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On a nonlinear flux-limited equation arising in the transport of morphogens

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ABSTRACT

Motivated by a mathematical model for the transport of morphogens in biological systems, we study existence and uniqueness of entropy solutions for a mixed initial–boundary value problem associated with a nonlinear flux-limited diffusion system. From a mathematical point of view the problem behaves more as a hyperbolic system than a parabolic one.

1. Introduction

The aim of this paper is to analyze the mixed initial–boundary value problem associated with a nonlinear flux-limited reaction–diffusion system

$$\begin{cases} \dfrac{\partial u}{\partial t} = \big(\mathbf{a}(u,u_x)\big)_x - f\big(t-\tau, u(t,x)\big)u(t,x) + g\big(t, u(t,x)\big) & \text{in }]0,T[\times]0,L[, \\ -\mathbf{a}\big(u(t,0), u_x(t,0)\big) = \beta > 0 \quad \text{and} \quad u(t,L) = 0 & \text{on } t \in]0,T[, \\ u(0,x) = u_0(x) & \text{in } x \in]0,L[, \end{cases} \tag{1.1}$$

being

$$\mathbf{a}(z,\xi) := \nu \frac{|z|\xi}{\sqrt{z^2 + \frac{\nu^2}{c^2}|\xi|^2}},$$

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where the boundary conditions must be interpreted in a weak sense to be precised, and the functions f and g are nonlinear with respect to u and depend on u through a coupled system of ordinary differential equations. This problem arises in the modelization of the transport of morphogens and the parameter τ represents a delay in the process of signaling pathway cell internalization.

The nonlinear diffusion equation

$$\frac{\partial u}{\partial t} = \big(\mathbf{a}(u, u_x)\big)_x \tag{1.2}$$

was introduced in different contexts as an alternative to the linear diffusion equation with the ideas of limiting the flux and reproducing a system with finite speed of propagation. The flux-limited type equations were motivated previously in [21], but they were firstly deduced by P. Rosenau formally, who proposed three alternative ways to introduce them [24]. Also, this equation was formally derived by Brenier [14] by means of Monge–Kantorovich's mass transport theory and named *relativistic heat equation* after him. As Brenier pointed out in [14], see also [34], the relativistic heat equation (1.2) is one among the various *flux-limited diffusion equations* used in the theory of radiation hydrodynamics [21]. A general class of flux-limited diffusion equations and the properties of the relativistic heat equation have been studied in a series of papers [5–10], where the well-posedness of the Cauchy, the Neumann and the Dirichlet problem for the relativistic heat equation is proved.

The above discussion on linear diffusion *versus* flux-limited diffusion leads to introduce the following change in the classical flux

$$\mathcal{F} = -\nu \nabla u, \quad \nu > 0, \tag{1.3}$$

associated with the heat equation (or the Fokker–Planck equation)

$$u_t = \nu \Delta u,$$

by a flux that saturates as the gradient becomes unbounded. To do that, it was proposed to link u to the flux $\mathcal{F}$ through the velocity $\mathbf{v}$ defined by the relation $\mathcal{F} = u\mathbf{v}$. Along with (1.3) this gives

$$\mathbf{v} = -\nu \frac{\nabla u}{u}. \tag{1.4}$$

According to (1.4), if $|\frac{\nabla u}{u}| \uparrow \infty$, so will do $\mathbf{v}$. However, the inertia effects impose a macroscopic upper bound on the allowed free speed, namely, the acoustic speed or light speed c. With this aim, Rosenau proposed to modify (1.4) by taking

$$\nu \frac{\nabla u}{u} = \frac{-\mathbf{v}}{\sqrt{1 - \frac{|\mathbf{v}|^2}{c^2}}}. \tag{1.5}$$

The postulate (1.5) forces $\mathbf{v}$ to stay in the subsonic regime (in the case c is the acoustic speed). The sonic limit is approached only if $|\frac{\nabla u}{u}| \uparrow \infty$. Solving (1.5) for $\mathbf{v}$, we obtain

$$\mathcal{F} = u\mathbf{v} = \frac{-u \nabla u}{\sqrt{1 + (\frac{\nu |\nabla u|}{cu})^2}}.$$

As we have mentioned before, the motivation for studying the system (1.1) comes from the transport of morphogens in biological systems. This is a classical problem since the pioneering work of Turing [31], Meinhard, Wolpert [35] or Lander [20]. Lander focused the question as a main problem in the understanding of the transport of proteins via signaling pathways: Do morphogen gradients arise by diffusion? The relevance of our study is founded on the analysis of the Hedgehog (Hh) signaling which has been found to play multiple roles in development, homeostasis and disease (reviewed in [22]). In vertebrates the Hh family comprises three proteins (Sonic, Desert and Indian), which act as secreted, intercellular factors that affect cell fate, differentiation, survival, and proliferation in the developing embryo and in many organs at one time or another. Sonic Hedgehog (Shh) signaling has also an important role in tumor formation: the deregulation of the Shh pathway leads to the development of various tumors, including those in skin, prostate and brain [25,26,30]. The idea is to analyze the morphogenetic patterning of the vertebrate embryonic neural tube along the dorsoventral (D-V) axis. The transport of the morphogen Shh along the D-V axis in the neural tube represents a natural privileged direction for the description of Shh propagation. Actually, the system is symmetric with respect to this axis and this justifies the reduction to one dimension. The discussion concerning whether the gradient formation of morphogens is produced or not by diffusion is a central and classic topic in developmental biology. This gives a continuous feedback between mathematical modeling and biological experiments, see [20,27,35,19]. Recent results in biology provide some findings that really call into question the hypothesis of diffusion which has been so often used to model these phenomena: 1) Concerning the cellular differentiation, the role of the quantity of morphogen received is as least as relevant as the time of exposure. With linear diffusion models every point (cell) of the neural tube receives instantaneously the information of the morphogen [17,27,23]. 2) Morphogens are transported in aggregates of several molecules that also include other morphogens or molecules. Then, the typical size of the cluster aggregates is big (of order 1/10) in comparison with the extracellular matrix where they are moving [33]. Also their concentration is quite dilute [16,33,17]. Therefore, Brownian motion does not seem to be the more appropriate choice. 3) In some cases, such as with the Hh morphogen, it has been proved that in absence of another cell-surface protein, called Ihog, there is neither propagation nor gradient function of Hh [16]. 4) There do exist privileged ways/paths of propagation in the extracellular matrix, a fact that makes the system resemble a traffic map, more than a linear diffusion system [13,16].

In this setting, the present paper tries to give some insight on this biological problem where the model here studied is a first step towards a complete model consisting in

$$\frac{\partial u(t,x)}{\partial t} = \mathbf{a}\big(u(t,x), u(t,x)_x\big)_x - f\big(t-\tau, u(t,x)\big)u(t,x) + g\big(t, u(t,x)\big),$$

where f stands for the concentration of transmembrane receptor in the cells, g represents the concentration of the complex binding the morphogen to the receptor, and where the dependence on u is given through a coupling with a system of seven ODE's modeling the rates of change of the concentrations of the proteins participating in the signaling pathway coming from the biochemical cascade inside the cells, see [32]. In that work it was also proved that numerical evidence fully agrees with the experiments from a quantitative as well as qualitative (propagation of fronts instead of linear diffusion behavior) point of view, see [16,29].

In addition to the biological or physical motivations, the mathematical analysis of this equation poses several difficulties, making even more interesting its study, such as the existence and evolution of fronts as well as the study of its finite speed of propagation, the related lack of regularity and the set-up of an appropriate functional framework to give a meaning to the differential operator and the boundary conditions. In fact, this flux-limited equation provides a behavior more related to hyperbolic systems than to usual diffusive (Fokker–Planck) systems. To deal with these mathematical problems we need to combine and extend the applicability of different techniques stemming from parabolic and hyperbolic contexts such as Crandall–Liggett's Theorem, Minty–Browder's technique, the concept of entropy solution, and the method of doubling variables due to S. Kruzhkov.

This paper deals with a preliminary study of (1.1) consisting in the analysis of the following system

$$\begin{cases} \dfrac{\partial u}{\partial t} = \big(\mathbf{a}(u, u_x)\big)_x & \text{in }]0,T[\times]0,L[,\\ -\mathbf{a}\big(u(t,0), u_x(t,0)\big) = \beta > 0 \quad \text{and} \quad u(t,L) = 0 & \text{on } t \in]0,T[,\\ u(0,x) = u_0(x) & \text{in } x \in]0,L[, \end{cases} \tag{1.6}$$

where the boundary conditions must be considered in a weak sense. Our main result is

Theorem 1.1. *For any initial datum $0 \leqslant u_0 \in L^\infty(]0,L[)$, there exists a unique bounded entropy solution u of (1.6) in $Q_T =]0,T[\times]0,L[$ for every $T > 0$ such that $u(0) = u_0$.*

The paper is structured as follows. In the next section we introduce all the tools needed to develop the theory: a suitable integration by parts formula, lower semi-continuity results and a functional calculus, in order to be able to give a sense to the differential operator. In Section 3 we discuss the associated elliptic problem: we define what a solution is, and then we prove existence and uniqueness of such a solution. Next, this material is used to define an accretive operator and construct a nonlinear semigroup, which accounts for solving (1.6) in a mild sense; all this is the content of Section 4. In Section 5 we prove that the mild solution previously constructed can be characterized in more operative terms, as a so-called entropy solution – a concept which is also introduced in this section, and we prove a comparison criterion which in particular entails uniqueness of entropy solutions, thus proving Theorem 1.1.

