whenever X is an $\mathcal{L}_p$-space when $1 \leqslant p \leqslant \infty$. Thus in Theorem A, we have that $K(Z)$ is amenable. By Proposition 2.8.58(i) of [7], the algebra obtained by adjoining an identity to a non-unital amenable Banach algebra is again amenable. Thus the algebra of all bounded operators on Z is amenable.

Theorem A also implies that every separable uniformly convex Banach space embeds into an indecomposable Banach space with separable dual. It is thus worth noting that the first and fifth named authors have recently proved that every separable reflexive Banach space embeds into a separable reflexive indecomposable Banach space [3].

The proof of Theorem A relies heavily on the Bourgain–Delbaen construction [6], the framework and the notation of which are reviewed in Section 2. In Theorem 2.9 we give a general criteria that will ultimately yield that the space Z, constructed in Section 4 where Theorem A is proved, satisfies the “scalar plus compact” property.

Theorem A will actually hold for any reflexive Banach space with Szlenk index ω_0. The theorem relies on X being reflexive as we will show that for any operator T on Z, there exists a scalar λ such that $T - \lambda\,\mathrm{Id}$ factors through X. Thus, $T - \lambda\,\mathrm{Id}$ is weakly compact as X is reflexive, and is hence norm compact as Z^* is isomorphic to ℓ_1 which has the Schur property. Our construction uses the mixed Tsirelson space given in [2], and we rely on block sequences in X being dominated by the unit vector basis for the mixed Tsirelson space. Requiring that X have Szlenk index ω_0 guarantees this property. There is work in progress by the authors, which will yield further results concerning spaces with very few operators. This is based on a different and very involved approach using higher complexity saturation methods, among other techniques [1].

2. The generalized Bourgain–Delbaen construction

In this section we review the general framework and notation of the construction of *Bourgain–Delbaen spaces*. We follow, with slight changes and some notational differences, the presentation in [2] and start by introducing *Bourgain–Delbaen sets*.

Definition 2.1 *(Bourgain–Delbaen sets).* A sequence of finite disjoint sets $(\Delta_n\colon n \in \mathbb{N})$ is called a *Sequence of Bourgain–Delbaen Sets* if it satisfies the following recursive conditions:

Δ_1 is any finite set, and assuming that for some $n \in \mathbb{N}$ the sets $\Delta_1, \Delta_2, \ldots, \Delta_n$ have been defined, we let $\Gamma_n = \bigcup_{j=1}^{n} \Delta_j$. We denote the unit vector basis of $\ell_1(\Gamma_n)$ by $(e^*_\gamma\colon \gamma \in \Gamma_n)$, and consider the spaces $\ell_1(\Gamma_j)$ and $\ell_1(\Gamma_n \setminus \Gamma_j)$, $j < n$, to be, in the natural way, embedded into $\ell_1(\Gamma_n)$.

For $n \geqslant 1$, Δ_{n+1} will then be the union of two sets $\Delta_{n+1}^{(0)}$ and $\Delta_{n+1}^{(1)}$, where $\Delta_{n+1}^{(0)}$ and $\Delta_{n+1}^{(1)}$ satisfy the following conditions.

The set $\Delta_{n+1}^{(0)}$ is finite and

$$\Delta_{n+1}^{(0)} \subset \{(n+1, \beta, b^*): \beta \in [0,1],\ b^* \in B_{\ell_1(\Gamma_n)}\}. \tag{1}$$

The set $\Delta_{n+1}^{(1)}$ is finite and

$$\Delta_{n+1}^{(1)} \subset \{(n+1, \alpha, k, \xi, \beta, b^*): \alpha, \beta \in [0,1],\ k \in \{1,2,\dots,n-1\},\ \xi \in \Delta_k,\ b^* \in B_{\ell_1(\Gamma_n \setminus \Gamma_k)}\}. \tag{2}$$

If (Δ_n) is a sequence of Bourgain–Delbaen sets we put $\Gamma = \bigcup_{j=1}^{\infty} \Delta_j$. For $n \in \mathbb{N}$, and $\gamma \in \Delta_n$ we call n the *rank of* γ and denote it by $\mathrm{rk}(\gamma)$. If $\gamma \in \Delta_n^{(0)}$, we say that γ is *of type* 0, and, in the case that $\gamma \in \Delta_n^{(1)}$, we say that γ is *of type* 1. In both cases we call β *the weight of* γ and denote it by $\mathrm{wt}(\gamma)$. In our application of the Bourgain–Delbaen construction to prove Theorem A, we will always have $\alpha = 1$ for $\gamma = (n, \alpha, k, \xi, \beta, b^*) \in \Delta_n^{(1)}$, and hence we will then suppress the α and use the notation $\gamma = (n, k, \xi, \beta, b^*) \in \Delta_n^{(1)}$. In [10], there was a special case when $\alpha \neq 1$, and we discuss later how the construction in [10] can be recoded to avoid this.

Given a sequence of Bourgain–Delbaen sets $\Delta = (\Delta_n: n \in \mathbb{N})$ we will always assume the sets $\Delta_n^{(0)}$, $\Delta_n^{(1)}$, Γ_n and Γ have been defined satisfying the conditions above. We consider the spaces $\ell_\infty(\bigcup_{j \in A} \Delta_j)$ and $\ell_1(\bigcup_{j \in A} \Delta_j)$, for $A \subset \mathbb{N}$, to be naturally embedded into $\ell_\infty(\Gamma)$ and $\ell_1(\Gamma)$, respectively.

We denote by $c_{00}(\Gamma)$ the real vector space of families $x = (x(\gamma): \gamma \in \Gamma) \subset \mathbb{R}$ for which the *support*, $\mathrm{supp}(x) = \{\gamma \in \Gamma: x(\gamma) \neq 0\}$, is finite. The unit vector basis of $c_{00}(\Gamma)$ is denoted by $(e_\gamma: \gamma \in \Gamma)$, or, if we think of c_{00} to be as being subspace of a dual space, such as $\ell_1(\Gamma)$, by $(e_\gamma^*: \gamma \in \Gamma)$. If $\Gamma = \mathbb{N}$ we write c_{00} instead of $c_{00}(\mathbb{N})$.

Definition 2.2 *(Bourgain–Delbaen families of functionals).* Assume that $(\Delta_n: n \in \mathbb{N})$ is a sequence of Bourgain–Delbaen sets. By induction on n we will define for all $\gamma \in \Delta_n$, elements $c_\gamma^* \in \ell_1(\Gamma_{n-1})$ and $d_\gamma^* \in \ell_1(\Gamma_n)$, with $d_\gamma^* = e_\gamma^* - c_\gamma^*$.

For $\gamma \in \Delta_1$ we define $c_\gamma^* = 0$, and thus $d_\gamma^* = e_\gamma^*$.

Assume that for some $n \in \mathbb{N}$ we have defined $(c_\gamma^*: \gamma \in \Gamma_n)$, with $c_\gamma^* \in \ell_1(\Gamma_{j-1})$, if $j \leqslant n$ and $\mathrm{rk}(\gamma) = j$. It follows therefore that $(d_\gamma^*: \gamma \in \Gamma_n) = (e_\gamma^* - c_\gamma^*: \gamma \in \Gamma_n)$ is a basis for $\ell_1(\Gamma_n)$ and thus for $k \leqslant n$ we have the projections:

$$P_{(k,n]}^*: \ell_1(\Gamma_n) \to \ell_1(\Gamma_n), \quad \sum_{\gamma \in \Gamma_n} a_\gamma d_\gamma^* \to \sum_{\gamma \in \Gamma_n \setminus \Gamma_k} a_\gamma d_\gamma^*. \tag{3}$$

For $\gamma \in \Delta_{n+1}$ we then define

$$c_\gamma^* = \begin{cases} \beta b^* & \text{if } \gamma = (n+1, \beta, b^*) \in \Delta_{n+1}^{(0)}, \\ \alpha e_\xi^* + \beta P_{(k,n]}^*(b^*) & \text{if } \gamma = (n+1, \alpha, k, \xi, \beta, b^*) \in \Delta_{n+1}^{(1)}. \end{cases} \tag{4}$$

We call $(c^*_\gamma\colon \gamma \in \Gamma)$, the *Bourgain–Delbaen family of functionals associated to* $(\Delta_n\colon n \in \mathbb{N})$. We will in this case consider the projections $P^*_{(k,n]}$ to be defined on all of $c_{00}(\Gamma)$, which is possible since $(d^*_\gamma\colon \gamma \in \Gamma)$ forms a vector basis of $c_{00}(\Gamma)$ and, (as we will observe later) under further assumptions, a Schauder basis of $\ell_1(\Gamma)$.

Remarks. The reason for using $*$ in the notation for $P^*_{(k,m]}$ is that later we will observe that the $P^*_{(k,m]}$ are the adjoints of some coordinate projections $P_{(k,m]}$ on a space Y with a finite-dimensional decomposition (FDD) $\mathbf{F} = (F_j)$ onto $\bigoplus_{j\in(k,m]} F_j$.

The next proposition is based on results in [2] and [6]. It follows from a more general theorem in [10].

Proposition 2.3. *(See [10, Proposition 2.4].) Assume that* $(\Delta_n\colon n \in \mathbb{N})$ *is a sequence of Bourgain–Delbaen sets and let* $(c^*_\gamma\colon \gamma \in \Gamma)$ *be the corresponding family of associated functionals. For* $n \in \mathbb{N}$ *put* $F^*_n = \operatorname{span}(d^*_\gamma\colon \gamma \in \Delta_n)$. *If for every* $\gamma = (n+1, \alpha, k, \xi, \beta, b^*) \in \Gamma$ *of type* 1 *it holds that* $\beta \leqslant \frac{1}{4}$, *then* $(F^*_n)_{n=1}^\infty$ *is an FDD for* $\ell_1(\Gamma)$ *whose decomposition constant* M *is not larger than* 2.

Remarks. Let Γ be linearly ordered as $(\gamma_j\colon j \in \mathbb{N})$ in such a way that $\operatorname{rk}(\gamma_i) \leqslant \operatorname{rk}(\gamma_j)$, if $i \leqslant j$. Under the assumption $\beta \leqslant \frac{1}{4}$ stated in Proposition 2.3, $(d^*_{\gamma_j})$ is actually a Schauder basis of ℓ_1 [2]. But for our purpose the FDD is the more natural coordinate system.

Assume we are given a sequence of Bourgain–Delbaen sets $(\Delta_n\colon n \in \mathbb{N})$, which satisfy the assumptions of Proposition 2.3, and let M be the decomposition constant of the FDD (F^*_n) in $\ell_1(\Gamma)$. We now define the *Bourgain–Delbaen space associated to* $(\Delta_n\colon n \in \mathbb{N})$. For a finite or cofinite set $A \subset \mathbb{N}$ we let P^*_A be the projection onto the subspace $\bigoplus_{j\in A} F^*_j$ of $\ell_1(\Gamma)$ given by

$$P^*_A : \ell_1(\Gamma) \to \ell_1(\Gamma), \quad \sum_{\gamma\in\Gamma} a_\gamma d^*_\gamma \mapsto \sum_{\gamma\in A} a_\gamma d^*_\gamma.$$

If $A = \{m\}$, for some $m \in \mathbb{N}$, we write P^*_m instead of $P^*_{\{m\}}$. For $m \in \mathbb{N}$ we denote by R_m the restriction operator from $\ell_1(\Gamma)$ onto $\ell_1(\Gamma_m)$ (in terms of the basis (e^*_γ)) as well the usual restriction operator from $\ell_\infty(\Gamma)$ onto $\ell_\infty(\Gamma_m)$. Since $R_m \circ P^*_{[1,m]}$ is a projection from $\ell_1(\Gamma)$ onto $\ell_1(\Gamma_m)$, for $m \in \mathbb{N}$, it follows that the map

$$J_m : \ell_\infty(\Gamma_m) \to \ell_\infty(\Gamma), \quad x \mapsto P^{**}_{[1,m]} \circ R^*_m(x),$$

is an isomorphic embedding ($P^{**}_{[1,m]}$ is the adjoint of $P^*_{[1,m]}$ and, thus, defined on $\ell_\infty(\Gamma)$). Since R^*_m is the natural embedding of $\ell_\infty(\Gamma_m)$ into $\ell_\infty(\Gamma)$ it follows for all $m \in \mathbb{N}$ that

$$R_m \circ J_m(x) = x, \quad \text{for } x \in \ell_\infty(\Gamma_m), \text{ thus } J_m \text{ is an extension operator,} \tag{5}$$

$$J_n \circ R_n \circ J_m(x) = J_m(x), \quad \text{whenever } m \leqslant n \text{ and } x \in \ell_\infty(\Gamma_m), \tag{6}$$

and by Proposition 2.3,

$$\|J_m\| \leqslant M. \tag{7}$$

Hence the spaces $Y_m = J_m(\ell_\infty(\Gamma_m))$, $m \in \mathbb{N}$, are finite-dimensional nested subspaces of $\ell_\infty(\Gamma)$ which (via J_m) are M-isomorphic images of $\ell_\infty(\Gamma_m)$. Therefore

$$Y = \overline{\bigcup_{m\in\mathbb{N}} Y_n}^{\ell_\infty} \tag{8}$$

is an $\mathcal{L}_{\infty,M}$ space. We call Y the *Bourgain–Delbaen space associated to* (Δ_n).