2. Preliminaries

2.1. BV functions and integration by parts

For bounded variation function of one variable we follow [1]. Let $I \subset \mathbb{R}$ be an interval, we say that a function $u \in L^1(I)$ is of bounded variation if its distributional derivative Du is a Radon measure on I with bounded total variation $|Du|(I) < +\infty$. We denote by $BV(I)$ the space of all functions of bounded variation in I. It is well know (see [1]) that given $u \in BV(I)$ there exists $\bar{u}$ in the equivalence class of u, called a good representative of u with the following properties. If J_u is the set of atoms of Du, i.e., $x \in J_u$ if and only if $Du(\{x\}) \neq 0$, then $\bar{u}$ is continuous in $I \setminus J_u$ and has a jump discontinuity at any point of J_u:

$$\bar{u}(x_-) := \lim_{y \uparrow x} \bar{u}(y) = Du\big(]0,x[\big), \qquad \bar{u}(x_+) := \lim_{y \downarrow x} \bar{u}(y) = Du\big(]0,x]\big) \quad \forall x \in J_u,$$

where by simplicity we are assuming that $I =]0,L[$. Consequently,

$$\bar{u}(x_+) - \bar{u}(x_-) = Du\big(\{x\}\big) \quad \forall x \in J_u.$$

Moreover, $\bar{u}$ is differentiable at $\mathcal{L}^1$ a.e. point of I, and the derivative $\bar{u}'$ is the density of Du with respect to $\mathcal{L}^1$, being $\mathcal{L}^d$ the d-dimensional Lebesgue's measure. For $u \in BV(I)$, the measure Du decomposes into its absolutely continuous and singular parts $Du = D^{ac}u + D^s u$. Then $D^{ac}u = \bar{u}'\mathcal{L}^1$. Obviously, if $u \in BV(I)$ then $u \in W^{1,1}(I)$ if and only if $D^s u \equiv 0$, and in this case we have $Du = \bar{u}'\mathcal{L}^1$. From now on, when we deal with pointwise valued BV-functions we always shall use the good representative. Hence, in the case $u \in W^{1,1}(I)$, we shall assume that $u \in C(\bar{I})$.

Given $\mathbf{z} \in W^{1,1}(I)$ and $u \in BV(I)$, by $\mathbf{z}Du$ we mean the Radon measure in I defined as

$$\langle \varphi, \mathbf{z}Du \rangle := \int_0^L \varphi \mathbf{z} Du \quad \forall \varphi \in C_c\big(]0,L[\big).$$

We need the following integration by parts formula, which can be proved using a suitable regularization of $u \in BV(I)$ as in the proof of Theorem C.9 in [3].

Lemma 2.1. *If $\mathbf{z} \in W^{1,1}(I)$ and $u \in BV(I)$, then*

$$\int_0^L \mathbf{z} Du + \int_0^L u(x)\mathbf{z}'(x)\,dx = \mathbf{z}(L)u(L_-) - \mathbf{z}(0)u(0_+).$$

2.2. Properties of the Lagrangian

Hereafter C denotes a generic constant, its value may change from line to line. We define

$$\mathbf{a}(z,\xi) := \frac{\nu|z|\xi}{\sqrt{z^2 + \frac{\nu^2}{c^2}|\xi|^2}}. \tag{2.7}$$

We assume $\mathbf{a}(z,0) = 0$ for all $z \in \mathbb{R}$. Then $\mathbf{a}(z,\xi) = \partial_\xi F(z,\xi)$, being the Lagrangian

$$F(z,\xi) := \frac{c^2}{\nu}|z|\sqrt{z^2 + \frac{\nu^2}{c^2}\xi^2}.$$

By the convexity of F,

$$\mathbf{a}(z,\xi)(\eta - \xi) \leqslant F(z,\eta) - F(z,\xi) \quad \text{for all } z,\xi,\eta \in \mathbb{R}. \tag{2.8}$$

Note that we have

$$c|z||\xi| - \frac{c^2}{\nu}z^2 \leqslant \mathbf{a}(z,\xi)\xi \leqslant cM|\xi| \quad \text{for all } z,\xi \in \mathbb{R},\ |z| \leqslant M. \tag{2.9}$$

Moreover, using (2.8) it is easy to see that

$$\big(\mathbf{a}(z,\xi) - \mathbf{a}(z,\hat{\xi})\big)\cdot(\xi - \hat{\xi}) \geqslant 0 \tag{2.10}$$

for any $(z,\xi), (z,\hat{\xi}) \in \mathbb{R}\times\mathbb{R}$, $|z| \leqslant M$.

We introduce the following notation to ease the way in which our functional calculus is written: for any function q let $J_q(r)$ denote its primitive, i.e., $J_q(r) = \int_0^r q(s)\,ds$.

Assume that $f : \mathbb{R}\times\mathbb{R} \to [0,\infty[$ is a continuous function convex in its last variable such that

$$0 \leqslant f(z,\xi) \leqslant C\big(1 + |\xi|\big) \quad \forall (z,\xi) \in \mathbb{R}\times\mathbb{R},\ |z| \leqslant M \tag{2.11}$$

for some constant $C \geqslant 0$ which may depend on M. Given $f(z,\xi)$, we define its recession function as

$$f^0(z,\xi) = \lim_{t\to 0^+} tf\left(z, \frac{\xi}{t}\right).$$

We assume that $f^0(z,\xi)=\varphi(z)\psi^0(\xi)$, with φ Lipschitz continuous, ψ^0 homogeneous of degree 1. Then, working as in [5], if for a fixed function $\phi\in C_c(]0,L[)$ we define the operator $\mathcal{R}_{\phi f}: BV(]0,L[)\to\mathbb{R}$ by

$$\mathcal{R}_{\phi f}(u):=\int_0^L \phi(x) f\big(u(x),u'(x)\big)\,dx+\int_0^L \phi(x)\psi^0\left(\frac{Du}{|Du|}\right)\big|D^s J_\varphi(u)\big|, \tag{2.12}$$

we have that $\mathcal{R}_{\phi f}$ is lower semi-continuous with respect to the L^1-convergence.

For instance, we discuss here for future usage one of the most recurrent cases: define $\theta(z)=c|z|$, and note that $F^0(z,\xi)=\theta(z)\psi^0(\xi)$, with $\psi^0(\xi)=|\xi|$. Therefore,

$$\mathcal{R}_{\phi F}(u):=\int_0^L \phi(x) F\big(u(x),u'(x)\big)\,dx+\frac{c}{2}\int_0^L \phi(x)\big|D^s\big(u^2\big)\big|$$

is lower semi-continuous in $BV(]0,L[)$ with respect to the L^1-convergence.

We shall consider the function $h:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ defined by $h(z,\xi):=\mathbf{a}(z,\xi)\cdot\xi$. Note that

$$h(z,\xi)\geqslant 0 \quad \forall \xi,z\in\mathbb{R}. \tag{2.13}$$

We will make use of the following property:

$$h^0(z,\xi)=F^0(z,\xi) \quad \forall \xi,z\in\mathbb{R}. \tag{2.14}$$

As for the Dirichlet problem (see [10]), in general, the data in L is not taken pointwise; we need to introduce functionals that take into account the boundary. The following result is a particular case of Theorem 2.4 in [10].

Theorem 2.2. *Let f be verifying* (2.11) *and $f^0(z,\xi)=\varphi(z)|\xi|$, with φ Lipschitz continuous, let $\phi\in C([0,L])^+$ be given. Then, the functional $\mathcal{F}^0_{\phi f}: BV(]0,L[)\to\mathbb{R}$ defined by*

$$\mathcal{F}^0_{\phi f}(u):=\mathcal{R}_{\phi f}(u)+\phi(L)\big|J_\varphi(u)(L_-)\big|$$

is lower semi-continuous with respect to the L^1-convergence.

2.3. Spaces of truncated functions and associated calculus

We need to consider the following truncature functions. For $a<b$, let $T_{a,b}(r):=\max(\min(b,r),a)$. As usual, we denote $T_k=T_{-k,k}$. We also consider the truncature functions $T^l_{a,b}(r):=T_{a,b}(r)-l$ ($l\in\mathbb{R}$). We denote

$$\mathcal{T}_r:=\{T_{a,b}\colon 0<a<b\},\qquad \mathcal{T}^+:=\big\{T^l_{a,b}\colon 0<a<b,\ l\in\mathbb{R},\ T^l_{a,b}\geqslant 0\big\}.$$

Given any truncature function T_k, we define

$$T_k(r)^+:=\max\big\{T_k(r),0\big\} \quad\text{and}\quad T_k(r)^-:=\min\big\{T_k(r),0\big\}=-T_k(-r)^+,\quad r\in\mathbb{R}.$$

Consider the function space

$$TBV^+(I):=\big\{u\in L^1(I)^+\colon\ T(u)\in BV(I),\ \forall T\in\mathcal{T}_r\big\};$$

we want to give a sense to the Radon–Nikodym derivative u' of a function $u \in TBV^+(I)$. Using chain's rule for BV functions (see, for instance, [1]), and with a similar proof to the one given in Lemma 2.1 of [11], we obtain the following result.

Lemma 2.3. *For every $u \in TBV^+(I)$ there exists a unique measurable function $v : I \to \mathbb{R}$ such that*

$$\big(T_{a,b}(u)\big)' = v\chi_{[a<u<b]} \quad \mathcal{L}^1\text{-a.e.}, \ \forall T_{a,b} \in \mathcal{T}_r. \tag{2.15}$$

Thanks to this result we define u' for a function $u \in TBV^+(I)$ as the unique function v which satisfies (2.15). This notation will be used throughout in the sequel. The notation ∂_x will also be used in the case of functions of several variables (say t and x), for the same purposes, whenever there is some risk of confusion.

We denote by $\mathcal{P}$ the set of Lipschitz continuous function $p : [0, +\infty[\to \mathbb{R}$ satisfying $p'(s) = 0$ for s large enough, and write $\mathcal{P}^+ := \{p \in \mathcal{P}: \ p \geqslant 0\}$. We recall the following result [2, Lemma 2].

Lemma 2.4. *If $u \in TBV^+(I)$, then $p(u) \in BV(I)$ for every $p \in \mathcal{P}$ such that there exists $a > 0$ with $p(r) = 0$ for all $0 \leqslant r \leqslant a$. Moreover, with the above notation $[p(u)]' = p'(u)u'$ $\mathcal{L}^1$-a.e.*

For $u \in TBV^+(]0, L[)$ we will define

$$u(0_+) := \lim_{n\to\infty} T_{\frac{1}{n},n}(u)(0_+) \quad \text{and} \quad u(L_-) := \lim_{n\to\infty} T_{\frac{1}{n},n}(u)(L_-).$$

It is easy to see that the above limits exist.