Define for $m \in \mathbb{N}$

$$P_{[1,m]} : Y \to Y, \quad x \mapsto J_m \circ R_m(x).$$

We claim that $P_{[1,m]}$ coincides with the restriction of the adjoint $P^{**}_{[1,m]}$ of $P^*_{[1,m]}$ to the space Y. Indeed, if $n \in \mathbb{N}$, with $n \geqslant m$, and $x = J_n(\tilde{x}) \in Y_n$, and $b^* \in \ell_1(\Gamma)$ we have that

$$\begin{aligned}\langle P^{**}_{[1,m]}(x), b^*\rangle &= \langle x, P^*_{[1,m]}(b^*)\rangle \\ &= \langle R_m(x), R_m \circ P^*_{[1,m]}(b^*)\rangle \quad \big(\text{since } P^*_{[1,m]}(b^*) \in \operatorname{span}(e^*_\gamma\colon \gamma \in \Gamma_m)\big) \\ &= \langle P^{**}_{[1,m]} \circ R^*_m \circ R_m(x), b^*\rangle = \langle P_{[1,m]}(x), b^*\rangle.\end{aligned}$$

Thus our claim follows since $\bigcup_n Y_n$ is dense in Y.

We therefore deduce that Y has an FDD (F_m), with $F_m = (P_{[1,m]} - P_{[1,m-1]})(Y)$ and $Y_m = \bigoplus_{j=1}^m F_j \sim_M \ell_\infty(\Gamma_m)$ for $m \in \mathbb{N}$. Moreover, denoting by P_A the coordinate projections from Y onto $\bigoplus_{j\in A} F_j$, for all finite or cofinite sets $A \subset \mathbb{N}$, it follows that P_A is the adjoint of P^*_A restricted to Y, and P^*_A is the adjoint of P_A restricted to the subspace of Y^* generated by the F^*_n's.

Denote by $\|\cdot\|_*$ the dual norm of Y^* restricted to the subspace $\bigoplus_{j=1}^\infty F^*_j = \ell_1$. We claim that for all $b^* \in \ell_1(\Gamma)$

$$\|b^*\|_* \leqslant \|b^*\|_{\ell_1} \leqslant M\|b^*\|_*. \tag{9}$$

The first inequality follows from the fact that $\|e^*_\gamma\|_* \leqslant \|e^*_\gamma\|_{\ell^*_\infty} = 1$, for $\gamma \in \Gamma$, and the triangle inequality. To show the second inequality we let $b^* \in \ell_1(\Gamma_n)$ for some $n \in \mathbb{N}$ and choose $x \in S_{\ell_\infty(\Gamma_n)}$ so that $\langle b^*, x\rangle = \|b^*\|_{\ell_1}$. Then it follows from (7) and (5)

$$\|b^*\|_* \geqslant \left\langle b^*, \frac{1}{M} J_n(x)\right\rangle = \frac{1}{M}\|b\|_{\ell_1}.$$

We now recall some notation introduced in [2]. Assume that we are given a Bourgain–Delbaen sequence (Δ_n), the corresponding Bourgain–Delbaen family $(c^*_\gamma\colon \gamma \in \Gamma)$, and the resulting Bourgain–Delbaen space Y, which admits a decomposition constant $M < \infty$. As above we denote its FDD by (F_n). For the remainder of the section, we restrict ourselves to considering sets Γ such that $\alpha = 1$ for all $\gamma = (n, \alpha, k, \xi, \beta, b^*) \in \Delta^{(1)}_n$. We suppress the α, and use the notation $\gamma = (n, k, \xi, \beta, b^*)$. For $n \in \mathbb{N}$ and $\gamma \in \Delta_n$, we write

$$e^*_\gamma = d^*_\gamma + c^*_\gamma = d^*_\gamma + \begin{cases} \beta b^* & \text{if } \gamma = (n, \beta, b^*) \in \Delta^{(0)}_n, \\ e^*_\xi + \beta P^*_{(k,\infty)}(b^*) & \text{if } \gamma = (n, k, \xi, \beta, b^*) \in \Delta^{(1)}_n. \end{cases}$$

In the second case, we can write $e^*_\xi = d^*_\xi + c^*_\xi$, and, then we can insert for c^*_ξ its definition. We can proceed this way and eventually arrive (after finitely many steps) to a functional of type 0. By an easy induction argument we therefore deduce the following

Proposition 2.4. *For all* $n \in \mathbb{N}$ *and* $\gamma \in \Delta_n$, *there are* $a \in \mathbb{N}$, $\beta_1, \beta_2, \dots, \beta_a \in [0,1]$, *numbers* $0 = p_0 < p_1 < p_2 - 1 < p_2 < p_3 - 1 < p_3 < \dots < p_{a-1} < p_a - 1 < p_a = n$ *in* $\mathbb{N}_0$, *vectors* $b^*_j \in B_{\ell_1}(\Gamma) \cap \operatorname{span}(e^*_\eta\colon \eta \in \Gamma_{p_j - 1} \setminus \Gamma_{p_{j-1}})$, *and* $(\xi_j) \subset \Gamma_n$, *with* $\xi_j \in \Delta_{p_j}$, *for* $j = 1, 2, \dots, a$, *and* $\xi_a = \gamma$, *so that*

$$e^*_\gamma = \sum_{j=1}^{a} d^*_{\xi_j} + \beta_j P^*_{(p_{j-1},\infty)}\big(b^*_j\big) = \sum_{j=1}^{a} d^*_{\xi_j} + \beta_j P^*_{(p_{j-1},p_j)}\big(b^*_j\big). \tag{10}$$

We call the representation in (10) *the evaluation analysis of* γ and define for $\gamma \in \Gamma$, $\operatorname{age}(\gamma) = a$ to be the *age of* γ. We define the *cuts of* γ to be $\operatorname{cuts}(\gamma) = \{p_1, p_2, \dots, p_a\}$. The sequence of triples, $(p_j, b^*_j, \xi_j)_{1 \leqslant j \leqslant a}$ is called the *analysis of* γ.

Remark. From now on, we assume that there is a strictly increasing sequence of natural numbers $(m_j)_{j=1}^\infty$ such that for each $\gamma \in \Gamma$, we have $m_j^{-1} = \beta_1 = \beta_2 = \dots = \beta_a$ for some $j \in \mathbb{N}$. Furthermore, we assume that $b^*_j \in \bigoplus_{i \in (p_{j-1}, p_j)} F^*_i$ and hence $P^*_{(p_{j-1},p_j)}(b^*_j) = b^*_j$ for all $1 \leqslant j \leqslant a$. Thus the evaluation analysis of γ simplifies to the following equation:

$$e^*_\gamma = \sum_{i=1}^{a} d^*_{\xi_i} + m_j^{-1} \sum_{i=1}^{a} b^*_i. \tag{11}$$

Definition 2.5. If $x \in Y$, we define the *support* of x to be $\operatorname{supp}(x) = \{n \in \mathbb{N}\colon\ P_n(x) \neq 0\}$. We say that a sequence $(x_n) \subset X$ is a *block sequence* with respect to (F_i) if $\max\operatorname{supp}(x_1) < \min\operatorname{supp}(x_2) \leqslant \max\operatorname{supp}(x_2) < \min\operatorname{supp}(x_3) < \cdots$. If in addition, $\max\operatorname{supp}(x_n) + 1 < \max\operatorname{supp}(x_{n+1})$ for all n then we say that $(x_n) \subset X$ is a *skipped block sequence*.

If $x \in \bigoplus_{i=1}^n F_i$, then there exists a unique $y \in \ell_\infty(\Gamma_n)$ such that $x = J_n(y)$. We define the *range* of x to be the smallest interval $\operatorname{rg}(x) = [k, m]$ such that $y(\gamma) = 0$ for all $\gamma \notin \Gamma_m \setminus \Gamma_{k-1}$.

Definition 2.6. Let (x_n) be a block sequence in Y, $(m_j)_{j=1}^\infty$ be a strictly increasing sequence of natural numbers, and $C > 0$. We say (x_n) is a *C-Rapidly Increasing Sequence*, or C-RIS, if for $k \in \mathbb{N}$

$$\begin{aligned} &\|x_k\| \leqslant C \quad \text{and} \quad \big|e^*_\gamma(x_k)\big| \leqslant C\operatorname{wt}(\gamma) \\ &\qquad \text{if } k \geqslant 2 \text{ and } \gamma \in \Gamma \text{ with } \operatorname{wt}(\gamma) \geqslant m^{-1}_{\max\operatorname{rg}(x_{k-1})}. \end{aligned} \tag{12}$$

We say that (x_n) is an *RIS*, if (x_n) is a C-RIS for some $C > 0$.

It is easy to see that Rapidly Increasing Sequences satisfy the following permanence properties.

Proposition 2.7. *Let $(m_j)_{j=1}^{\infty}$ be a strictly increasing sequence of natural numbers, and $C > 0$.*

(a) *Every subsequence of a C-RIS is a C-RIS.*
(b) *If (x_n) and (y_n) are C-RIS's and $\alpha, \beta > 0$, then there is a subsequence (k_n) of $\mathbb{N}$ so that $(\alpha x_{k_n} + \beta y_{k_n})_{n\in\mathbb{N}}$, is a $C(\alpha+\beta)$-RIS.*

Proposition 2.8. *(See [2, Proposition 5.11].) Let $T : Y \to W$ be a bounded linear operator, W being a Banach space. Then $\|T(x_k)\| \to 0$ whenever (x_k) is a bounded block sequence if and only if $\|T(x_k)\| \to 0$ whenever (x_k) is an RIS.*

We finish this section by stating a criterion which implies that all operators $T : Y \to Y$ are compact perturbations of a multiplication operator. Most of the proof is based on the proof of a similar statement in [2].

Theorem 2.9. *Let (Δ_n) be a sequence of Bourgain–Delbaen sets, with finite decomposition constant M. Assume furthermore that the FDD (F_n) of Y, which we defined above, is shrinking.*

Let X be a reflexive subspace of Y and assume that $T : Y \to Y$ is a bounded linear operator satisfying for every $C > 0$ and C-RIS (x_n) the condition that

$$\liminf_{n\to\infty} \operatorname{dist}\bigl(T(x_n), [x_n] + X\bigr) = 0. \tag{13}$$

Then there is a $\lambda \in \mathbb{R}$ and a compact operator K on Y so that $T = \lambda\,\mathrm{Id} + K$.

Proof. Assume that (x_n) is a C-RIS which is seminormalized in the quotient space Y/X. By our assumption we can choose a subsequence (x'_n) of (x_n) and a bounded sequence $(\lambda_n) \subset \mathbb{R}$ so that $\lim_{n\to\infty} \|T(x'_n) - \lambda_n x'_n\|_{Y/X} = 0$. After passing again to a subsequence we can assume that $\lambda = \lim_{n\to\infty} \lambda_n$ exists and, thus, that $\lim_{n\to\infty} \|T(x'_n) - \lambda x'_n\|_{Y/X} = 0$.

Secondly, we claim that there is a universal $\lambda \in \mathbb{R}$ so that for all $C > 0$ every C-RIS (x_n), which is seminormalized in Y/X, has a subsequence (x'_n) so that $\lim_{n\to\infty}\|T(x'_n) - \lambda x'_n\|_{Y/X} = 0$. Indeed, assume that (x_n) and (y_n) are such sequences, and assume that λ and μ are in $\mathbb{R}$ so that for some subsequences (x'_n) of (x_n) and (y'_n) of (y_n), $\lim_{n\to\infty} \|T(x'_n) - \lambda x'_n\|_{Y/X} = 0$ and $\lim_{n\to\infty} \|T(y'_n) - \mu y'_n\|_{Y/X} = 0$. For each $n \in \mathbb{N}$, choose $f_n^* \in S_{(Y/X)^*}$ such that $f_n^*(Q(x'_n)) = \|x'_n\|_{Y/X}$, where $Q : Y \to Y/X$ is the quotient map. The sequence (y'_n) is weakly null, thus after passing to a subsequence if necessary, we may assume that $|f_n^*(Q(y'_n))| < 2^{-n}$ for all $n \in \mathbb{N}$. Hence,

$$\begin{aligned}\liminf_{n\to\infty}\bigl\|\alpha x'_n + \beta y'_n\bigr\|_{Y/X} &\geqslant \bigl|Q^* f_n^*\bigl(\alpha x'_n + \beta y'_n\bigr)\bigr| \\ &\geqslant |\alpha| \inf_{n\in\mathbb{N}} \bigl\|x'_n\bigr\|_{Y/X} \quad \text{for every } \alpha, \beta \in \mathbb{R}.\end{aligned} \tag{14}$$

Using Proposition 2.7 we can, after passing to subsequences, if necessary, assume that $(x'_n + y'_n)$ is a $(2C)$-RIS. After passing to subsequences again, we assume that there is a $\rho \in \mathbb{R}$ so that $\lim_{n\to\infty} \|T(x'_n + y'_n) - \rho(x'_n + y'_n)\|_{Y/X} = 0$.