Let $S \in \mathcal{P}^+$ and $T = T^a_{a,b}$. Given $u \in TBV^+(]0, L[)$, Lemma 2.4 assures that $S(u)T(u)$, $J_{T'S}(u)$, $J_{TS'}(u) \in BV(]0, L[)$. Moreover, $D(S(u)T(u)) = DJ_{T'S}(u) + DJ_{TS'}(u)$ and hence, if $\mathbf{z} \in W^{1,1}(]0, L[)$,

$$\mathbf{z}D\big(T(u)S(u)\big) = \mathbf{z}DJ_{T'S}(u) + \mathbf{z}DJ_{TS'}(u).$$

For $u \in TBV^+(]0, L[)$, $\phi \in C_c(]0, L[)$, $T = T_{a,b} - l \in \mathcal{T}^+$ and f as in the previous subsection – see (2.11) – we define the functional

$$\mathcal{R}(\phi f, T)(u) := \mathcal{R}_{\phi f}\big(T_{a,b}(u)\big) + \int_{[u\leqslant a]} \phi(x)\big(f\big(u(x), 0\big) - f(a, 0)\big)\,dx$$
$$- \int_{[u\geqslant b]} \phi(x)\big(f\big(u(x), 0\big) - f(b, 0)\big)\,dx.$$

We have that $\mathcal{R}(\phi f, T)(\cdot)$ is lower semi-continuous in $TBV^+(]0, L[)$ with respect to the L^1-convergence.

Given $S, T \in \mathcal{T}^+$ and $u \in TBV^+(]0, L[)$, we define the following Radon measures in $]0, L[$,

$$\big\langle F\big(u, DT(u)\big), \phi\big\rangle := \mathcal{R}(\phi F, T)(u),$$
$$\big\langle F_S\big(u, DT(u)\big), \phi\big\rangle := \mathcal{R}(\phi SF, T)(u),$$
$$\big\langle h\big(u, DT(u)\big), \phi\big\rangle := \mathcal{R}(\phi h, T)(u), \qquad \big\langle h_S\big(u, DT(u)\big), \phi\big\rangle := \mathcal{R}(\phi Sh, T)(u),$$

for $\phi \in C_c(]0, L[)$. Using (2.12) and (2.14), we compute

$$F\big(u, DT(u)\big)^s = \frac{c}{2}\big|D^s\big(T(u)\big)^2\big| = h\big(u, DT(u)\big)^s,$$
$$F_S\big(u, DT(u)\big)^s = \big|D^s J_{S\theta}\big(T(u)\big)\big| = h_S\big(u, DT(u)\big)^s,$$
$$h\big(u, DT(u)\big)^{ac} = h\big(u, \big(T(u)\big)'\big), \qquad h_S\big(u, DT(u)\big)^{ac} = S(u)h\big(u, \big(T(u)\big)'\big).$$

3. The elliptic problem

Given $v \in L^1(]0, L[)$, we are interested in the following problem:

$$\begin{cases} -\big(\mathbf{a}\big(u, u'\big)\big)' = v \quad \text{in }]0, L[, \\ -\mathbf{a}\big(u, u'\big)\big|_{x=0} = \beta > 0 \quad \text{and} \quad u(L) = 0, \end{cases} \tag{3.16}$$

where $\mathbf{a}$ is given by (2.7). We introduce the following concept of solution for problem (3.16).

Definition 3.1. Given $v \in L^1(]0, L[)$, we say that $u \geqslant 0$ is an *entropy solution* of (3.16) if $u \in TBV^+(]0, L[)$ and $\mathbf{a}(u, u') \in C([0, L])$ both satisfy

$$v = -D\mathbf{a}\big(u, u'\big) \quad \text{in } \mathcal{D}'\big(]0, L[\big),$$
$$-\mathbf{a}\big(u, u'\big)(0) = \beta \quad \text{and} \quad \mathbf{a}\big(u, u'\big)(L) = -cu(L_-),$$
$$h\big(u, DT(u)\big) \leqslant \mathbf{a}\big(u, u'\big)DT(u) \quad \text{as measures } \forall T \in \mathcal{T}^+, \tag{3.17}$$
$$h_S\big(u, DT(u)\big) \leqslant \mathbf{a}\big(u, u'\big)DJ_{T'S}(u) \quad \text{as measures } \forall S \in \mathcal{P}^+,\ T \in \mathcal{T}^+. \tag{3.18}$$

Note that (3.17) can be rewritten as $h(u, DT(u))^s \leqslant [\mathbf{a}(u, u')DT(u)]^s$, and thus it is equivalent to

$$\frac{c}{2}\big|D^s\big(\big(T(u)\big)^2\big)\big| \leqslant \mathbf{a}\big(u, u'\big)D^sT(u) \quad \text{as measures } \forall T \in \mathcal{T}^+.$$

Also we have that (3.18) can be rewritten as $h_S(u, DT(u))^s \leqslant [\mathbf{a}(u, u')DJ_{T'S}(u)]^s$, and is equivalent to

$$\big|D^s J_{S\theta}\big(T(u)\big)\big| \leqslant \mathbf{a}\big(u, u'\big)D^s J_{T'S}(u) \quad \text{as measures } \forall S \in \mathcal{P}^+,\ T \in \mathcal{T}^+.$$

Observe that since $-\mathbf{a}(u, u')(0) = \beta$, we have

$$u(0_+) \geqslant \frac{\beta}{c} > 0. \tag{3.19}$$

We introduce now the main result of this section.

Theorem 3.2. *For any* $0 \leqslant f \in L^\infty(]0, L[)$ *there exists a unique entropy solution* $u \in TBV^+(]0, L[)$ *of the problem*

$$\begin{cases} u - \big(\mathbf{a}\big(u, u'\big)\big)' = f \quad \textit{in }]0, L[, \\ -\mathbf{a}\big(u, u'\big)\big|_{x=0} = \beta > 0, \quad u(L) = 0, \end{cases} \tag{3.20}$$

which satisfies $\|u\|_\infty \leqslant M(\beta, c, \nu, \|f\|_\infty)$.

Moreover, let $u, \bar{u}$ be two entropy solutions of (3.20) *associated to $f, \overline{f} \in L^1(]0, L[)^+$, respectively. Then,*

$$\int_0^L (u - \bar{u})^+ \, dx \leqslant \int_0^L (f - \overline{f})^+ \, dx.$$

Proof. Let $0 \leqslant f \in L^\infty(]0, L[)$. For every $n \in \mathbb{N}$, consider $\mathbf{a}_n(z, \xi) := \mathbf{a}(z, \xi) + \frac{1}{n}\xi$. As a consequence of the results about pseudo-monotone operators in [15] we know that $\forall n \in \mathbb{N}$ there exists a unique $u_n \in W^{1,2}(]0, L[)$ such that $u_n(L) = 0$ and

$$\int_0^L u_n v \, dx + \int_0^L \mathbf{a}\big(u_n, u_n'\big) v' \, dx + \frac{1}{n} \int_0^L u_n' v' \, dx - \beta v(0) = \int_0^L f v \, dx \tag{3.21}$$

for all $v \in W^{1,2}(]0, L[)$, $v(L) = 0$.

The following result can be easily obtained by multiplication by u_n^- and integration over $[0, L]$.

Lemma 3.3. *The functions u_n are non-negative $\forall n \in \mathbb{N}$.*

Now we give a bound for the sequence u_n at zero.

Lemma 3.4. *The sequence $\{u_n(0)\}$ is bounded. More precisely,*

$$0 \leqslant u_n(0) \leqslant \begin{cases} \frac{4\beta c}{\nu} + \sqrt{\frac{2cL}{\nu}} \|f\|_\infty & \text{if } c > \sqrt{\nu}, \\ \frac{4\beta}{c} + \sqrt{\frac{2L}{c}} \|f\|_\infty & \text{if } c \leqslant \sqrt{\nu}. \end{cases}$$

Proof. Taking $v = u_n$ in (3.21), we get

$$\int_0^L \left(u_n^2 + \mathbf{a}\big(u_n, u_n'\big) u_n' + \frac{1}{n} \big((u_n)'\big)^2 \right) dx = \beta u_n(0) + \int_0^L f u_n \, dx. \tag{3.22}$$

Then, dropping non-negative terms and using Young's inequality, we get

$$\int_0^L u_n^2 \, dx \leqslant \int_0^L f^2 \, dx + 2\beta u_n(0). \tag{3.23}$$

Now we can write $u_n |u_n'| = \frac{1}{2} |(u_n^2)'|$, and taking into account (2.9) we have $u_n' \mathbf{a}(u_n, u_n') \geqslant c u_n |u_n'| - \frac{c^2}{\nu} u_n^2$. Then, from (3.22), we obtain

$$\int_0^L \left(\frac{c}{2} \big|(u_n^2)'\big| + \frac{((u_n)')^2}{n} \right) dx \leqslant \int_0^L \left(\left(\frac{c^2}{\nu} - 1 \right) u_n^2 + f u_n \right) dx + \beta u_n(0). \tag{3.24}$$

Assuming now that $\frac{c^2}{\nu} - 1 > 0$, we apply Young's inequality in the right-hand side of (3.24), which now reads

$$\left(\frac{c^2}{\nu} - \frac{1}{2}\right) \int_0^L u_n^2\,dx + \frac{1}{2}\int_0^L f^2\,dx + \beta u_n(0).$$

As $c > \sqrt{\nu}$ we have $\frac{c^2}{\nu} - \frac{1}{2} > 0$, which allows us to bring in (3.23), thus obtaining

$$\frac{c}{2}\int_0^L \left|\left(u_n^2\right)'\right|dx + \frac{1}{n}\int_0^L \left((u_n)'\right)^2 dx \leqslant \frac{c^2}{\nu}\int_0^L f^2\,dx + \frac{2\beta c^2}{\nu} u_n(0). \tag{3.25}$$