This implies that $\lim_{n\to\infty} \|\lambda x'_n + \mu y'_n - \rho(x'_n + y'_n)\|_{Y/X} = 0$. Hence $0 = \lim_{n\to\infty} \|(\lambda-\rho)x'_n + (\mu-\rho)y'_n\|_{Y/X} \geqslant (\lambda-\rho)\inf_{n\in\mathbb{N}} \|x'_n\|$. Thus, $\lambda = \rho$, as (x'_n) is seminormalized in Y/X. Thus $\lim_{n\to\infty} \|\mu y'_n - \rho y'_n\|_{Y/X} = 0$, and hence $\mu = \rho$ as (y'_n) is seminormalized in Y/X.

We claim now that $S = T - \lambda\,\mathrm{Id}$ is a weakly compact operator, which finishes our proof since by Schauder's theorem S is (weakly) compact if and only if S^* is (weakly) compact and since by Schur's theorem all weakly compact operators on ℓ_1 are norm compact.

First note, using Proposition 2.8, it follows that the operator $\tilde{S} : Y \to Y/X$, $x \mapsto Q \circ S(x)$, where $Q : Y \to Y/X$ is the quotient mapping, is norm compact. Hence for a given $\varepsilon > 0$ there is an $N = N_\varepsilon$ so that $\mathrm{dist}(S(x), 2\|S\|B_X) < \varepsilon$, whenever $x \in \bigoplus_{j=N+1}^\infty F_j \cap B_Y$. Thus $S(B_Y) \subset W_\varepsilon + \varepsilon B_Y$, where $W_\varepsilon = 2MS(B_{\bigoplus_{j=1}^N F_j}) + 4M\|S\|B_X$. We thus showed that for every $\varepsilon > 0$ there is a relatively weakly compact set $W_\varepsilon \subset Y$ so that $S(B_Y) \subset W_\varepsilon + \varepsilon B_Y$. We therefore deduce our claim from a well-known characterization of weakly compactness (cf. [9]). $\square$

3. Mixed Tsirelson spaces

Our method for constructing the space Z in Theorem A combines the technique of embedding a Banach space into a Bourgain–Delbaen space given in [10] with the technique of producing a Bourgain–Delbaen space with very few operators given in [2]. All existing methods of creating hereditarily indecomposable spaces or spaces with very few operators, involve the use of mixed Tsirelson spaces, the first example of such being given in [20]. We will use the same mixed Tsirelson space for constructing Z as was used in [2]. We recall the notation and terminology from [4]. Let $(\ell_j)_{j\in\mathbb{N}}$ be a sequence of positive integers and let $(\theta_j)_{j\in\mathbb{N}}$ be a sequence of real numbers with $0 < \theta_j < 1$. We define $W[(\mathcal{A}_{\ell_j}, \theta_j)_{j\in\mathbb{N}}]$ to be the smallest set W of c_{00} with the following properties:

(1) $\pm e_k^* \in W$ for all $k \in \mathbb{N}$, where such functionals are said to be of type 0;
(2) whenever $f_1^*, \dots, f_n^* \in W$ are successive vectors and $n \leqslant \ell_j$,

$$f^* = \theta_j \sum_{i=1}^n f_i^* \in W, \quad \text{where } f^* \text{ is said to be of type 1 with weight } \theta_j.$$

It is possible for a given f^* to have more than one weight, but this shall not pose a problem. The reader who is familiar with [4] should note that our W is there denoted by W_0'.

The *mixed Tsirelson space* $T[(\mathcal{A}_{\ell_j}, \theta_j)_{j\in\mathbb{N}}]$ is defined to be the completion of c_{00} with respect to the norm

$$\|x\| = \sup\bigl\{f^*(x) \bigm| f^* \in W\bigl[(\mathcal{A}_{\ell_j}, \theta_j)_{j\in\mathbb{N}}\bigr]\bigr\}.$$

The norm $\|\cdot\|$ on $T[(\mathcal{A}_{\ell_j}, \theta_j)_{j\in\mathbb{N}}]$ satisfies the following recursive relationship

$$\|x\| = \|x\|_\infty \vee \sup_{j\in\mathbb{N},\ E_1<\cdots<E_{\ell_j}} \theta_j \sum_{i=1}^{\ell_j} \|x\chi_{E_i}\| \quad \text{for all } x \in T\bigl[(\mathcal{A}_{\ell_j}, \theta_j)_{j\in\mathbb{N}}\bigr].$$

As in [2] we will work with two sequences (m_j) and (n_j) in $\mathbb{N}$ satisfying the following properties

$$\sqrt{n_1/2} > m_1 > 16, \qquad m_{j+1} \geqslant m_j^2,$$
$$n_{j+1} \geqslant (16n_j)^{\log_2(m_{j+1})} \geqslant m_{j+1}^2 (4n_j)^{\log_2 m_{j+1}}. \tag{15}$$

A simple way to achieve this is to have $(m_j, n_j)_{j\in\mathbb{N}}$ be a subsequence of $(2^{2^j}, 2^{2^{j^2+1}})_{j>4}$. The sequences (m_j) and (n_j) will be used to construct the space Z in a way somewhat analogous to the construction of $T[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\in\mathbb{N}}]$. In the sequel, we will need upper norm estimates for certain vectors in Z, and this will be achieved through the following upper norm estimates for certain vectors in $T[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\in\mathbb{N}}]$.

Proposition 3.1. *(See [2, Proposition 2.5].) If $j_0 \in \mathbb{N}$ and $f^* \in W[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\in\mathbb{N}}]$ is an element of weight m_h^{-1}, then*

$$\left| f^*\left(n_{j_0}^{-1} \sum_{\ell=1}^{n_{j_0}} e_\ell \right) \right| \leqslant \begin{cases} 2m_h^{-1} m_{j_0}^{-1} & \text{if } h < j_0, \\ m_h^{-1} & \text{if } h \geqslant j_0. \end{cases}$$

In particular, the norm of $n_{j_0}^{-1} \sum_{\ell=1}^{n_{j_0}} e_\ell$ in $W[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\in\mathbb{N}}]$ is exactly $m_{j_0}^{-1}$. If we make the additional assumption that $f^* \in W[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\neq j_0}$ then

$$\left| f^*\left(n_{j_0}^{-1} \sum_{\ell=1}^{n_{j_0}} e_\ell \right) \right| \leqslant \begin{cases} 2m_h^{-1} m_{j_0}^{-2} & \text{if } h < j_0, \\ m_h^{-1} & \text{if } h > j_0. \end{cases}$$

In particular, the norm of $n_{j_0}^{-1} \sum_{\ell=1}^{n_{j_0}} e_\ell$ in $W[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\neq j_0}]$ is at most $m_{j_0}^{-2}$.

4. Construction of the space Z

Our goal in this section is to prove the following theorem.

Theorem 4.1. *Let X be a separable reflexive Banach space with Szlenk index ω_0. Then X embeds in a Banach space Z, whose dual space is isomorphic to ℓ_1, and which has the property that all operators T on Z are of the form $T = \lambda\,\mathrm{Id} + K$, where* Id *denotes the identity, λ is a scalar and K is a compact operator on Z.*

Every separable uniformly convex infinite-dimensional Banach space is both reflexive and has Szlenk index ω_0. That every uniformly convex Banach space is reflexive is well known. That every uniformly convex Banach space has Szlenk index ω_0 actually follows immediately from the definition of the Szlenk index [21]. The Szlenk index of a Banach space X is essentially the supremum over $\varepsilon > 0$ of how long it takes to remove all the points from B_{X^*} by successively removing all relatively w^* open subsets of B_{X^*} with diameter at most ε. If X is uniformly convex, then X can be renormed so that B_{X^*} is also uniformly convex. Removing all relatively w^* open subsets of B_{X^*} of diameter at most ε will then leave a subset of $(1 - \delta(\varepsilon))B_{X^*}$, where $\delta(\varepsilon)$ is the modulus of convexity of B_{X^*}. Repeating the procedure n times will leave a subset of $(1 - \delta(\varepsilon))^n B_{X^*}$, which will eventually have diameter less than ε. Thus all of B_{X^*} will be removed in a finite number of steps, and then taking the supremum over $\varepsilon > 0$ gives that X has Szlenk index ω_0. Thus Theorem A is an immediate corollary of Theorem 4.1.

We will recall one of the embedding results established in [10], and discuss how the construction can be slightly modified to be better incorporated into the construction of [2]. The construction in [10] is quite technical and combines some of the upper estimate results from [11] with the general Bourgain–Delbaen construction. For the sake of simplicity, we will avoid getting too deep into the details. By [19], we may assume that X is reflexive with an FDD (E_i) which satisfies block upper ℓ_p estimates for some $1 < p < \infty$. The following theorem is one of the major results in [10] together with some of the useful properties given by the construction.

Theorem 4.2. *(See [10].) Let X be a Banach space with separable dual and a shrinking FDD (E_i). Then X embeds into an $\mathcal{L}_\infty$ space Y with Y^* isomorphic to ℓ_1. Furthermore, Y may be constructed to have the following properties*:

(a) *Y has a shrinking FDD denoted by (F_i) with decomposition constant at most 2 and projection operators denoted by $P_{[k,n]}$ for each $1 \leqslant k \leqslant n$.*
(b) *There exists a subsequence (k_i) of $\mathbb{N}$ such that $E_i \subset F_{k_i}$ for every $i \in \mathbb{N}$, and hence $P_{[k,n]}x \in X$ for all $x \in X$ and $k \leqslant n$.*

The space Y is constructed using Bourgain–Delbaen families of functionals as discussed in Section 2, and we keep the same notation. Before discussing how the construction in [10] may be adapted, we note the following.

Remark. In any Bourgain–Delbaen space, whenever $\gamma \in \Gamma$ is of the form $\gamma = (n, \beta, b^*)$ or $\gamma = (n, k, \xi, \beta, b^*)$ such that $0 \leqslant \beta \leqslant c$ for some constant c, we may recode γ to be $\gamma = (n, c, (\beta/c)b^*)$ or $\gamma = (n, k, \xi, c, (\beta/c)b^*)$. This keeps the equation for c^*_γ unchanged. Thus recoding in this manner will produce the same Bourgain–Delbaen space. This will make notation easier for us, and allow us to be more consistent with the construction in [2].

We now claim that the construction in [10] can be adapted to satisfy the following proposition.

Proposition 4.3. *Let X be a Banach space with separable dual and a shrinking FDD (E_i). Then X embeds into an $\mathcal{L}_\infty$ space Y with Y^* isomorphic to ℓ_1. Furthermore, Y can be constructed to satisfy Theorem 4.2 as well as the following*:

(c) *There is a constant $0 < c < \frac{1}{16}$ such that the weight of γ is c for every $\gamma \in \Gamma$ and $\frac{1}{c} \in \mathbb{N}$.*
(d) *For each $\gamma \in \Delta_n^{(1)}$, there exist $k < n - 1$, $\xi \in \Delta_k$, $\eta \in \Gamma_{n-1} \setminus \Gamma_k$, and $\beta \in [0, c]$ such that*

$$e^*_\gamma = e^*_\xi + \beta e^*_\eta + d^*_\gamma.$$

(e) *There is a constant $\ell_0 \in \mathbb{N}$ such that the age of γ is at most ℓ_0 for every $\gamma \in \Gamma$.*

Proof. We will not recall the full construction of [10], but will just give a brief discussion about how it can be simply modified. We let $0 < c < \frac{1}{16}$ be any constant satisfying $\frac{1}{c} \in \mathbb{N}$. Lemma 4.1 in [10] is used to obtain a countable set $D \subset B_{X^*}$ which $(1-\varepsilon)$-norms X (among other properties). We replace each $f^* \in D \setminus \bigcup E_i^*$ with $(c/2)f^*$. The construction of [10] then goes through, except that we obtain a $(c/2 - c\varepsilon/2)^{-1}$-embedding of X into Y instead of a $(1-\varepsilon)^{-1}$-embedding. Furthermore, this will result in the weight of γ being at most c for every $\gamma \in \Gamma$. Thus property (c) is satisfied by the remark after Theorem 4.2.