Then, we have

$$\frac{c}{2}\left|u_n^2(0)\right| = \frac{c}{2}\left|u_n^2(L) - u_n^2(0)\right| = \frac{c}{2}\left|\int_0^L \left(u_n^2\right)' dx\right| \leqslant \frac{c^2}{\nu}\int_0^L f^2\,dx + \frac{2\beta c^2}{\nu} u_n(0),$$

from where we get that $u_n^2(0) - \frac{4\beta c}{\nu} u_n(0) - \frac{2c}{\nu}\|f\|_2^2 \leqslant 0$. Hence, for all $n \in \mathbb{N}$,

$$0 \leqslant u_n(0) \leqslant \frac{1}{2}\left(\frac{4\beta c}{\nu} + \sqrt{\left(\frac{4\beta c}{\nu}\right)^2 + \frac{8c}{\nu}\|f\|_2^2}\right) \leqslant \frac{4\beta c}{\nu} + \sqrt{\frac{2c}{\nu}}\|f\|_2.$$

In case that $c^2/\nu - 1 \leqslant 0$, from (3.24) we obtain

$$\frac{c}{2}\int_0^L \left|\left(u_n^2\right)'\right|dx + \frac{1}{n}\int_0^L \left((u_n)'\right)^2 dx \leqslant \int_0^L f u_n\,dx + \beta u_n(0).$$

Then, using Young's inequality and having in mind (3.23), we get

$$\frac{c}{2}\int_0^L \left|\left(u_n^2\right)'\right|dx + \frac{1}{n}\int_0^L \left((u_n)'\right)^2 dx \leqslant \int_0^L f^2\,dx + 2\beta u_n(0). \tag{3.26}$$

Thus, we have that $u_n^2(0) - \frac{4}{c}\beta u_n(0) - \frac{2}{c}\int_0^L f^2 \leqslant 0$, from where it follows that for all $n \in \mathbb{N}$,

$$0 \leqslant u_n(0) \leqslant \frac{1}{2}\left(\frac{4\beta}{c} + \sqrt{\left(\frac{4\beta}{c}\right)^2 + \frac{8}{c}\|f\|_2^2}\right) \leqslant \frac{4\beta}{c} + \sqrt{\frac{2}{c}}\|f\|_2. \qquad \square$$

By (3.25), (3.26) and Lemma 3.4, we get

$$\frac{c}{2}\int_0^L \left|\left(u_n^2\right)'\right|dx + \frac{1}{n}\int_0^L \left((u_n)'\right)^2 dx \leqslant C \quad \forall n \in \mathbb{N}. \tag{3.27}$$

Lemma 3.5. *The sequence $\{u_n \colon n \in \mathbb{N}\}$ is uniformly bounded in $L^\infty(0, L)$.*

Proof. By Lemma 3.4, we know that $M = \max\{\|f\|_\infty, \max\{u_n(0)\colon\ n \in \mathbb{N}\}\}$ is finite. Then, taking $v = (u_n - M)^+$ as test function in (3.21), it is easy to see that $\|u_n\|_\infty \leqslant M$ and Lemma 3.5 holds. □

Lemma 3.6. *The sequence $\{u_n\}$ is uniformly bounded in $TBV^+(]0, L[)$. Furthermore, there exists a function $0 \leqslant u \in TBV^+(]0, L[) \cap L^\infty(]0, L[)$ such that (up to subsequence) $u_n \to u$ a.e. and strongly in $L^1(]0, L[)$.*

Proof. By Lemma 3.5, extracting a subsequence if necessary, we may assume that u_n converges weakly in $L^2(]0, L[)$ to some non-negative function u as $n \to +\infty$. Moreover, by Lemma 3.5 again, we have that $0 \leqslant u \in L^\infty(]0, L[)$. On the other hand, if $0 < a < b$, by the co-area formula and (3.27), we have

$$\int_0^L \left|\left(T_{a,b}(u_n)\right)'\right| dx = \int_a^b |D\chi_{[u_n \leqslant t]}|\big(]0, L[\big)\, dt = \int_a^b |D\chi_{[u_n^2 \leqslant t^2]}|\big(]0, L[\big)\, dt$$
$$= \int_{a^2}^{b^2} |D\chi_{[u_n^2 \leqslant s]}|\big(]0, L[\big) \frac{ds}{2\sqrt{s}} \leqslant \frac{1}{2a} \int_0^L \left|\left(u_n^2\right)'\right| dx \leqslant \frac{C}{a}.$$

Consequently, we may assume that u_n converges almost everywhere to u. Then, by the Vitali Convergence Theorem, we get that $u_n \to u$ in $L^1(]0, L[)$, and using the above estimate on the gradients we obtain that $u \in TBV^+(]0, L[)$. □

Since $|\mathbf{a}(u_n, u_n')| \leqslant c|u_n|$, by Lemma 3.5 we may assume that

$$\mathbf{a}\big(u_n, u_n'\big) \rightharpoonup \mathbf{z} \quad \text{as } n \to \infty, \text{ weakly}^* \text{ in } L^\infty\big(]0, L[\big). \tag{3.28}$$

By assumption we have that $\mathbf{a}(u_n, u_n') = c|u_n|\mathbf{b}(u_n, u_n')$ with $|\mathbf{b}(u_n, u_n')| \leqslant 1$ (independent of n), $\|u_n\|_\infty \leqslant M$, and $u_n \to u$ a.e. as $n \to \infty$, so we may assume that $\mathbf{b}(u_n, u_n') \rightharpoonup \mathbf{z}_b$ as $n \to \infty$, weakly* in $L^\infty(]0, L[)$, and

$$\mathbf{z} = cu\mathbf{z}_b, \quad \text{with } \|\mathbf{z}_b\|_\infty \leqslant 1. \tag{3.29}$$

On the other hand, by (3.27),

$$\frac{1}{n} u_n' \to 0 \quad \text{in } L^2\big(]0, L[\big). \tag{3.30}$$

Given $\phi \in \mathcal{D}(]0, L[)$, taking $v = \phi$ in (3.21) we obtain

$$\int_0^L u_n \phi\, dx + \int_0^L \mathbf{a}\big(u_n, u_n'\big)\phi'\, dx + \frac{1}{n} \int_0^L u_n' \phi'\, dx = \int_0^L f\phi\, dx.$$

Letting $n \to +\infty$, having in mind (3.28) and (3.30), we obtain

$$\int_0^L (f - u)\phi\, dx = \int_0^L \mathbf{z} \cdot \phi'\, dx,$$

that is,

$$f - u = -D\mathbf{z} \quad \text{in } \mathcal{D}'\big(]0, L[\big) \tag{3.31}$$

and

$$\big(\mathbf{a}_n\big(u_n, u_n'\big)\big)' \rightharpoonup D\mathbf{z} \quad \text{weakly in } L^2\big(]0, L[\big).$$

Note that by (3.31), we have $\mathbf{z} \in W^{1,1}(]0, L[)$ and $D\mathbf{z} = \mathbf{z}'$.

Working as in the proof of Lemma 4.2 of [5], we can prove the identification

$$\mathbf{z}(x) = \mathbf{a}\big(u(x), u'(x)\big) \quad \text{a.e. } x \in]0, L[. \tag{3.32}$$

From (3.32) and (3.31) it follows that

$$f - u = -D\mathbf{a}\big(u, u'\big) \quad \text{in } \mathcal{D}'\big(]0, L[\big).$$

Lemma 3.7. *The flux $-\mathbf{a}(u, u')$ verifies the Neumann condition at $x = 0$.*

Proof. Let $w \in W^{1,1}(]0, L[)$ such that $w(L) = 0$ and consider $w_k \in W^{1,2}(]0, L[)$ with $w_k(L) = 0$ for all $k \in \mathbb{N}$, $w_k \to \hat{w}$ pointwise and $w_k' \to w'$ in $L^1(]0, L[)$. Taking in (3.21) w_k as test function and letting $n \to +\infty$, we get

$$\int_0^L u w_k \, dx + \int_0^L \mathbf{z} w_k' \, dx - \beta w_k(0) = \int_0^L f w_k \, dx.$$

Then, letting $k \to +\infty$ we arrive to

$$\int_0^L u w \, dx + \int_0^L \mathbf{z} w' \, dx - \beta w(0) = \int_0^L f w \, dx. \tag{3.33}$$

Fixed $w \in BV(]0, L[)$ such that $w(L_-) = 0$, let $w_m \in W^{1,1}(]0, L[)$ with $w_m(L) = 0$, $w_m(0) = w(0_+)$, and such that $w_m \to w$ in $L^1(]0, L[)$. Taking in (3.33) w_m as test functions and integrating by parts we get

$$\int_0^L (f - u) w_m \, dx = \int_0^L \mathbf{z} w_m' \, dx - \beta w(0_+) = -\int_0^L \mathbf{z}' w_m \, dx - w(0_+)\big(\mathbf{z}(0) + \beta\big),$$

and letting $m \to +\infty$, we obtain $-\mathbf{z}(0) = \beta$. □

Lemma 3.8. *Let $S \in \mathcal{P}^+$, $T \in \mathcal{T}^+$ and $\phi \in C^1([0, L])$, $\phi \geqslant 0$, with $\phi(0) = 0$. Then*

$$\begin{aligned} &\int_0^L \phi F\big(u, DT(u)\big) + \phi(L)\frac{c}{2}\big|\big(T(u)\big)^2(L_-)\big| \\ &\quad \leqslant \int_0^L \phi \mathbf{z} DT(u) + \int_0^L \phi F(u, 0)\, dx - \phi(L) T(u)(L_-) + \phi(L)\big|J_\theta\big(T(0)\big)\big| \end{aligned} \tag{3.34}$$

and

$$\int_0^L \phi F_S\big(u, DT(u)\big) + \phi(L)\big|J_{\theta S}\big(T(u)(L_-)\big)\big|$$

$$\leqslant \int_0^L \phi \mathbf{z} D J_{T'S}(u) + \int_0^L \phi S(u) F(u,0)\,dx - \phi(L)\mathbf{z}(L) J_{T'S}\big(u(L_-)\big) + \phi(L)\big|J_{\theta S}\big(T(0)\big)\big|. \tag{3.35}$$

In particular,

$$F\big(u, DT(u)\big) \leqslant \mathbf{z} DT(u) + F(u,0)\mathcal{L}^1 \quad \textit{as measures in }]0, L[, \tag{3.36}$$

$$F_S\big(u, DT(u)\big) \leqslant \mathbf{z} D\big(J_{T'S}(u)\big) + S(u)F(u,0)\mathcal{L}^1 \quad \textit{as measures in }]0, L[. \tag{3.37}$$

Proof. We will only prove (3.35), the proof of (3.34) being similar. Let $0 \leqslant \phi \in C^1([0, L])$ with $\phi(0) = 0$.

Since $\mathcal{F}^0_{\phi SF}$ is l.s.c. with respect to the L^1-convergence, letting $n \to \infty$ we obtain

$$\int_0^L \phi F_S\big(u, DT(u)\big) + \phi(L)\big|J_{\theta S}\big(T(u)(L_-)\big)\big|$$

$$\leqslant \liminf_{n\to\infty} \int_0^L \phi S(u_n) F\big(u_n, T(u_n)'\big)\,dx + \phi(L)\big|J_{\theta S}\big(T(0)\big)\big|$$

$$\leqslant \limsup_{n\to\infty} \int_0^L \phi S(u_n) F\big(u_n, T(u_n)'\big)\,dx + \phi(L)\big|J_{\theta S}\big(T(0)\big)\big|.$$

By the convexity (2.8) of F and using that $\mathbf{a}(u_n, T(u_n)')T(u_n)' = \mathbf{a}(u_n, u_n')T(u_n)'$, we have

$$\int_0^L \phi S(u_n) F\big(u_n, T(u_n)'\big)\,dx \leqslant \int_0^L \phi S(u_n)\mathbf{a}\big(u_n, T(u_n)'\big)T(u_n)'\,dx + \int_0^L \phi S(u_n) F(u_n, 0)\,dx$$

$$= \int_0^L \phi \mathbf{a}\big(u_n, u_n'\big)\big(J_{T'S}(u_n)\big)'\,dx + \int_0^L \phi S(u_n) F(u_n, 0)\,dx.$$

Now we take $v = J_{T'S}(u_n)\phi$ as test function in (3.21) and we obtain

$$\int_0^L \phi \mathbf{a}\big(u_n, u_n'\big)\big(J_{T'S}(u_n)\big)'\,dx + \frac{1}{n}\int_0^L \phi u_n'\big(J_{T'S}(u_n)\big)'\,dx$$

$$= \int_0^L (f - u_n) J_{T'S}(u_n)\phi\,dx - \int_0^L J_{T'S}(u_n)\mathbf{a}\big(u_n, u_n'\big)\phi'\,dx - \frac{1}{n}\int_0^L J_{T'S}(u_n) u_n' \phi'\,dx.$$

Letting $n \to \infty$ we get

$$\limsup_n \int_0^L \phi \mathbf{a}\big(u_n, u_n'\big)\big(J_{T'S}(u_n)\big)' \, dx \leqslant \int_0^L \phi (f-u) J_{T'S}(u)\, dx - \int_0^L J_{T'S}(u)\mathbf{z}\phi' \, dx$$
$$= \int_0^L \phi \mathbf{z} D\big(J_{T'S}(u)\big) - \phi(L)\mathbf{z}(L) J_{T'S}\big(u(L_-)\big).$$

Finally,

$$\int_0^L \phi F_S\big(u, DT(u)\big) + \phi(L)\big|J_{\theta S}\big(T(u)\big)(L_-)\big|$$
$$\leqslant \int_0^L \phi \mathbf{z} D J_{T'S}(u) + \phi(L)\big|J_{\theta S}\big(T(0)\big)\big| - \phi(L)\mathbf{z}(L) J_{T'S}\big(u(L_-)\big) + \int_0^L \phi S(u) F(u,0)\, dx$$

and (3.35) holds. □

Lemma 3.9. *The inequalities* (3.17) *and* (3.18) *hold.*

Proof. Using (3.36) and the fact that $h(u, DT(u))$ is a measure concentrated in $]0, L[$, it follows that

$$h\big(u, DT(u)\big)^s = F\big(u, DT(u)\big)^s \leqslant \big(\mathbf{z} DT(u)\big)^s.$$

Hence,

$$\mathbf{z} DT(u) = \mathbf{z} T(u)' \mathcal{L}^1 + \big(\mathbf{z} DT(u)\big)^s \geqslant \mathbf{z} T(u)' \mathcal{L}^1 + h\big(u, DT(u)\big)^s = h\big(u, DT(u)\big),$$

and (3.17) holds.

Using (3.37) we have

$$\mathbf{z} D\big(J_{T'S}(u)\big) = \big(\mathbf{z} D\big(J_{T'S}(u)\big)\big)^{ac} + \big(\mathbf{z} D\big(J_{T'S}(u)\big)\big)^s \geqslant \mathbf{z}\big(J_{T'S}(u)\big)' + \big(F_S\big(u, DT(u)\big)\big)^s$$
$$= \mathbf{z}\big(J_{T'S}(u)\big)' \mathcal{L}^N + \big(h_S\big(u, DT(u)\big)\big)^s = h_S\big(u, DT(u)\big),$$

and we obtain (3.18). □

Lemma 3.10. *The Dirichlet condition* $\mathbf{a}(u, u')(L) = -cu(L_-)$ *holds.*

Proof. Firstly, observe that by (3.29) we have

$$\big|\mathbf{z}(L)\big| \leqslant cu(L_-).$$

Then, it is enough to prove the lemma in the case $u(L_-) > 0$. In that case, again by (3.29) and having in mind that $\mathbf{z}$ is continuous in $[0, L]$, we have

$$\mathbf{z}(L) = cu(L_-)\xi, \quad \text{with } |\xi| \leqslant 1. \tag{3.38}$$

Given $T \in T^+$, for $m > 1$ we consider $S := T^{m-1} \in \mathcal{P}^+$. Taking singular parts in (3.35) we have

$$\left|J_{\theta T^{m-1}}\big(T(u)\big)(L_-)\right| \leqslant -\mathbf{z}(L) J_{T^{m-1}T'}\big(u(L_-)\big) + \left|J_{\theta T^{m-1}}\big(T(0)\big)\right|. \tag{3.39}$$

Consider now $T = T_{d,d'}$ with $0 < d \leqslant u(L_-) \leqslant \|u-\|_\infty \leqslant d'$. Using (3.38), the inequality (3.39) particularizes to

$$\frac{c}{2}d^{m+1} + \frac{c}{m+1}\big(u^{m+1}(L_-) - d^{m+1}\big) \leqslant \frac{c}{2}d^{m+1} - \frac{c}{m}\xi u(L_-)\big(u^m(L_-) - d^m\big)$$

and letting $d \to 0^+$ we have

$$\frac{c}{m+1}u^{m+1}(L_-) \leqslant -\frac{c}{m}u(L_-)\xi u^m(L_-).$$

Then, since $u(L_-) > 0$, we get $\frac{m}{m+1} \leqslant -\xi$ for all $1 < m$. Therefore, since $|\xi| \leqslant 1$, we have $\xi = -1$. Consequently, by (3.38) we finish the proof. □

Proof of uniqueness. Let $u, \bar{u}$ be entropy solutions of (3.20) associated with $f, \overline{f} \in L^1(]0,L[)^+$, respectively.

Let ρ_n be a classical mollifier in $]0,L[$, $\psi \in \mathcal{D}(]0,L[)$ and $b > a > 2\epsilon > 0$. Let us write

$$\xi_n(x,y) = \rho_n(x-y)\psi\left(\frac{x+y}{2}\right) \quad \text{and} \quad T = T^a_{a,b}.$$

We need to consider truncature functions of the form $S_{\epsilon,l}(r) := T_\epsilon(r-l)^+ = T_{l,l+\epsilon}(r) - l \in \mathcal{T}^+$ and $S^l_\epsilon(r) := T_\epsilon(r-l)^- + \epsilon = T_{l-\epsilon,l}(r) + \epsilon - l \in \mathcal{T}^+$, where $l \geqslant 0$. Observe that $S^l_\epsilon(r) = -T_\epsilon(l-r)^+ + \epsilon$.

If we denote $\mathbf{z}(y) = \mathbf{a}(u(y), \partial_y u(y))$ and $\bar{\mathbf{z}}(x) = \mathbf{a}(\bar{u}(x), \partial_x \bar{u}(x))$, we have

$$u - \mathbf{z}' = f \quad \text{and} \quad \bar{u} - \bar{\mathbf{z}}' = \overline{f} \quad \text{in } \mathcal{D}'\big(]0,L[\big).$$

Then, multiplying the equation for u by $T(u(y))S_{\epsilon,\bar{u}(x)}(u(y))\xi_n(x,y)$, that for $\bar{u}$ by $T(\bar{u}(x)) \times S^{u(y)}_\epsilon(\bar{u}(x))\xi_n(x,y)$, integrating in both variables we obtain

$$\int_0^L\int_0^L \big[u(y)T\big(u(y)\big) - \bar{u}(x)T\big(\bar{u}(x)\big)\big]T_\epsilon\big(u(y) - \bar{u}(x)\big)^+\xi_n(x,y)\,dx\,dy$$

$$+\epsilon\int_0^L\int_0^L \big(\bar{u}(x) - \overline{f}(x)\big)T\big(\bar{u}(x)\big) + \xi_n(x,y)\,dx\,dy$$

$$+\int_0^L\int_0^L \xi_n(x,y)\big(\mathbf{z}D_y\big[T(u)S_{\epsilon,\bar{u}(x)}(u)\big]\,dx + \bar{\mathbf{z}}D_x\big[T(\bar{u})S^{u(y)}_\epsilon(\bar{u})\big]\,dy\big)$$

$$+\int_0^L\int_0^L T\big(u(y)\big)S_{\epsilon,\bar{u}(x)}\big(u(y)\big)\mathbf{z}(y)\cdot\partial_y\xi_n(x,y)\,dx\,dy$$

$$+\int_0^L\int_0^L T\big(\bar u(x)\big)S_\epsilon^{u(y)}\big(\bar u(x)\big)\bar{\mathbf z}(x)\cdot\partial_x\xi_n(x,y)\,dx\,dy$$

$$=\int_0^L\int_0^L\big[f(y)T\big(u(y)\big)-\bar f(x)T\big(\bar u(x)\big)\big]T_\epsilon\big(u(y)-\bar u(x)\big)^+\xi_n\,dx\,dy. \tag{3.40}$$

Let I denote all the terms at the left-hand side of the above identity, but the first one. From now on, since u, $\mathbf z$ are always functions of y, and $\bar u$, $\bar{\mathbf z}$ are always functions of x, to make our expressions shorter, we shall omit the arguments except in some cases where we find useful to remind them.