After our modification, we will have that for each $\gamma \in \Delta_{n+1}^{(1)}$, there exist $\eta, \xi \in \Gamma_n$ such that $c^*_\gamma = \alpha_\gamma e^*_\xi + \beta_\gamma e^*_\eta$ for some constants $\alpha_\gamma, \beta_\gamma \geqslant 0$ with $\beta_\gamma \leqslant c/2$ and either $\alpha_\gamma = 1$ or $\alpha_\gamma \leqslant c/2$.

In the case that $\alpha_\gamma \leqslant c/2$ we set $b^* = (\alpha_\gamma/c)e^*_\xi + (\beta_\gamma/c)e^*_\eta \in B_{\ell_1(\Gamma_n)}$, and hence $c^*_\gamma = cb^*$. In this case, we may then consider γ to have type 0, and hence we have that $\alpha = 1$ for every $\gamma \in \Gamma$ of type 1. Thus property (d) is satisfied.

In the construction of [10], the age of an element $\gamma \in \Gamma$ will be equal to the length of the optimal c-decomposition of a particular functional $f_\gamma \in B_{X^*}$. By optimal c-decomposition, we mean a block sequence $(f_i)_{i=1}^k$ in X^* such that $\sum_{i=1}^k f_i = f_\gamma$, $\|f_i + f_{i+1}\| \geqslant c$, and for each $1 \leqslant i \leqslant k$, either $\|f_i\| < c$ or $f_i \in E^*_j$ for some $j \in \mathbb{N}$. The condition that X satisfies block upper ℓ_p estimates implies that X^* satisfies block lower ℓ_q estimates. Hence, there exists a constant $C > 0$ such that if $(f_i)_{i=1}^k \subset X^*$ is a c-decomposition then $1 \geqslant \|f\| \geqslant \|\sum_{i=1}^{\lfloor k/2 \rfloor} f_{2i-1} + f_{2i}\| \geqslant Cc(\lfloor k/2 \rfloor)^{1/q}$. Thus there is a constant $\ell_0 \in \mathbb{N}$ such that the length of any c-decomposition of an element of X^* is at most ℓ_0, and hence the age of γ is at most ℓ_0 for every $\gamma \in \Gamma$. Thus property (e) is satisfied. □

In our next step we will use the construction in [2], and increase the sets Δ_n to sets $\overline{\Delta}_n = \Delta_n \cup \Theta_n$ in such a way, that X will still embed into a Bourgain–Delbaen space Z corresponding to the Bourgain–Delbaen sets $(\overline{\Delta}_n)$, and will have the additional property that all operators on Z will be compact perturbations of a scalar multiple of the identity.

Recall that we have two sequences (m_j) and (n_j) in $\mathbb{N}$ satisfying the properties of (15). We now also require that $1/c = m_1$ and $n_1 \geqslant 2\ell_0$.

By induction, we define for every $n \in \mathbb{N}$, sets $\Theta_n^{(0)}$ and $\Theta_n^{(1)}$. In the notation of Definition 2.1 we will always have $\alpha = 1$, and we will therefore suppress this dependency. We will write (n, m_i^{-1}, b^*) for elements in $\Theta_n^{(0)}$ and $(n, k, \xi, m_i^{-1}, b^*)$ for elements in $\Theta_n^{(1)}$.

We let $\Theta_1^{(0)} = \Theta_1^{(1)} = \emptyset$, and assuming we defined $\Theta_j^{(0)}$ and $\Theta_j^{(1)}$, for $j = 1, 2, \ldots, n$, we let $\overline{\Delta}_j^{(0)} = \Delta_j^{(0)} \cup \Theta_j^{(0)}$, $\overline{\Delta}_j^{(1)} = \Delta_j^{(1)} \cup \Theta_j^{(1)}$, $\Theta_j = \Theta_j^{(0)} \cup \Theta_j^{(1)}$, $\overline{\Delta}_j = \overline{\Delta}_j^{(0)} \cup \overline{\Delta}_j^{(1)}$, $\Lambda_j = \bigcup_{i=1}^j \Theta_j$ and $\overline{\Gamma}_j = \bigcup_{i=1}^j \overline{\Delta}_i$, for $j = 1, \ldots, n$. We also assume that, so far $(\overline{\Delta}_j)_{j=1}^n$ satisfies the conditions of Bourgain–Delbaen sets in Definition 2.1. The terms rank, type, weight, analysis and age of γ are therefore defined for all $\gamma \in \overline{\Gamma}_n$. Also the functionals $c^*_\gamma \in \ell_1(\overline{\Gamma}_n)$, as well as the projections $\overline{P}^*_{[p,n]}$ (on $\ell_1(\overline{\Gamma}_n)$) for $1 \leqslant p \leqslant n$, and the FDD $(\overline{F}_i)_{i=1}^n$ is defined.

Recall that in our original Bourgain–Delbaen space Y, we have $X \subseteq Y$ such that the FDD (E_i) for X fits nicely with the FDD (F_i) for Y. That is, for $i \in \mathbb{N}$ with $k_i \leqslant n$, we have that $E_i \subset F_{k_i}$. This provides the following (natural) embeddings

$$\bigoplus_{i, k_i \leqslant n} E_i \hookrightarrow \bigoplus_{j \leqslant n} F_j \equiv_M \ell_\infty(\Gamma_n).$$

We will identify $X \subset \ell_\infty(\Gamma)$ with $X \oplus 0 \subset \ell_\infty(\Gamma) \oplus \ell_\infty(\Lambda) = \ell_\infty(\overline{\Gamma})$. Specifically, if $b^* \in \ell_1(\overline{\Gamma}_n)$, we denote by $b^*|_X$ the functional defined by the restriction of b^* onto the space $\bigoplus_{i, k_i \leqslant n} E_i \oplus 0 \subset \ell_\infty(\Gamma_n) \oplus \ell_\infty(\Lambda_n) = \ell_\infty(\overline{\Gamma}_n)$.

Let $\varepsilon > 0$ and $\varepsilon_n \searrow 0$ such that $\sum \varepsilon_n < \varepsilon$. For $0 \leqslant p \leqslant n$, we choose finite sets $\overline{B}^*_{(p,n]}$ which form an ε_n net in the $\ell_1(\Gamma_n)$ norm for the set

$$\{b^* \in \ell_1(\overline{\Gamma}_n)\colon \|b^*\|_{\ell_1} \leqslant 1,\ b^*|_X \equiv 0,\ b^*|_{\overline{\Gamma}_p} \equiv 0\}.$$

We assume, without loss of generality, that $\overline{B}^*_{(p,n]} \subset \overline{B}^*_{(q,m]}$ if $q \leqslant p < n \leqslant m$. We will also require that if $\gamma \in \overline{\Gamma}_n \setminus \Gamma_n$ and $e^*_\gamma|_{\overline{\Gamma}_p} \equiv 0$ then $e^*_\gamma \in \overline{B}^*_{(p,n]}$. Note that the conditions $\|e^*_\gamma\|_{\ell_1} \leqslant 1$ and $e^*_\gamma|_X \equiv 0$ are automatically satisfied for $\gamma \in \overline{\Gamma} \setminus \Gamma$. For $f^* \in c_{00}(\overline{\Gamma})$ we define the *range* of f^* to be the smallest interval $\operatorname{range}(f^*) = [p, n]$ such that $f^*|_{\overline{\Gamma}_{p-1}} \equiv 0$ and $f^*|_{\overline{\Gamma} \setminus \overline{\Gamma}_n} \equiv 0$.

Our sets $\Theta^{(0)}_{n+1}$ and $\Theta^{(1)}_{n+1}$ will be divided into elements of even weight, $\frac{1}{m_{2j}}$ for some $1 \leqslant j \leqslant \lfloor \frac{n+1}{2} \rfloor$, and elements of odd weight, $\frac{1}{m_{2j-1}}$ for some $1 \leqslant j \leqslant \lfloor \frac{n+2}{2} \rfloor$. We will put extra constraints on elements of odd weight by using a coding function $\sigma : \Gamma \to \mathbb{N}$ as first introduced in [18]. This will allow us to use the elements of odd weight in a similar manner to how special functionals are used in [12] and other HI constructions. For our purposes, all we need is an injective function $\sigma : \Gamma \to \mathbb{N}$ satisfying $\sigma(\gamma) > \operatorname{rank}(\gamma)$ for all $\gamma \in \Gamma$. This function σ will be incorporated in to our recursive construction of Γ.

The sets $\Theta^{(0)}_n$ and $\Theta^{(1)}_n$ are now defined as follows

$$\Theta^{(0)}_{n+1} = \bigcup_{p=0}^{n-1} \bigcup_{j=1}^{\lfloor \frac{n+1}{2} \rfloor} \left\{ \left(n+1, p, \frac{1}{m_{2j}}, b^* \right) : \ b^* \in \overline{B}^*_{(p,n]} \right\}$$
$$\cup \bigcup_{p=0}^{n-1} \bigcup_{j=1}^{\lfloor \frac{n+2}{2} \rfloor} \left\{ \left(n+1, p, \frac{1}{m_{2j-1}}, e^*_\eta \right) : \begin{array}{l} \eta \in \Lambda_n, \ \min \operatorname{range}(e^*_\eta) > p, \\ \operatorname{wt}(\eta) = \frac{1}{m_{4i-2}}, \ \text{with } m_{4i-2} > n^2_{2j-1} \end{array} \right\} \tag{16}$$

and

$$\Theta^{(1)}_{n+1} = \bigcup_{p=2}^{n-1} \bigcup_{j=1}^{\lfloor \frac{p}{2} \rfloor} \left\{ \left(n+1, p, \xi, \frac{1}{m_{2j}}, b^* \right) : \ \xi \in \Theta_p, \ \operatorname{wt}(\xi) = \frac{1}{m_{2j}}, \ \operatorname{age}(\xi) < n_{2j}, \ b^* \in \overline{B}^*_{(p,n]} \right\}$$
$$\cup \bigcup_{p=2}^{n-1} \bigcup_{j=1}^{\lfloor \frac{p+1}{2} \rfloor} \left\{ \left(n+1, p, \xi, \frac{1}{m_{2j-1}}, e^*_\eta \right) : \begin{array}{l} \xi \in \Theta_p, \ \operatorname{wt}(\xi) = \frac{1}{m_{2j-1}}, \ \operatorname{age}(\xi) < n_{2j-1}, \\ \eta \in \Lambda_n, \ \min \operatorname{range}(e^*_\eta) > p, \ \operatorname{wt}(\eta) = \frac{1}{m_{4\sigma(\xi)}} \end{array} \right\}. \tag{17}$$

The sets $\overline{\Delta}_n = \Delta_n \cup \Theta_n$ form Bourgain–Delbaen sets as in Definition 2.1. By Proposition 2.3, we have that $(\bar{F}^*_n)$ is an FDD for $\ell_1(\overline{\Gamma})$ with decomposition constant not larger than 2. We let Z be the Bourgain–Delbaen space associated to $(\overline{\Delta}_n : n \in \mathbb{N})$, which again by Proposition 2.3 is an $\mathcal{L}_{\infty,2}$ space.

In our construction of $\overline{B}^*_{(p,n]}$ we required that $e^*_\gamma \in \overline{B}^*_{(p,n]}$ if $\gamma \in \overline{\Gamma} \setminus \Gamma$ and $\operatorname{range}(e^*_\gamma) \subset (p, n]$. In some circumstances, this allows us to conveniently combine all four possible cases for $\gamma \in \overline{\Gamma} \setminus \Gamma$ into one general case. For instance, if $\gamma \in \overline{\Gamma} \setminus \Gamma$ has age a and weight m_j^{-1}, then the evaluation analysis of γ is given by

$$e^*_\gamma = \sum_{i=1}^{a} d^*_{\xi_i} + m_j^{-1} \sum_{i=1}^{a} b^*_i,$$

where $b_i^* \in \overline{B}_{(p_{i-1},p_i]}$ and $\xi_i \in \Theta_{p_i}$ for some sequence of non-negative integers $(p_i)_{i=0}^a \subset \mathbb{N}_0$. It is important to point out that we do not include the projection operators $P^*_{(p_{i-1},p_i)}$ as we have guaranteed that $\min \operatorname{range}(b_i^*) > p_{i-1}$ and hence $P^*_{(p_{i-1},\infty)} b_i^* = b_i^*$.

Our first goal is to show that X is naturally isomorphic to a subspace of Z. We are given that $X \subset Y \subset \ell_\infty(\Gamma)$ and that $Z \subset \ell_\infty(\Gamma \cup \Lambda) = \ell_\infty(\Gamma) \oplus \ell_\infty(\Lambda)$. We identify X with $X \oplus 0 \subset \ell_\infty(\Gamma) \oplus \ell_\infty(\Lambda)$.

Lemma 4.4. *If $\gamma \in \overline{\Gamma} \setminus \Gamma$ then $e_\gamma^*|_X = c_\gamma^*|_X = d_\gamma^*|_X = 0$.*

Proof. We have that $e_\gamma^* = d_\gamma^* + c_\gamma^*$, and thus it will be sufficient for us to just prove that $c_\gamma^*|_X = 0$. We will prove this by induction on the rank of γ.

There are two possible cases for $\gamma \in \overline{\Gamma} \setminus \Gamma$. In the first case $\gamma = (n+1, p, m_j^{-1}, b^*) \in \Theta_{n+1}^{(0)}$ for some $b^* \in \overline{B}^*_{(p,n]}$. Thus $c_\gamma^*|_X = m_j^{-1} b^*|_X = 0$ as $b^*|_X = 0$ for all $b^* \in \overline{B}^*_{(p,n]}$.

In the second case, $\gamma = (n+1, p, \xi, m_j^{-1}, b^*) \in \Theta_{n+1}^{(1)}$ for some $b^* \in \overline{B}^*_{(p,n]}$ and $\xi \in \Theta_p$. We assume that $c_\eta^*|_X = 0$ for all $\eta \in \overline{\Gamma}_n \setminus \Gamma_n$. Thus $c_\gamma^*|_X = e_\xi^*|_X + m_{2j}^{-1} b^*|_X = 0$ as $e_\xi^*|_X = 0$ and $b^*|_X = 0$ for all $b^* \in \overline{B}^*_{(p,n]}$. $\square$

The following theorem, whose proof we omit, now follows from Lemma 4.4.

Theorem 4.5. *The space $X \oplus 0 \subset \ell_\infty(\Gamma) \oplus \ell_\infty(\Lambda)$ is a subspace of $Z \subset \ell_\infty(\Gamma) \oplus \ell_\infty(\Lambda) = \ell_\infty(\overline{\Gamma})$.*

As in [2], we have the following proposition.

Proposition 4.6. *Let $\gamma, \gamma' \in \Theta_{n+1}^{(1)}$ each have the same odd weight $\frac{1}{m_{2j-1}}$ and have respective analyses $(p_i, e^*_{\eta_i}, \xi_i)_{1 \leqslant i \leqslant a}$ and $(p_i', e^*_{\eta_i'}, \xi_i')_{1 \leqslant i \leqslant a'}$. If $a \geqslant a'$ then there exists $1 \leqslant \ell \leqslant a$ such that $\xi_i' = \xi_i$ for all $i < \ell$ and $\mathrm{wt}(\eta_j) \neq \mathrm{wt}(\eta_i')$ for all j and all $\ell < i \leqslant a'$.*

Proof. We choose $1 \leqslant \ell \leqslant a'$ to be maximal such that $\xi_i' = \xi_i$ for all $i < \ell$. If $\ell = a'$ then the proposition holds. If $\ell < a'$ it must be that $\xi_\ell' \neq \xi_\ell$, and hence $\mathrm{wt}(\eta_\ell) = m_{4\sigma(\xi_\ell)} \neq m_{4\sigma(\xi_\ell')} = \mathrm{wt}(\eta_\ell')$. In this setup, ages are simply given by $\mathrm{age}(\xi_i) = i$ and $\mathrm{age}(\xi_j') = j$ for all $1 \leqslant j \leqslant a$ and $1 \leqslant i \leqslant a'$. Thus whenever $i \neq j$ we have that $\mathrm{wt}(\eta_j) = m_{4\sigma(\xi_j)} \neq m_{4\sigma(\xi_i')} = \mathrm{wt}(\eta_i')$. If $j > \ell$ then the analysis of ξ_j is $(p_i, e^*_{\eta_i}, \xi_i)_{1 \leqslant i \leqslant j-1}$ and the analysis of ξ_j' is $(p_i', e^*_{\eta_i'}, \xi_i')_{1 \leqslant i \leqslant j-1}$. The elements ξ_j and ξ_j' clearly have different analyses as $\xi_\ell' \neq \xi_\ell$, and thus $\xi_j \neq \xi_j'$. We then have that $\mathrm{wt}(\eta_j) = m_{4\sigma(\xi_j)} \neq m_{4\sigma(\xi_j')} = \mathrm{wt}(\eta_j')$. We have covered all the cases, and thus the proposition is proven. $\square$

If we are given some $\gamma \in \overline{\Gamma}$ then we can find the analysis of γ through simple iteration. Conversely, it will be important for us to be able to choose an element $\gamma \in \overline{\Gamma}$ which has some specified analysis. The following lemmas state essentially that if we satisfy some important conditions, then we are able to choose such a γ.

Lemma 4.7. *Let a, j be positive integers such that $a \leqslant n_{2j}$. If $p_0 < p_0 + 1 < p_1 < p_1 + 1 < p_2 < \cdots < p_a$ are natural numbers with $2j \leqslant p_1$ and b_r^* is a functional in $\overline{B}^*_{(p_{r-1},p_r-1]}$ for all*

$1 \leqslant r \leqslant a$, *then there are elements* $\xi_r \in \Theta_{p_r}$ *each with weight* $\frac{1}{m_{2j}}$ *such that the analysis of* $\gamma = \xi_a$ *is* $(p_r, b_r^*, \xi_r)_{r=1}^a$.

Proof. It is specified that $2j \leqslant p_1$, and thus $(p_1, p_0, \frac{1}{m_{2j}}, b_1^*) \in \Theta_{p_1}^{(0)}$. We now assume that $1 \leqslant k < a$ and that ξ_k has been found with analysis $(p_r, b_r^*, \xi_r)_{r=1}^k$ and weight $\frac{1}{m_{2j}}$. We have that $\mathrm{age}(\xi_k) = k < a \leqslant n_{2j}$, and so $\xi_{k+1} = (p_{k+1}, p_k, \xi_k, \frac{1}{m_{2j}}, b_{k+1}^*) \in \Theta_{p_{k+1}}^{(1)}$. Thus $e_{\xi_{k+1}}^* = d_{\xi_{k+1}}^* + e_{\xi_k}^* + \frac{1}{m_{2j}} b_{k+1}^*$, and hence the analysis of ξ_{k+1} is $(p_r, b_r^*, \xi_r)_{r=1}^{k+1}$. The proof is then complete by induction. □

A similar proof yields the following lemma.

Lemma 4.8. *Let* a, j_0 *be positive integers such that* $a \leqslant n_{2j_0-1}$. *Let* $p_0 < p_0 + 1 < p_1 < p_1 + 1 < p_2 < \cdots < p_a$ *be natural numbers with* $2j_0 - 1 \leqslant p_1$. *Let* $(\eta_r)_{r=1}^a \subset \Lambda$ *with* $\mathrm{range}(e_{\eta_r}^*) \subset (p_{r-1}, p_r]$ *such that* $\mathrm{wt}(\eta_1) = \frac{1}{m_{4j_1-2}}$ *for some* $j_1 \in \mathbb{N}$ *with* $m_{4j_1-2} > n_{2j_0-1}^2$ *and* $\mathrm{wt}(\eta_r) = \frac{1}{m_{4\sigma(\eta_{r-1})}}$ *for all* $2 \leqslant r \leqslant a$. *Then there exist elements* $\xi_r \in \Theta_{p_r}$ *each with weight* $\frac{1}{m_{2j_0-1}}$ *such that the analysis of* $\gamma = \xi_a$ *is* $(p_r, e_{\eta_r}^*, \xi_r)_{r=1}^a$.

Lemma 4.9. *If* $p < q$ *and* $x \in \bigoplus_{i=p+1}^q F_i$ *such that* $\|x\|_{Z/X} = 1$ *then there exists* $b^* \in \overline{B}_{(p,q]}^*$ *with* $b^*(x) > \frac{1}{8} - \varepsilon_q$.

Proof. As $\|x\|_{Z/X} = 1$, there exists $x^* \in S_{Z^*}$ such that $x^*(x) = 1$ and $x^*|_X = 0$. We then set $b_0^* = \frac{1}{8} \overline{P}_{[p+1,q]}^* x^*$. We have that $\|b_0^*\|_{\ell_1(\Gamma_q)} \leqslant 2\|b_0^*\| \leqslant \frac{1}{4} \|\overline{P}_{[p+1,q]}^*\| \|x^*\| \leqslant 1$. If $x_0 \in X$, then our particular embedding of X into Z results in $\overline{P}_{[p+1,q]} x_0 \in X$. Thus $b_0^*(x_0) = \frac{1}{8} x^*(\overline{P}_{[p+1,q]} x) = 0$, as $x^*|_X = 0$. Combining these properties gives that $b_0^* \in \{b^* \in \bigoplus_{i=p+1}^n \bar{F}_i^*\colon \|b^*\|_{\ell_1} \leqslant 1,\ b^*|_X \equiv 0\}$, and hence there exists $b^* \in \overline{B}_{(p,q]}^*$ such that $\|b^* - b_0^*\| \leqslant \varepsilon_q$. Thus we have that $b^*(x) > b_0^*(x) - \varepsilon_q \geqslant \frac{1}{8} - \varepsilon_q$. □

Lemma 4.10. *Let* $(x_r)_{r=1}^a$ *be a skipped block sequence in* Z *with* $a \leqslant n_{2j}$ *and* $2j \leqslant \min ran(x_2)$. *Then there exists* $\gamma \in \Lambda$ *of weight* $\frac{1}{m_{2j}}$ *such that* $\sum_{i=1}^a x_i(\gamma) \geqslant \frac{1-8\varepsilon}{8m_{2j}} \sum_{i=1}^a \|x_i\|_{Z/X}$.

Proof. Choose $p_0 < p_1 < p_2 < \cdots < p_a$ such that $ran(x_r) \subset (p_{r-1}, p_r)$ for all $1 \leqslant r \leqslant a$. By Lemma 4.9 we may choose $b_r^* \in \overline{B}_{(p_{r-1}, p_r)}^*$ such that $b_r^*(x_r) \geqslant \frac{1-8\varepsilon_{p_r-1}}{8} \|x\|_{Z/X}$. By Lemma 4.7 there exists $\xi_r \in \Theta_{p_r}$ for each $1 \leqslant r \leqslant a$ with weight $\frac{1}{m_{2j}}$ such that the analysis of $\gamma = \xi_a$ is $(p_r, b_r^*, \xi_r)_{r=1}^a$. We first note that $d_{\xi_i}^*(x_r) = 0$ for all i, r because $\xi_i \in \Theta_{p_i}$ and $p_i \notin ran(x_r)$ for all i, r. We further note that $b_i^*(x_r) = 0$ for all $i \neq r$ because $ran(x_r), ran(b_r^*) \subset (p_{r-1}, p_r)$. We now obtain the lower estimate for $\sum_{i=1}^a x_i(\gamma) = e_\gamma^*(\sum_{i=1}^a x_i)$ by using the evaluation analysis of γ

$$e_\gamma^*\left(\sum_{r=1}^a x_r\right) = \sum_{i=1}^a d_{\xi_i}^*\left(\sum_{r=1}^a x_r\right) + \frac{1}{m_{2j}} \sum_{i=1}^a b_i^*\left(\sum_{r=1}^a x_r\right) = \frac{1}{m_{2j}} \sum_{r=1}^a b_r^*(x_r)$$

$$> \frac{1}{m_{2j}} \sum_{r=1}^a \frac{1 - 8\varepsilon_{p_r-1}}{8} \|x_r\|_{Z/X}. \qquad \square$$

To simplify some proofs we will now add an additional condition to the definition of C-RIS.

Definition 4.11. Let (x_n) be a block basis in Z and $C > 0$. We say (x_n) is a *C-Rapidly Increasing Sequence*, or C-RIS, if for $k \in \mathbb{N}$:

(1) $\|x_k\| \leqslant C$.
(2) $|x_k(\gamma)| \leqslant C\, weight(\gamma)$ if $k \geqslant 2$ and $\gamma \in \overline{\Gamma}$ with $weight(\gamma) \geqslant m^{-1}_{\max \mathrm{rg}(x_{k-1})}$.
(3) $|x_k(\gamma)| \leqslant C m_1^{-1}$ if $\gamma \in \overline{\Gamma}$ with $weight(\gamma) = m_1^{-1}$.

Adding condition (3) is not a significant change for us since if $(x_k)_{k=1}^{\infty}$ is a C-RIS for Definition 2.6, then $(x_k)_{k=2}^{\infty}$ is a C-RIS for Definition 4.11.

It will be essential to obtain certain upper bounds on values of the form $|e^*_\gamma \sum_{k \in I} \lambda_k x_k|$ where (x_k) is a C-RIS. Estimating these bounds for $\gamma \in \overline{\Gamma} \setminus \Gamma$ will follow proofs similar to those in [2].