With slight modifications of the method used in the proof of uniqueness in [5] we can obtain the following result.

Lemma 3.11. *The following inequality is satisfied*

$$\frac{1}{\epsilon}I\geqslant o(\epsilon)-\int_0^L\left(\int_0^L\xi_n\bar{\mathbf z}D_xT(\bar u)\right)dy+\frac{1}{\epsilon}\int_0^L\int_0^L T_\epsilon(u-\bar u)^+\big(T(u)\mathbf z-T(\bar u)\bar{\mathbf z}\big)\cdot(\partial_x\xi_n+\partial_y\xi_n)\,dx\,dy,$$

where $o(\epsilon)$ denotes an expression such that $o(\epsilon)\to 0$ as $\epsilon\to 0$.

By the above lemma, dividing (3.40) by ϵ and letting $\epsilon\to 0$ we obtain

$$\int_0^L\int_0^L\xi_n(x,y)\big(u(y)T\big(u(y)\big)-\bar u(x)T\big(\bar u(x)\big)\big)\operatorname{sign}_0^+\big(u(y)-\bar u(x)\big)\,dx\,dy$$

$$+\int_0^L\int_0^L\rho_n(x-y)\operatorname{sign}_0^+\big(u(y)-\bar u(x)\big)\big(T\big(u(y)\big)\mathbf z(y)-T\big(\bar u(x)\big)\bar{\mathbf z}(x)\big)\cdot\psi'\left(\frac{x+y}{2}\right)dx\,dy$$

$$\leqslant\int_0^L\int_0^L\xi_n(x,y)\big(f(y)T\big(u(y)\big)-\bar f(x)T\big(\bar u(x)\big)\big)\operatorname{sign}_0^+\big(u(y)-\bar u(x)\big)\,dx\,dy$$

$$+\int_0^L\left(\int_0^L\xi_n(x,y)\bar{\mathbf z}D_xT(\bar u)\right)dy,$$

where

$$\operatorname{sign}_0^+(r)=\begin{cases}1 & \text{if } r>0,\\ 0 & \text{if } r\leqslant 0.\end{cases}$$

Letting $n\to\infty$, we find

$$\int_0^L\psi(x)\big(u(x)T\big(u(x)\big)-\bar u(x)T\big(\bar u(x)\big)\big)\operatorname{sign}_0^+\big(u(x)-\bar u(x)\big)\,dx$$

$$+\int_0^L \operatorname{sign}_0^+\big(u(x)-\bar{u}(x)\big)\big(T\big(u(x)\big)\mathbf{z}(x)-T\big(\bar{u}(x)\big)\bar{\mathbf{z}}(x)\big)\cdot\psi'(x)\,dx$$

$$\leqslant \int_0^L \psi(x)\big[f(x)T\big(u(x)\big)-\bar{f}(x)T\big(\bar{u}(x)\big)\big]\operatorname{sign}_0^+\big(u(x)-\bar{u}(x)\big)\,dx+\int_0^L \psi(x)\bar{\mathbf{z}}DT(\bar{u}).$$

Taking now a sequence $\psi_m \uparrow \chi_{]0,L[}$, $\psi_m \in \mathcal{D}(]0,L[)$ in the above formula, we have

$$\int_0^L \big(u(x)T\big(u(x)\big)-\bar{u}(x)T\big(\bar{u}(x)\big)\big)\operatorname{sign}_0^+\big(u(x)-\bar{u}(x)\big)\,dx$$

$$+\lim_{m\to\infty}\int_0^L \operatorname{sign}_0^+\big(u(x)-\bar{u}(x)\big)\big(T\big(u(x)\big)\mathbf{z}(x)-T\big(\bar{u}(x)\big)\bar{\mathbf{z}}(x)\big)\cdot\psi_m'(x)\,dx$$

$$\leqslant \int_0^L \big(f(x)T\big(u(x)\big)-\bar{f}(x)T\big(\bar{u}(x)\big)\big)\operatorname{sign}_0^+\big(u(x)-\bar{u}(x)\big)\,dx+\int_0^L \bar{\mathbf{z}}DT(\bar{u}).$$

Now we deal with the second term in the above expression:

$$\lim_{m\to\infty}\int_0^L \operatorname{sign}_0^+\big(u(x)-\bar{u}(x)\big)\big(T\big(u(x)\big)\mathbf{z}(x)-T\big(\bar{u}(x)\big)\bar{\mathbf{z}}(x)\big)\cdot\psi_m'(x)\,dx$$

$$=-\lim_{m\to\infty}\int_0^L \psi_m(x)\big\{\mathbf{z}D\big[\operatorname{sign}_0^+(u-\bar{u})T(u)\big]-\bar{\mathbf{z}}(x)D\big[\operatorname{sign}_0^+(u-\bar{u})T(\bar{u})\big]\big\}$$

$$+\lim_{m\to\infty}\int_0^L \psi_m(x)\big\{\operatorname{sign}_0^+\big(u(x)-\bar{u}(x)\big)T\big(\bar{u}(x)\big)\bar{\mathbf{z}}'(x)-\operatorname{sign}_0^+\big(u(x)-\bar{u}(x)\big)T\big(u(x)\big)\mathbf{z}'(x)\big\}\,dx,$$

which leads to

$$=-\int_0^L \operatorname{sign}_0^+\big(u(x)-\bar{u}(x)\big)T\big(u(x)\big)\mathbf{z}'(x)\,dx-\int_0^L \mathbf{z}D\big[\operatorname{sign}_0^+(u-\bar{u})T(u)\big]$$

$$+\int_0^L \operatorname{sign}_0^+\big(u(x)-\bar{u}(x)\big)T\big(\bar{u}(x)\big)\bar{\mathbf{z}}'(x)\,dx+\int_0^L \bar{\mathbf{z}}D\big[\operatorname{sign}_0^+(u-\bar{u})T(\bar{u})\big]$$

$$=\big[\mathbf{z}(0)T\big(u(0_+)\big)-\bar{\mathbf{z}}(0)T\big(\bar{u}(0_+)\big)\big]\operatorname{sign}_0^+\big(u(0_+)-\bar{u}(0_+)\big)$$

$$-\big[\mathbf{z}(L)T\big(u(L_-)\big)-\bar{\mathbf{z}}(L)T\big(\bar{u}(L_-)\big)\big]\operatorname{sign}_0^+\big(u(L_-)-\bar{u}(L_-)\big).$$

Therefore,

$$\int_0^L \big(u(x)T\big(u(x)\big) - \bar{u}(x)T\big(\bar{u}(x)\big)\big)\,\mathrm{sign}_0^+\big(u(x)-\bar{u}(x)\big)\,dx$$
$$+\big[\mathbf{z}(0)T\big(u(0_+)\big) - \bar{\mathbf{z}}(0)T\big(\bar{u}(0_+)\big)\big]\,\mathrm{sign}_0^+\big(u(0_+)-\bar{u}(0_+)\big)$$
$$-\big[\mathbf{z}(L)T\big(u(L_-)\big) - \bar{\mathbf{z}}(L)T\big(\bar{u}(L_-)\big)\big]\,\mathrm{sign}_0^+\big(u(L_-)-\bar{u}(L_-)\big)$$
$$\leqslant \int_0^L \big[f(x)T\big(u(x)\big) - \bar{f}(x)T\big(\bar{u}(x)\big)\big]\,\mathrm{sign}_0^+\big(u(x)-\bar{u}(x)\big)\,dx + \int_0^L \bar{\mathbf{z}}DT(\bar{u}).$$

Dividing by $b>0$, and letting $a \to 0^+$, and then $b \to 0^+$ in this order, we obtain

$$\int_0^L (u\chi_{[u>0]} - \bar{u}\chi_{[\bar{u}>0]})\,\mathrm{sign}_0^+(u-\bar{u})\,dx$$
$$+\big[\mathbf{z}(0)\,\mathrm{sign}_0^+\big(u(0_+)\big) - \bar{\mathbf{z}}(0)\,\mathrm{sign}_0^+\big(\bar{u}(0_+)\big)\big]\,\mathrm{sign}_0^+\big(u(0_+)-\bar{u}(0_+)\big)$$
$$-\big[\mathbf{z}(L)\,\mathrm{sign}_0^+\big(u(L_-)\big) - \bar{\mathbf{z}}(L)\,\mathrm{sign}_0^+\big(\bar{u}(L_-)\big)\big]\,\mathrm{sign}_0^+\big(u(L_-)-\bar{u}(L_-)\big)$$
$$\leqslant \int_0^L (f\chi_{[u>0]} - \bar{f}\chi_{[\bar{u}>0]})\,\mathrm{sign}_0^+(u-\bar{u})\,dx + \lim_{b\to 0}\frac{1}{b}\left(\lim_{a\to 0}\int_0^L \bar{\mathbf{z}}DT(\bar{u})\right).$$