Lemma 4.12. *Let (x_k) be a C-RIS in Z. If $\gamma \in \overline{\Gamma} \setminus \Gamma$ and $\mathrm{wt}(\gamma) = m_i^{-1}$ then,*

$$\left|e^*_\gamma(x_k)\right| \leqslant C m_i^{-1} \quad \text{if } i < \max ran(x_{k-1}) \text{ or if } i > \max ran(x_k).$$

Proof. By the definition of C-RIS, the case $i < \max ran(x_{k-1})$ is immediate. The evaluation analysis for γ is given by

$$e^*_\gamma = \sum_{r=1}^{a} d^*_{\xi_r} + \frac{1}{m_i} b^*_r,$$

for some $(\xi_r)_{r=1}^a \subset \overline{\Gamma} \setminus \Gamma$ and $(b^*_r)_{r=1}^a \subset B_{\ell_1(\Gamma)}$. The element ξ_r has weight m_i^{-1}, for each $1 \leqslant r \leqslant a$. This is important as the set Θ_p contains elements of weight m_i^{-1} only if $p \geqslant i$. Thus if we consider p_1 so that $\xi_1 \in \Theta_{p_1}$ then $p_1 \geqslant i > \max ran(x_k)$. Hence, $\min \mathrm{range}(d^*_{\xi_r}) \geqslant p_1 > \max \mathrm{range}(x_k)$ for all $1 \leqslant r \leqslant a$ and $\min \mathrm{range}(b^*_r) > p_1 > \max ran(x_k)$ for all $1 < r \leqslant a$. Thus we have that $d^*_{\xi_r}(x_k) = 0$ for all $1 \leqslant r \leqslant a$, and $b^*_r(x_k) = 0$ for all $1 < r \leqslant a$. Applying this to the evaluation analysis for e^*_γ gives the following desired result,

$$\left|e^*_\gamma(x_k)\right| = \left|\sum_{r=1}^{a} d^*_{\xi_r}(x_k) + \frac{1}{m_i} b^*_r(x_k)\right| = \left|\frac{1}{m_i} b^*_1(x_k)\right| \leqslant \frac{1}{m_i}\left\|b^*_1\right\| \|x_k\| \leqslant \frac{C}{m_i}. \qquad \square$$

Lemma 4.13. *Let $(x_k)_{k \in I}$ be a C-RIS in Z for some interval $I \subseteq \mathbb{N}$, let λ_k be real numbers, and let γ be an element of Γ. There exists a functional $g^* \in W[(\mathcal{A}_{n_1}, m_1^{-1})]$ such that*:

(1) $\mathrm{supp}(g^*) \subset I$.
(2) $|e^*_\gamma(\sum_{k \in I} \lambda_k x_k)| \leqslant C g^*(\sum_{k \in I} |\lambda_k| e_k)$.

Proof. We proceed by induction on the rank of $\gamma \in \Gamma$. If $\mathrm{rank}(\gamma) = 1$ then

$$e^*_\gamma\left(\sum_{k \in I} \lambda_k x_k\right) = \lambda_{k_0} e^*_\gamma(x_{k_0}) \quad \text{where } k_0 = \min(I).$$

We may thus simply take $g^* = e^*_{k_0} \in W[(\mathcal{A}_{n_1}, m_1^{-1})]$.

We now assume that $\gamma \in \Gamma$ has rank greater than 1 and age a. We assume that the lemma holds for all elements of Γ with rank less than that of γ. We consider the evaluation analysis of e^*_γ, which is given by

$$e^*_\gamma = \sum_{i=1}^{a} d^*_{\xi_i} + \sum_{i=1}^{a} \beta_i e^*_{\eta_i},$$

for some sequences $(\xi_i)_{i=1}^a, (\eta_i)_{i=1}^a \subset \Gamma$ and $(\beta_i) \subset [0, m_1^{-1}]$. Recall that ℓ_0 was chosen so that $\mathrm{age}(\eta) \leqslant \ell_0$ for all $\eta \in \Gamma$. Let $(p_i)_{i=0}^a \subset \mathbb{N}$ be the sequence such that $\xi_i \in \Delta_{p_i}$ for all $1 \leqslant i \leqslant a$ and $p_0 = 0$. Let $I_0 = \{k \in I\colon\ p_r \in \mathrm{range}(x_k)$ for some $1 \leqslant r \leqslant a\}$. As (x_k) is a block sequence, for each $1 \leqslant r \leqslant a$ there is at most one $k \in I$ such that $p_r \in \mathrm{range}(x_k)$. Thus, $|I_0| \leqslant a \leqslant \ell_0$. We then set $I_r = \{k \in I\colon \mathrm{range}(x_k) \subset (p_{r-1}, p_r)\}$. Note that each I_r is an interval, and if $k \notin \bigcup_{r=0}^a I_r$, then $e^*_\gamma(x_k) = 0$. We now have the following equality,

$$e^*_\gamma\Big(\sum \lambda_k x_k\Big) = e^*_\gamma\Big(\sum_{k\in I_0} \lambda_k x_k\Big) + \sum_{r=1}^{a} \beta_r e^*_{\eta_r}\Big(\sum_{k\in I_r} \lambda_k x_k\Big).$$

For $k \in I_0$, we apply condition (3) in the definition of C-RIS to get the estimate $|e^*_\gamma(x_k)| \leqslant Cm_1^{-1}$. Thus we now have that,

$$\Big|e^*_\gamma\Big(\sum \lambda_k x_k\Big)\Big| \leqslant Cm_1^{-1} \sum_{k\in I_0} |\lambda_k| + \sum_{r=1}^{a} \Big|\beta_r e^*_{\eta_r}\Big(\sum_{k\in I_r} \lambda_k x_k\Big)\Big|. \tag{18}$$

For each $1 \leqslant r \leqslant a$, we apply the induction hypothesis to $\eta_r \in \Gamma$ to obtain $g^*_r \in W[(\mathcal{A}_{n_1}, m_1^{-1})]$ with $\mathrm{supp}(g^*_r) \subset I_r$, such that

$$\Big|e^*_{\eta_r}\Big(\sum_{k\in I_r} \lambda_k x_k\Big)\Big| \leqslant C g^*_r\Big(\sum_{k\in I_r} |\lambda_k| e_k\Big). \tag{19}$$

We now define g^* by setting $g^* = m_1^{-1}(\sum_{k\in I_0} e^*_k + \sum_{r=1}^a g^*_r)$. This is a sum, weighted by m_1^{-1} of at most $2\ell_0 \leqslant n_1$ functionals in $W[(\mathcal{A}_{n_1}, m_1^{-1})]$ which are each supported on successive intervals. Thus $g^* \in W[(\mathcal{A}_{n_1}, m_1^{-1})]$. We now use (18) and (19) to obtain the following

$$\begin{aligned}
\Big|e^*_\gamma\Big(\sum \lambda_k x_k\Big)\Big| &\leqslant Cm_1^{-1} \sum_{k\in I_0} |\lambda_k| + \sum_{r=1}^{a} \Big|\beta_r e^*_{\eta_r}\Big(\sum_{k\in I_r} \lambda_k x_k\Big)\Big| \quad \text{by (18)} \\
&\leqslant Cm_1^{-1} \sum_{k\in I_0} |\lambda_k| + \sum_{r=1}^{a} |\beta_r| C g^*_r\Big(\sum_{k\in I_r} |\lambda_k| e_k\Big) \quad \text{by (19)} \\
&\leqslant Cm_1^{-1} \sum_{k\in I_0} |\lambda_k| + \sum_{r=1}^{a} m_1^{-1} C g^*_r\Big(\sum_{k\in I_r} |\lambda_k| e_k\Big) \quad \text{as } |\beta_r| \leqslant m_1^{-1}
\end{aligned}$$

$$
\begin{aligned}
&= Cm_1^{-1}\Bigg(\sum_{k\in I_0} e_k^* + \sum_{r=1}^{a} g_r^*\Bigg)\\
&\quad \times \Bigg(\sum_{k\in I_0} |\lambda_k| e_k + \sum_{r=1}^{a}\sum_{k\in I_r} |\lambda_k| e_k\Bigg) \quad \text{as } \operatorname{supp}\big(g_r^*\big) \subset I_r\\
&= Cg^*\Bigg(\sum_{k\in I} |\lambda_k| e_k\Bigg). \qquad \square
\end{aligned}
$$

Proposition 4.14 *(Basic inequality). Let $(x_k)_{k\in I}$ be a C-RIS in Z for some interval $I \subseteq \mathbb{N}$, let λ_k be real numbers, and let γ be an element of $\overline{\Gamma}$. There exist $k_0 \in I$ and a functional $g^* \in W[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\in\mathbb{N}}]$ such that*:

(1) *either $g^* = 0$ or weight(g^*) = weight(γ) and* $\operatorname{supp}(g^*) \subset \{k \in I\colon k > k_0\}$;
(2) $|e_\gamma^*(\sum_{k\in I} \lambda_k x_k)| \leqslant 2C|\lambda_{k_0}| + 2Cg^*(\sum_{k\in I} |\lambda_k| e_k)$.

Moreover, if j_0 is such that $|e_\xi^*(\sum_{k\in J} \lambda_k x_k)| \leqslant C \max_{k\in J} |\lambda_k|$ for all subintervals J of I and all $\xi \in \overline{\Gamma}$ of weight $m_{j_0}^{-1}$, then we may choose g^* to be in $W[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\neq j_0}]$.

Proof. We assume that $\gamma \in \Gamma$ and will show that the moreover part holds. For $\gamma \in \Gamma$, the rest of the proposition is an obvious corollary of Lemma 4.13. We first consider $j_0 \neq 1$. By Lemma 4.13, there exists $g^* \in W[(\mathcal{A}_{n_1}, m_1^{-1})]$ satisfying (1) and (2). Thus $g^* \in W[(\mathcal{A}_{n_1}, m_1^{-1})] \subset W[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\neq j_0}]$, which proves the moreover part. We now consider $j_0 = 1$, and assume that $|e_\xi^*(\sum_{k\in J} \lambda_k x_k)| \leqslant C \max_{k\in J} |\lambda_k|$ for all subintervals J of I and all $\xi \in \overline{\Gamma}$ of weight m_1^{-1}. However, γ has weight m_1^{-1}, and thus we may take $g^* = 0 \in W[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\neq j_0}]$. Thus the proposition is true for all $\gamma \in \Gamma$.

We now consider the case $\gamma \in \overline{\Gamma} \setminus \Gamma$, and will proceed by induction on the rank of γ. There are no $\gamma \in \overline{\Gamma} \setminus \Gamma$ of rank 1, and thus we first consider the case that rank$(\gamma) = 2$. We then have that

$$
e_\gamma^*\Bigg(\sum_{k\in I} \lambda_k x_k\Bigg) = \lambda_{k_0} e_\gamma^*(x_{k_0}) + \lambda_{k_1} e_\gamma^*(x_{k_1}),
$$

where k_0 and k_1 are the first two elements of I. Thus setting $g^* = e_{k_1}^*$ gives the desired inequality.

We now assume that $\gamma \in \overline{\Gamma} \setminus \Gamma$ has rank greater than 2, age a, and weight m_h^{-1}. We suppose that there is some $\ell \in I$ such that $\max\operatorname{range}(x_{\ell-1}) < h \leqslant \max\operatorname{range}(x_\ell)$. The simpler cases of $h \leqslant \max\operatorname{range}(x_1)$ or $\max\operatorname{range}(x_k) < h$ for all $k \in I$ can be proved in the same way, and so will not be considered. We will split the following summation into three parts, and estimate each part separately

$$
e_\gamma^*\Bigg(\sum_{k\in I} \lambda_k x_k\Bigg) = \sum_{k\in I,\ k<\ell} \lambda_k e_\gamma^*(x_k) + \lambda_\ell e_\gamma^*(x_\ell) + e_\gamma^*\Bigg(\sum_{k\in I,\ k>\ell} \lambda_k x_k\Bigg). \tag{20}
$$

For the first part, we have by our choice of ℓ that $h > \max\operatorname{range}(x_k)$ for all $k < \ell$. Thus for $k < \ell$, Lemma 4.12 gives us $|e_\gamma^*(x_k)| \leqslant Cm_h^{-1}$. Furthermore, the inequality $\max\operatorname{range}(x_\ell) < h$ implies that $\sharp\{k \in I\colon k < \ell\} < h$, and thus trivially $\sharp\{k \in I\colon k < \ell\} m_h^{-1} < 1$. We now have the following upper bound

$$\left|\sum_{k\in I,\ k<\ell} \lambda_k e_\gamma^*(x_k)\right| \leqslant C \sum_{k<\ell} m_h^{-1}|\lambda_k| \leqslant C\sharp\{k \in I\colon k < \ell\} m_h^{-1} \max_{k<\ell}|\lambda_k| < C \max_{k<\ell}|\lambda_k|.$$

For the second term we have the trivial bound

$$\left|\lambda_\ell e_\gamma^*(x_\ell)\right| \leqslant C|\lambda_\ell|.$$

Thus combining the first two terms gives the inequality

$$\left|e_\gamma^*\left(\sum_{k\in I,\ k\leqslant\ell} \lambda_k x_k\right)\right| \leqslant C \max_{k\in I,\ k<\ell}|\lambda_k| + C|\lambda_\ell| \leqslant 2C|\lambda_{k_0}|, \tag{21}$$

for some particular $k_0 \leqslant \ell$.