Now, since $\mathbf{z}(0) = \bar{\mathbf{z}}(0) = -\beta \neq 0$, and $u(0_+) \geqslant \frac{\beta}{c} > 0$ and $\bar{u}(0_+) \geqslant \frac{\beta}{c} > 0$ by (3.19), we have that the second term in the above expression vanishes. On the other hand, since $\mathbf{z}(L) = -cu(L_-)$ and $\bar{\mathbf{z}}(L) = -c\bar{u}(L_-)$, the third term in the above expression is non-negative. Consequently,

$$\int_0^L (u\chi_{[u>0]} - \bar{u}\chi_{[\bar{u}>0]})\,\mathrm{sign}_0^+(u-\bar{u})\,dx$$
$$\leqslant \int_0^L (f\chi_{[u>0]} - \bar{f}\chi_{[\bar{u}>0]})\,\mathrm{sign}_0^+(u-\bar{u})\,dx + \lim_{b\to 0}\frac{1}{b}\left(\lim_{a\to 0}\int_0^L \bar{\mathbf{z}}DT(\bar{u})\right). \tag{3.41}$$

Next we claim that

$$f = 0 \quad \text{a.e. on } [u=0] \quad \text{and} \quad \bar{f} = 0 \quad \text{a.e on } [\bar{u}=0]. \tag{3.42}$$

Let $0 \leqslant \phi \in \mathcal{D}(]0,L[)$ and $a>0$, $\epsilon>0$. Multiplying $f-u = -\mathbf{z}'$ in $\mathcal{D}'(]0,L[)$ by $T^a_{a,a+\epsilon}(u)\phi$ and integrating by parts and having in mind (3.17) and (2.13), we have

$$\int_0^L (f-u)T^a_{a,a+\epsilon}(u)\phi\,dx = \int_0^L \phi\mathbf{z}DT^a_{a,a+\epsilon}(u) + \int_0^L \mathbf{z}\cdot\phi' T^a_{a,a+\epsilon}(u)\,dx \geqslant \int_0^L \mathbf{z}\cdot\phi' T^a_{a,a+\epsilon}(u)\,dx.$$

Dividing by ϵ and letting $\epsilon \to 0^+$, we get

$$\int_0^L (f-u)\chi_{[u>a]}\phi\,dx \geqslant \int_0^L \mathbf{z}\cdot\phi'\chi_{[u>a]}\,dx.$$

Hence

$$\int_0^L (f-u)\chi_{[u\leqslant a]}\phi\,dx = \int_0^L (f-u)\phi\,dx - \int_0^L (f-u)\chi_{[u>a]}(x)\phi\,dx$$
$$\leqslant \int_0^L (f-u)\phi\,dx - \int_0^L \mathbf{z}\cdot\phi'\chi_{[u>a]}\,dx = \int_0^L \mathbf{z}\cdot\phi'\chi_{[u\leqslant a]}\,dx.$$

Then, letting $a \to 0^+$, since $\mathbf{z}=0$ in $[u=0]$, we have

$$\int_0^L f\chi_{[u=0]}\phi\,dx = \int_0^L (f-u)\chi_{[u=0]}\phi\,dx \leqslant 0,$$

for all $0 \leqslant \phi \in \mathcal{D}(]0,L[)$, from where it follows that $f\chi_{[u=0]}=0$ a.e. in $]0,L[$. Similarly, $\overline{f}\chi_{[\bar{u}=0]}=0$ a.e. in $]0,L[$ and (3.42) holds.

On the other hand, by (3.42) we have

$$\lim_{b\to 0}\frac{1}{b}\left(\lim_{a\to 0}\int_0^L \bar{\mathbf{z}}DT(\bar{u})\right) = -\lim_{b\to 0}\frac{1}{b}\lim_{a\to 0}\left(\bar{\mathbf{z}}(0)T\big(\bar{u}(0_+)\big) - \bar{\mathbf{z}}(L)T\big(\bar{u}(L_-)\big) + \int_0^L T(\bar{u})\bar{\mathbf{z}}'\,dx\right)$$
$$= -\lim_{b\to 0}\frac{1}{b}\left(\bar{\mathbf{z}}(0)T_{0,b}\big(\bar{u}(0_+)\big) - \bar{\mathbf{z}}(L)T_{0,b}\big(\bar{u}(L_-)\big) + \int_0^L T_{0,b}(\bar{u})\bar{\mathbf{z}}'\,dx\right)$$
$$= -\bar{\mathbf{z}}(0)\,\mathrm{sign}_0^+\big(\bar{u}(0_+)\big) + \bar{\mathbf{z}}(L)\,\mathrm{sign}_0^+\big(\bar{u}(L_-)\big) - \int_0^L \chi_{[\bar{u}>0]}\bar{\mathbf{z}}'\,dx$$
$$= -\bar{\mathbf{z}}(0)\,\mathrm{sign}_0^+\big(\bar{u}(0_+)\big) + \bar{\mathbf{z}}(L)\,\mathrm{sign}_0^+\big(\bar{u}(L_-)\big) - \int_0^L \bar{\mathbf{z}}'\,dx$$
$$= \bar{\mathbf{z}}(0)\big(1-\mathrm{sign}_0^+\big(\bar{u}(0_+)\big)\big) + \bar{\mathbf{z}}(L)\big(\mathrm{sign}_0^+\big(\bar{u}(L_-)\big)-1\big) = 0.$$

Then, from (3.41), it follows that

$$\int_0^L (u\chi_{[u>0]} - \bar{u}\chi_{[\bar{u}>0]})\,\mathrm{sign}_0^+(u-\bar{u})\,dx \leqslant \int_0^L (f\chi_{[u>0]} - \overline{f}\chi_{[\bar{u}>0]})\,\mathrm{sign}_0^+(u-\bar{u})\,dx.$$

Hence, using (3.42), we obtain

$$\int_0^L (u-\bar{u})^+ \, dx \leqslant \int_0^L (f-\overline{f}) \operatorname{sign}_0^+(u-\bar{u}) \, dx \leqslant \int_0^L (f-\overline{f})^+ \, dx.$$

This concludes the proof of the uniqueness part of Theorem 3.2. □

4. Semigroup solution

In this section we shall associate an accretive operator in $L^1(]0,L[)$ to the problem (3.16).

Definition 4.1. $(u,v) \in \mathcal{B}_\beta$ if and only if $0 \leqslant u \in TBV^+(]0,L[)$, $v \in L^1(]0,L[)$ and u is the entropy solution of problem (3.16).

From Theorem 3.2, it follows that the operator $\mathcal{B}_\beta$ is T-accretive in $L^1(]0,L[)$ and verifies

$$L^\infty\big(]0,L[\big)^+ \subset R(I+\lambda\mathcal{B}_\beta) \quad \text{for all } \lambda > 0. \tag{4.43}$$

In order to get an L^∞-estimate of the resolvent, we need to find a steady state solution, that is, a function u_β which is an entropy solution of the problem

$$\begin{cases} -\big(\mathbf{a}\big(u_\beta, u'_\beta\big)\big)' = 0 & \text{in }]0,L[, \\ -\mathbf{a}\big(u_\beta, u'_\beta\big)\big|_{x=0} = \beta > 0 & \text{and} \quad u_\beta(L) = 0. \end{cases} \tag{4.44}$$

Proposition 4.2. *There is a non-increasing function $u_\beta \in C^1(]0,L[)$, with $u_\beta \geqslant \frac{\beta}{c}$, that is an entropy solution of the stationary problem* (4.44)*. Moreover, there exists a constant $M := M(c,\beta,\nu,L)$ such that*

$$\|u_\beta\|_\infty \leqslant M.$$

Proof. Integrating (4.44) over $]0,L[$ we find that $\mathbf{a}(u_\beta, u'_\beta)(L) = -\beta$. Now, if u_β has to fulfill the weak Dirichlet condition $\mathbf{a}(u_\beta, u'_\beta)(L) = -cu_\beta(L_-)$ then we must have $u_\beta(L_-) = \beta/c$. We will follow this prescription hereafter.

If u_β is a solution of the problem (4.44), we have

$$-\big(\mathbf{a}\big(u_\beta, u'_\beta\big)\big)' = 0 \quad \Longleftrightarrow \quad \nu \frac{u_\beta u'_\beta}{\sqrt{u_\beta^2 + \frac{\nu^2}{c^2}(u'_\beta)^2}} = -\beta.$$

Then, assuming that $u'_\beta < 0$, we get

$$u'_\beta = -\frac{\beta u_\beta}{\nu\sqrt{u_\beta^2 - (\frac{\beta}{c})^2}}.$$

Thus, we get that u_β satisfies the ordinary differential equation

$$\frac{u'_\beta\sqrt{u_\beta^2 - (\frac{\beta}{c})^2}}{u_\beta} = -\frac{\beta}{\nu}.$$

By means of the change of variable $v^2 = u_\beta^2 - (\frac{\beta}{c})^2$, we arrive to the ODE

$$-\frac{\beta}{\nu} = \left(1 - \frac{1}{1 + (\frac{v}{\beta/c})^2}\right) v'.$$

Then,

$$\int_x^L \left(-\frac{\beta}{\nu}\right) dy = \int_x^L v'(y)\, dy - \int_x^L \frac{v'(y)}{1 + (\frac{v(y)}{\beta/c})^2}\, dy$$
$$= v(L) - v(x) - \frac{\beta}{c} \arctan\left(\frac{v(L)}{\beta/c}\right) + \frac{\beta}{c} \arctan\left(\frac{v(x)}{\beta/c}\right).$$

Hence, we get

$$x = L - \frac{\nu}{\beta}\sqrt{u_\beta(x)^2 - \left(\frac{\beta}{c}\right)^2} + \frac{\nu}{c}\arctan\left[\frac{c}{\beta}\sqrt{u_\beta(x)^2 - \left(\frac{\beta}{c}\right)^2}\right]. \tag{4.45}$$

If $x = u_\beta^{-1}(y)$, then we can write (4.45) as

$$u_\beta^{-1}(y) = L - \frac{\nu}{\beta}\sqrt{y^2 - \left(\frac{\beta}{c}\right)^2} + \frac{\nu}{c}\arctan\left[\frac{c}{\beta}\sqrt{y^2 - \left(\frac{\beta}{c}\right)^2}\right].$$

Thus,

$$\left(u_\beta^{-1}\right)'(y) = \frac{y}{\sqrt{y^2 - (\frac{\beta}{c})^2}}\left(\frac{\nu}{\beta}\right)\left(-1 + \frac{\beta^2}{c^2 y^2}\right),$$

and consequently, since $(u_\beta)(L_-) = \frac{\beta}{c}$, we obtain that

$$(u_\beta)'(L_-) = \lim_{y \searrow \frac{\beta}{c}} \frac{1}{(u_\beta^{-1})'(y)} = \lim_{y \searrow \frac{\beta}{c}} \frac{\sqrt{y^2 - (\frac{\beta}{c})^2}}{y}\left(\frac{\beta}{\nu}\right)\left(\frac{c^2 y^2}{\beta^2 - c^2 y^2}\right) = -\infty.$$

Finally, since u_β satisfies $-(\mathbf{a}(u_\beta(x), u'_\beta(x)))' = 0$ if $x \in]0, L[$ and satisfies the boundary conditions also, we have that u_β is an entropy solution of the problem (4.44). □

The following homogeneity of the operator $\mathcal{B}_\beta$ will be important to get the L^∞-estimate of the resolvent.