We now set $I' = \{k \in I\colon k > \ell\}$, and focus on estimating the last term: $|e_\gamma^*(\sum_{k\in I'} \lambda_k x_k)|$. The evaluation analysis of e_γ^* is given by

$$e_\gamma^* = \sum_{r=1}^{a} d_{\xi_r}^* + m_h^{-1} \sum_{r=1}^{a} b_r^*,$$

for some $(\xi_r)_{r=1}^a \subset \overline{\Gamma} \setminus \Gamma$ and $(b_r^*)_{r=1}^a \subset B_{\ell_1(\overline{\Gamma})}$. Let $(p_r)_{r=1}^a \subset \mathbb{N}$ be the sequence such that $\xi_r \in \Theta_{p_r}$ for each $1 \leqslant r \leqslant a$. This implies that $b_r^* \in \overline{B}^*_{(p_{r-1},p_r)}$ for all $1 \leqslant r \leqslant a$, where we set $p_0 = 0$. Let $I'_0 = \{k \in I'\colon p_r \in \operatorname{range}(x_k)$ for some $1 \leqslant r \leqslant a\}$. As (x_k) is a block sequence, for each $1 \leqslant r \leqslant a$ there is at most one $k \in I'$ such that $p_r \in \operatorname{range}(x_k)$. We then set $I'_r = \{k \in I'\colon \operatorname{range}(x_k) \subset (p_{r-1}, p_r)\}$. Note that, if $k \in I' \setminus \bigcup_{r=0}^a I'_r$, then $e_\gamma^*(x_k) = 0$. We now have the following equality,

$$e_\gamma^*\left(\sum_{k\in I'} \lambda_k x_k\right) = \sum_{k\in I'_0} \lambda_k e_\gamma^*(x_k) + m_h^{-1} \sum_{r=1}^{a} b_r^*\left(\sum_{k\in I'_r} \lambda_k x_k\right). \tag{22}$$

As $b_r^* \in B_{\ell_1(\overline{\Gamma})} \cap \bigoplus_{p_{r-1}+1}^{p_r-1} F_i^*$, we have $b_r^* = \sum_{\eta\in\overline{\Gamma}_{p_r-1}} \alpha_\eta e_\eta^*$ for some α_η with $\sum|\alpha_\eta| \leqslant 1$. Thus we may choose $\eta_r \in \overline{\Gamma}_{p_r-1}$ such that

$$\left|b_r^*\left(\sum_{k\in I'_r} \lambda_k x_k\right)\right| \leqslant \left|e_{\eta_r}^*\left(\sum_{k\in I'_r} \lambda_k x_k\right)\right|. \tag{23}$$

Note that $e_\eta^*(x_k) = 0$ for all $\eta \in \overline{\Gamma}_{p_{r-1}}$ and $k \in I'_r$, and thus we may assume that $p_{r-1} < \operatorname{rank}(\eta_r) < p_r$. For each r, we apply the induction hypothesis to $\eta_r \in \overline{\Gamma}$ and the C-RIS $(x_k)_{k\in I'_r}$, obtaining $k_r \in I_r'$ and a functional $g_r^* \in W[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\in\mathbb{N}}]$ supported on $\{k \in I_r'\colon k > k_r\}$ satisfying

$$\left|e_{\eta_r}^*\left(\sum_{k\in I_r'}\lambda_k x_k\right)\right| \leqslant 2C|\lambda_{k_r}| + 2Cg_r^*\left(\sum_{k\in I_r'}|\lambda_k|e_k\right). \tag{24}$$

We now define g^* by setting

$$g^* = m_h^{-1}\left(\sum_{k\in I_0'} e_k^* + \sum_{r=1}^{a}\left(e_{k_r}^* + g_r^*\right)\right). \tag{25}$$

This is a sum, weighted by m_h^{-1} of at most $3n_h$ functionals in $W[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\in\mathbb{N}}]$ which are each supported on successive intervals. Thus $g^* \in W[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\in\mathbb{N}}]$. We now obtain the following estimates

$$\begin{aligned}
\left|e_\gamma^*\left(\sum\lambda_k x_k\right)\right| &\leqslant 2C|\lambda_{k_0}| + Cm_h^{-1}\sum_{k\in I_0'}|\lambda_k| \\
&\quad + m_h^{-1}\sum_{r=1}^{a}\left|b_r^*\left(\sum_{k\in I_r'}\lambda_k x_k\right)\right| \quad \text{by (20), (21) and (22)} \\
&\leqslant 2C|\lambda_{k_0}| + Cm_h^{-1}\sum_{k\in I_0'}|\lambda_k| + m_h^{-1}\sum_{r=1}^{a}\left|e_{\eta_r}^*\left(\sum_{k\in I_r'}\lambda_k x_k\right)\right| \quad \text{by (23)} \\
&\leqslant 2C|\lambda_{k_0}| + 2Cm_h^{-1}\left(\sum_{k\in I_0'}|\lambda_k| + \sum_{r=1}^{a}|\lambda_{k_r}| + g_r^*\left(\sum_{k\in I_r'}|\lambda_k|e_k\right)\right) \quad \text{by (24)} \\
&= 2C|\lambda_{k_0}| + 2Cg^*\left(\sum_{k\in I'}|\lambda_k|e_k\right) \quad \text{by (25).}
\end{aligned}$$

If j_0 satisfies the moreover condition in the statement of the proposition we proceed by the same induction. The base case is the same. When we prove the inductive step for γ with weight m_h^{-1} we need to consider separately the cases $h \neq j_0$ and $h = j_0$. For the case $h \neq j_0$, the proof remains unchanged as we are able to assume by induction that $g_r^* \in W[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\neq j_0}]$. Thus when we define g^* as in (25) we have that $g^* \in W[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\neq j_0}]$.

For the remaining case $h = j_0$, the moreover assumption gives automatically that

$$\left|e_\gamma^*\left(\sum_{k\in I}\lambda_k x_k\right)\right| \leqslant C\max_{k\in I}|\lambda_k|.$$

Thus we are able to take $g^* = 0$. □

The basic inequality relates the functionals e_γ^* with $\gamma \in \overline{\Gamma}$ to functionals in the mixed Tsirelson space $W[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\in\mathbb{N}}]$. Recall that Proposition 3.1 gives us good estimates for what happens when we apply functionals in mixed Tsirelson spaces to averages. We combine the basic inequality with Proposition 3.1 to obtain the following lemma which will be used extensively in proving our main result.

Lemma 4.15. *Let $\bar{x} = (x_k)_{k=1}^{n_{j_0}}$ be a skipped block C-RIS in Z with $j_0 > 1$. Then the vector $z(j_0, \bar{x}) := z := \frac{m_{j_0}}{n_{j_0}} \sum_{k=1}^{n_{j_0}} x_k$ has the following four properties.*

(1) $d_\xi^*(z) \leqslant \frac{3Cm_{j_0}}{n_{j_0}} < \frac{C}{m_{j_0}}$ *for all* $\xi \in \overline{\Gamma}$.
(2) $\|z\| < 3C$.
(3) *For all* $\gamma \in \overline{\Gamma} \setminus \Gamma$ *with weight* $\frac{1}{m_h}$ *such that* $h \neq j_0$ *we have*

$$\left|e_\gamma^*(z)\right| \leqslant \begin{cases} 5Cm_h^{-1} & \text{if } h < j_0, \\ 3Cm_{j_0}^{-1} & \text{if } h > j_0. \end{cases}$$

(4) $|e_\gamma^*(z)| < Cm_{j_0}^{-1}$ *for all* $\gamma \in \Gamma$.

Proof. Let $\xi \in \overline{\Gamma}$. We have that (x_k) is a block sequence and hence $d_\xi^*(\sum x_k) = d_\xi^*(x_m)$ for some $1 \leqslant m \leqslant n_{j_0}$. The inequality in (1) is given by $|d_\xi^*(\frac{m_{j_0}}{n_{j_0}} \sum_{k=1}^{n_{j_0}} x_k)| = |d_\xi^*(\frac{m_{j_0}}{n_{j_0}} x_m)| \leqslant \|d_\xi^*\| \frac{m_{j_0}}{n_{j_0}} \|x_k\| \leqslant \frac{3Cm_{j_0}}{n_{j_0}} < \frac{C}{m_{j_0}}$, since $\|d_\xi^*\| \leqslant 3$.

To obtain the inequality in (2), let $\gamma \in \overline{\Gamma}$ then apply the basic inequality to $e_\gamma^*(z)$ to obtain

$$\begin{aligned} \left|e_\gamma^*(z)\right| &\leqslant 2m_{j_0}n_{j_0}^{-1}C + 2Cm_{j_0}g^*\left(n_{j_0}^{-1}\sum_{i=1}^{n_{j_0}} e_k\right) \quad \text{for some } g^* \in W\left[\left(\mathcal{A}_{3n_j}, m_j^{-1}\right)_{j\in\mathbb{N}}\right] \\ &\leqslant 2m_{j_0}n_{j_0}^{-1}C + 2C \quad \text{by Proposition 3.1.} \end{aligned}$$

Thus we have that $\|z\| = \sup_{\gamma\in\overline{\Gamma}} |e_\gamma^*(z)| < 3C$.

To obtain the inequality in (3), consider $\gamma \in \overline{\Gamma}$ with weight m_h^{-1} such that $h \neq j_0$. Applying the basic inequality to $e_\gamma^*(z)$ gives $g^* \in W[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\in\mathbb{N}}]$ with $g^* = 0$ or $weight(g^*) = m_h^{-1}$ such that

$$\left|e_\gamma^*(z)\right| \leqslant 2m_{j_0}n_{j_0}^{-1}C + 2Cm_{j_0}g^*\left(n_{j_0}^{-1}\sum_{i=1}^{n_{j_0}} e_k\right).$$

We now apply Proposition 3.1 to g^* to obtain

$$g^*\left(n_{j_0}^{-1}\sum_{i=1}^{n_{j_0}} e_k\right) \leqslant \begin{cases} 2m_h^{-1}m_{j_0}^{-1} & \text{if } h < j_0, \\ m_h^{-1} & \text{if } h > j_0. \end{cases}$$

Combining the above two inequalities gives

$$\left|e_\gamma^*(z)\right| \leqslant \begin{cases} 2Cm_{j_0}n_{j_0}^{-1} + 4Cm_h^{-1} & \text{if } h < j_0, \\ 2Cm_{j_0}n_{j_0}^{-1} + 2Cm_{j_0}m_h^{-1} & \text{if } h > j_0. \end{cases}$$

Thus (3) follows as $2m_{j_0}n_{j_0}^{-1} \leqslant m_{j_0}^{-1}$ by (15) and $2Cm_{j_0}m_h^{-1} \leqslant Cm_{j_0}^{-1}$ when $h > j_0$.

For our final inequality (4) we apply Lemma 4.13 to $\gamma \in \Gamma$ to obtain $g^* \in W[(\mathcal{A}_{n_1}, m_1^{-1})]$ such that

$$\left|e_\gamma^*(z)\right| \leqslant Cm_{j_0}g^*\left(n_{j_0}^{-1}\sum_{i=1}^{n_{j_0}} e_k\right).$$

We now apply Proposition 3.1 to g^*, using that $g^* \in W[(\mathcal{A}_{n_1}, m_1^{-1})] \subset W[(\mathcal{A}_{3n_j}, m_j^{-1})_{j \neq j_0}]$, to obtain

$$g^*\left(n_{j_0}^{-1}\sum_{i=1}^{n_{j_0}} e_k\right) \leqslant 2m_1^{-1}m_{j_0}^{-2} \quad \text{as } 1 < j_0.$$

Combining the above two inequalities gives (4) as $2m_1^{-1} < 1$. $\square$

Proposition 4.16. *The FDD $(\bar{F}_i)$ of Z is shrinking and hence Z^* is isomorphic to ℓ_1.*

Proof. The Banach space Z is a separable $\mathcal{L}_\infty$ Banach space, and thus Z^* is isomorphic to ℓ_1 if and only if ℓ_1 does not embed into Z [15]. Thus if $(\bar{F}_i)$ is shrinking then Z^* is isomorphic to ℓ_1. We now prove that $(\bar{F}_i)$ is shrinking. Assume to the contrary that there exists a normalized block basis (b_k) which is not weakly null. Hence there exists $f \in Z^*$ such that $|f(b_k)| \nrightarrow 0$. By Proposition 2.8 there also exists some skipped block C-RIS (x_k) such that $|f(x_k)| \nrightarrow 0$. After passing to a subsequence we may assume that $|f(x_k)| > \delta$ for all $k \in \mathbb{N}$ and some $\delta > 0$. In particular we have that $|f(n_{2j}^{-1}\sum_{k=1}^{n_{2j}} x_k)| > \delta$ for all $j \in \mathbb{N}$. However, by Lemma 4.15(2) we have that $\|n_{2j}^{-1}\sum_{k=1}^{n_{2j}} x_k\| \leqslant 3Cm_{2j}^{-1}$. This is a contradiction if $j \in \mathbb{N}$ is chosen to be sufficiently large. $\square$

We are now prepared to prove our main result.

Proof of Theorem 4.1. By Theorem 2.9 we just need to prove that if T is an operator on Z and (x_n) is an RIS in Z then $\lim_{n\to\infty} \operatorname{dist}(T(x_n), [x_n] + X) = 0$. We assume to the contrary that there is some $C > 1$ and a C-RIS (x_n) with $\|T(x_n)\|_{Z/X+[x_n]} \geqslant 8 + 8\varepsilon$. As (x_n) is a block sequence of a shrinking FDD, we may pass to a subsequence of (x_n) and a compact perturbation of T so that there exists integers $0 = p_0 < p_1 < p_1 + 1 < p_2 < p_2 + 1 < p_3 < \cdots$ with $ran(x_n) \cup ran(T(x_n)) \subset (p_{n-1} + 1, p_n)$ for all $n \in \mathbb{N}$. Following the proof of Lemma 4.9 we may choose for each $n \in \mathbb{N}$ a functional $b_n^* \in \overline{B}^*_{(p_{n-1}+1, p_n)}$ such that $|b_n^*(x_n)| < \varepsilon_n$ and $b_n^*(T(x_n)) \geqslant 1$. We recall from Lemma 4.15 that we denote

$$z\big(j, (x_i)\big) = \frac{m_j}{n_j}\sum_{k=1}^{n_j} x_k \quad \text{for each } j \in \mathbb{N}.$$

We now fix some $i_0 \in \mathbb{N}$. The proof will proceed by constructing a block sequence $(u_i)_{i=1}^{n_{2i_0-1}}$ of (x_i) with each u_i being of the form $z(j, \bar{x}^i)$ for some $j \in \mathbb{N}$ and some subsequence $\bar{x}^i$ of (x_n). First we choose j_1 such that $m_{4j_1-2} > n_{2i_0-1}^2$ and $k_1 \in \mathbb{N}$ so that $4j_1 - 2 \leqslant p_{k_1}$. We then set $u_1 = z(4j_1 - 2, (x_i)_{i \geqslant k_1})$. By Lemma 4.7 we may choose $\gamma_1 \in \overline{\Gamma}$ with weight m_{4j_1-2} and $\operatorname{rank}(\gamma_1) > \max\{\operatorname{supp}(u_1) \cup \operatorname{supp}(T(u_1))\}$ such that the analysis of γ_1 is $(p_r, b_r^*, \xi_r)_{k_1 \leqslant r \leqslant k_1 + n_{4j_1-2}-1}$ for some $\xi_r \in \Theta_{p_r}$. A simple calculation shows that $e_{\gamma_1}^*(T(u_1)) \geqslant 1$ and $|e_{\gamma_1}^*(u_1)| < \varepsilon_1$. We now construct $u_r \in Z$ and $\gamma_r \in \overline{\Gamma} \setminus \Gamma$ for each $2 \leqslant r \leqslant n_{2i_0-1}$ according to the following procedure. Given $2 \leqslant r \leqslant n_{2i_0-1}$, we set $j_r = \sigma(\gamma_{r-1})$ and choose $k_r \in \mathbb{N}$ such that $4j_r < p_{k_r}$. We then set $u_r = z(4j_r, (x_i)_{i=k_r}^{\infty})$. Again by Lemma 4.7 we may choose $\gamma_r \in \overline{\Gamma}$ of weight $\frac{1}{m_{4j_r}}$ with $\operatorname{rank}(\gamma_r) > \max\{\operatorname{supp}(u_r) \cup \operatorname{supp}(T(u_r))\}$ such that $e_{\gamma_r}^*(T(u_r)) \geqslant 1$ and $|e_{\gamma_r}^*(u_r)| < \varepsilon_r$. This completes the construction of $(u_r)_{r=1}^{n_{2i_0-1}}$. We now set $u = z(2i_0 - 1, (u_r))$. Note that we have chosen $(\gamma_r)_{r=1}^{n_{2i_0-1}}$ and $(j_r)_{r=1}^{n_{2i_0-1}}$ to satisfy the conditions of Lemma 4.8, and thus there exists $\gamma \in \overline{\Gamma}$ with analysis $(p_{k_r} + 1, e_{\gamma_r}^*, e_{\xi_r}^*)$ for some $\xi_r \in \Theta_{p_{k_r}+1}$ with weight $\frac{1}{m_{2i_0-1}}$. A simple calculation shows that $e_\gamma^*(T(u)) \geqslant 1$ and $e_\gamma^*(u) < \varepsilon$. We will prove that actually $\|u\| \leqslant 20Cm_{2i_0-1}^{-1}$. Thus by choosing i_0 to be sufficiently large we reach a contradiction with $\|T\|$ being bounded.

The norm of u is given by $\|u\| = \max_{\gamma \in \overline{\Gamma}} |u(\gamma)|$. By part (4) of Lemma 4.15, we have that $|u(\gamma)| \leqslant Cm_{2i_0-1}^{-1}$ for all $\gamma \in \Gamma$. We will prove that $|u(\gamma)| \leqslant 20Cm_{2i_0-1}^{-1}$ for all $\gamma \in \overline{\Gamma} \setminus \Gamma$ using the moreover part of Proposition 4.14. We first note that parts (2), (3), and (4) of Lemma 4.15 imply that the sequence (u_r) is a $5C$-RIS. Assuming we are able to satisfy the moreover part of Proposition 4.14, we would have that

$$
\begin{aligned}
\|u\| &= \left\| \frac{m_{2i_0-1}}{n_{2i_0-1}} \sum_{r=1}^{n_{2i_0-1}} u_r \right\| \\
&\leqslant 10C \frac{m_{2i_0-1}}{n_{2i_0-1}} + 10Cg^* \left(\frac{m_{2i_0-1}}{n_{2i_0-1}} \sum_{r=1}^{n_{2i_0-1}} e_r \right) \quad \text{for some } g^* \in W\big[\big(\mathcal{A}_{3n_j}, m_j^{-1}\big)_{j \neq 2i_0-1}\big] \\
&\leqslant 10C \frac{m_{2i_0-1}}{n_{2i_0-1}} + \frac{10C}{m_{2i_0-1}} \quad \text{by Proposition 3.1} \\
&\leqslant \frac{10C}{m_{2i_0-1}} + \frac{10C}{m_{2i_0-1}}.
\end{aligned}
$$

Thus all that remains to be verified is the moreover part of Proposition 4.14. Given a subinterval J of $[1, n_{2i_0-1}]$ and an element $\gamma' \in \overline{\Gamma} \setminus \Gamma$ of weight $m_{2i_0-1}^{-1}$ we need to prove that $|e_{\gamma'}^*(\sum_{r \in J} u_r)| \leqslant 4C$. Without loss of generality we may assume that the age of γ' is the maximal value n_{2i_0-1}. We denote the analysis of γ' by $(q_r', e_{\gamma_r'}^*, e_{\xi_r'}^*)_{r \leqslant n_{2i_0-1}}$ and the analysis of γ by $(q_r, e_{\gamma_r}^*, e_{\xi_r}^*)_{r \leqslant n_{2i_0-1}}$. We thus have the following evaluation analysis for γ',

$$e_{\gamma'}^* = \sum_{r=1}^{n_{2i_0-1}} d_{\xi_r'}^* + \frac{1}{m_{2i_0-1}} e_{\gamma_r'}^*.$$

By the definition of Θ_n, it must be that $\mathrm{wt}(\gamma_r'), \mathrm{wt}(\gamma_r) < n_{2i_0-1}^{-2}$ for all $1 \leqslant r \leqslant n_{2i_0-1}$. This important fact will be used repeatedly in the remainder of the proof. Because (u_i) is a block sequence, there exists $1 \leqslant j \leqslant n_{2i_0-1}$ such that $d_{\xi_r}^*(\sum_{i\in J} u_i) = d_{\xi_r}^*(u_j)$. By applying this fact with part (1) of Lemma 4.15 we obtain the following inequality

$$\left| d_{\xi_r}^*\left(\sum_{i\in J} u_i\right)\right| = \left|d_{\xi_r}^*(u_j)\right| \leqslant C\mathrm{wt}(\gamma_j) < Cn_{2i_0-1}^{-2} \quad \text{for all } 1 \leqslant r \leqslant n_{2i_0-1}. \tag{26}$$

By Proposition 4.6 there exists $1 \leqslant \ell \leqslant n_{2i_0-1}$ such that $\xi_r' = \xi_r$ for all $r < \ell$ and $\mathrm{wt}(\gamma_j') \neq \mathrm{wt}(\gamma_r)$ for all j and all $\ell < r \leqslant n_{2i_0-1}$. In particular $\gamma_r' = \gamma_r$ and $q_r' = q_r$ for all $r < \ell$. Thus we have that

$$\left| e_{\gamma_r'}^*\left(\sum_{i\in J} u_i\right)\right| = \left| e_{\gamma_r}^*\left(\sum_{i\in J} u_i\right)\right| = \left|e_{\gamma_r}^*(u_r)\right| < \varepsilon_{p_{k_{r-1}}} \quad \text{for all } r < \ell. \tag{27}$$

Part (2) of Lemma 4.15 implies the following

$$\left|e_{\gamma_\ell'}^*(u_j)\right| \leqslant \|u_j\| \leqslant 3C \quad \text{if } \mathrm{wt}(\gamma_\ell') = \mathrm{wt}(\gamma_j). \tag{28}$$

We use part (3) of Lemma 4.15 with the fact that $\mathrm{wt}(\gamma_r'), \mathrm{wt}(\gamma_r) < n_{2i_0-1}^{-2}$ for all $1 \leqslant r \leqslant n_{2i_0-1}$ to achieve

$$\left|e_{\gamma_r'}^*(u_j)\right| \leqslant 5Cn_{2i_0-1}^{-2} \quad \text{if } \mathrm{wt}(\gamma_r') \neq \mathrm{wt}(\gamma_j). \tag{29}$$

We will apply inequality (29) for the case $r > \ell$ and for the case $r = \ell$ with $\mathrm{wt}(\gamma_\ell') \neq \mathrm{wt}(\gamma_j)$. The sequence $(e_{\gamma_r'}^*)_{1\leqslant r\leqslant n_{2i_0-1}}$ is a block sequence of $(\bar{F}_i^*)$ and $(u_i)_{1\leqslant i\leqslant n_{2i_0-1}}$ is a block sequence of $(\bar{F}_i)$. This implies the following simple combinatorial result

$$\sharp\left\{(r,j) \mid e_{\gamma_r'}^*(u_j) \neq 0\right\} < 2n_{2i_0-1}. \tag{30}$$

Combining all the inequalities (26), (27), (28), (29), and (30) gives our desired estimate

$$\begin{aligned}
\left| e_{\gamma'}^*\left(\sum_{i\in J} u_i\right)\right| &= \sum_{r=1}^{n_{2i_0-1}} d_{\xi_r'}^*\left(\sum_{i\in J} u_i\right) + m_{2i_0-1}^{-1} \sum_{r=1}^{n_{2i_0-1}} e_{\gamma_r'}^*\left(\sum_{i\in J} u_i\right) \\
&= \left(\sum_{r=1}^{n_{2i_0-1}} d_{\xi_r'}^* + m_{2i_0-1}^{-1}\left(\sum_{r<\ell} e_{\gamma_r'}^* + e_{\gamma_\ell'}^* + \sum_{r>\ell} e_{\gamma_r'}^*\right)\right)\left(\sum_{i\in J} u_i\right) \\
&< Cn_{2i_0-1}^{-1} + m_{2i_0-1}^{-1}\varepsilon + m_{2i_0-1}^{-1}3C + m_{2i_0-1}^{-1}2n_{2i_0-1}4Cn_{2i_0-1}^{-2} \\
&< C < 5C.
\end{aligned}$$

Thus the moreover part of Proposition 4.14 has been verified, and the proof is complete. □

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