Proposition 4.3. *For $\mu > 0$, $\lambda > 0$ and $\beta > 0$, we have*

$$(I + \lambda\mathcal{B}_\beta)^{-1}(\mu u) = \mu(I + \lambda\mathcal{B}_{\frac{\beta}{\mu}})^{-1}(u). \tag{4.46}$$

Moreover, for $\beta_1 \leqslant \beta_2$, $u \in L^\infty(]0, L[)^+$ and $\lambda > 0$ such that $(I + \lambda\mathcal{B}_{\beta_2})^{-1}(u) \in BV(]0, L[)$, we have

$$(I + \lambda\mathcal{B}_{\beta_1})^{-1}(u) \leqslant (I + \lambda\mathcal{B}_{\beta_2})^{-1}(u) \quad a.e.\ x \in]0, L[. \tag{4.47}$$

Proof. From the definition of the operator it is easy to see that if $u \in D(\mathcal{B}_{\frac{\beta}{\mu}})$, then $\mu u \in D(\mathcal{B}_\beta)$ and $\mathcal{B}_\beta(\mu u) = \mu \mathcal{B}_{\frac{\beta}{\mu}}(u)$. Then, we have

$$v := (I + \lambda\mathcal{B}_\beta)^{-1}(\mu u) \iff v + \lambda\mathcal{B}_\beta(v) = \mu u \iff \frac{1}{\mu}v + \frac{1}{\mu}\lambda\mathcal{B}_{\mu\frac{\beta}{\mu}}(v) = u$$
$$\iff \frac{1}{\mu}v + \lambda\mathcal{B}_{\frac{\beta}{\mu}}\left(\frac{v}{\mu}\right) = u \iff (I + \lambda\mathcal{B}_{\frac{\beta}{\mu}})^{-1}(u) = \frac{v}{\mu},$$

from where (4.46) follows.

Finally, let us see that (4.47) holds. Let $u_i := (I + \lambda\mathcal{B}_{\beta_i})^{-1}(u)$, $i = 1, 2$. Then, u_i is an entropy solution of the problem

$$\begin{cases} u_i - \lambda\big(\mathbf{a}(u_i, u_i')\big)' = u & \text{in }]0, L[, \\ -\mathbf{a}(u_i, u_i')\big|_{x=0} = \beta_i > 0 & \text{and} \quad u(L) = 0. \end{cases}$$

Therefore, if p_n are non-negative increasing functions that are an approximation of the sign_0^+ function, having in mind (2.10), since $p_n(u_1 - u_2) \in BV(]0, L[)$, we get

$$\begin{aligned}
&\int_0^L (u_1 - u_2)p_n(u_1 - u_2)\,dx \\
&\quad = \int_0^L \lambda\big(\big(\mathbf{a}(u_1, u_1')\big)' - \big(\mathbf{a}(u_2, u_2')\big)'\big)p_n(u_1 - u_2)\,dx \\
&\quad = -\int_0^L \lambda\big(\mathbf{a}(u_1, u_1') - \mathbf{a}(u_2, u_2')\big)D\big(p_n(u_1 - u_2)\big) + \lambda\big(\mathbf{a}(u_1, u_1')(L_-) - \mathbf{a}(u_2, u_2')(L_-)\big) \\
&\qquad \times p_n(u_1 - u_2)(L_-) - \lambda\big(\mathbf{a}(u_1, u_1')(0_+) - \mathbf{a}(u_2, u_2')(0_+)\big)p_n(u_1 - u_2)(0_+) \\
&\quad \leqslant \lambda\big(\mathbf{a}(u_1, u_1')(L_-) - \mathbf{a}(u_2, u_2')(L_-)\big)p_n(u_1 - u_2)(L_-) + \lambda(\beta_1 - \beta_2)p_n(u_1 - u_2)(0_+) \\
&\quad \leqslant \lambda\big(\mathbf{a}(u_1, u_1')(L_-) - \mathbf{a}(u_2, u_2')(L_-)\big)p_n(u_1 - u_2)(L_-).
\end{aligned}$$

Then, taking limit as $n \to +\infty$ we get

$$\int_0^L (u_1 - u_2)^+\,dx \leqslant \lambda\big(\mathbf{a}(u_1, u_1')(L_-) - \mathbf{a}(u_2, u_2')(L_-)\big)\operatorname{sign}_0^+(u_1 - u_2)(L_-) \leqslant 0,$$

since $\mathbf{a}(u_i, u_i')(L) = -cu_i(L_-)$, $i = 1, 2$. Therefore, $u_1 \leqslant u_2$, and we finish the proof. □

Proposition 4.4. *For $u \in L^\infty(]0, L[)^+$ and $\lambda > 0$, we have*

$$0 \leqslant (I + \lambda\mathcal{B}_\beta)^{-1}(u) \leqslant \mu u_\beta, \quad \textit{with } \mu = \max\left\{\frac{c\|u\|_\infty}{\beta}, 1\right\}.$$

Proof. Let u_β be the entropy solution of the stationary problem (4.44) given in Proposition 4.2. Then, $(u_\beta, 0) \in \mathcal{B}_\beta$, from where it follows that

$$(I + \lambda \mathcal{B}_\beta)^{-1}(u_\beta) = u_\beta. \tag{4.48}$$

On the other hand, since $u_\beta \geqslant \frac{\beta}{c}$, if $\mu := \max\{\frac{c\|u\|_\infty}{\beta}, 1\}$, we have $0 \leqslant u \leqslant \mu u_\beta$. Hence, by Proposition 4.3 and having in mind (4.48), we get

$$0 \leqslant (I + \lambda \mathcal{B}_\beta)^{-1}(u) \leqslant (I + \lambda \mathcal{B}_\beta)^{-1}(\mu u_\beta) = \mu (I + \lambda \mathcal{B}_{\frac{\beta}{\mu}})^{-1}(u_\beta) \leqslant \mu (I + \lambda \mathcal{B}_\beta)^{-1}(u_\beta) = \mu u_\beta. \qquad \square$$

Next we introduce the main result of this section, which paves the way for the operator $\mathcal{B}_\beta$ to generate an order-preserving semigroup [12].

Theorem 4.5. *$\mathcal{B}_\beta$ is T-accretive in $L^1(]0, L[)$ and verifies the range condition*

$$\overline{D(\mathcal{B}_\beta)}^{L^1(]0,L[)} = L^1\big(]0, L[\big)^+ \subset R(I + \lambda \mathcal{B}_\beta) \quad \textit{for all } \lambda > 0.$$

Proof. The T-accretivity of the operator $\mathcal{B}_\beta$ is known, and that it verifies (4.43) also. To prove the density of $D(\mathcal{B}_\beta)$ in $L^1(]0, L[)^+$, we prove that $\mathcal{D}(]0, L[)^+ \subseteq \overline{D(\mathcal{B}_\beta)}^{L^1(]0,L[)}$. Let $0 \leqslant v \in \mathcal{D}(]0, L[)$. By (4.43), $v \in R(I + \frac{1}{n}\mathcal{B}_\beta)$ for all $n \in \mathbb{N}$. Thus, for each $n \in \mathbb{N}$, there exists $u_n \in D(\mathcal{B})$ such that $(u_n, n(v - u_n)) \in \mathcal{B}$. Since $u_n = (I + \frac{1}{n}\mathcal{B}_\beta)^{-1}(v)$, by Proposition 4.4, we get

$$\|u_n\|_\infty \leqslant M := M\big(\beta, c, \nu, L, \|v\|_\infty\big). \tag{4.49}$$

Let $\epsilon > 0$. Since

$$n(v - u_n) = -D\mathbf{a}\big(u_n, u_n'\big) \quad \text{in } \mathcal{D}'\big(]0, L[\big),$$

multiplying by $v - S_\epsilon(u_n)$, with $S_\epsilon := T_{\epsilon, \|v\|_\infty}$, and integrating by parts, we get

$$\int_0^L \big(v - S_\epsilon(u_n)\big) n(v - u_n)\, dx$$
$$= \left[\int_0^L \mathbf{a}\big(u_n, u_n'\big)\big(Dv - DS_\epsilon(u_n)\big)\right] - c u_n(L_-) S_\epsilon(u_n)(L_-) + \beta S_\epsilon(u_n)(0_+).$$

Then, since

$$\int_0^L \mathbf{a}\big(u_n, u_n'\big) DS_\epsilon(u_n) \geqslant 0,$$

having in mind (4.49), we get

$$\int_0^L \big(v - S_\epsilon(u_n)\big)(v - u_n)\, dx \leqslant \frac{1}{n}\left[\int_0^L \mathbf{a}\big(u_n, u_n'\big) Dv\right] + \frac{1}{n}\beta S_\epsilon(u_n)(0_+) \leqslant \frac{C}{n}.$$

Letting $\epsilon \to 0^+$, we get

$$\int_0^L (v - u_n)^2 \, dx \leqslant \frac{C}{n},$$

and we obtain that $u_n \to v$ in $L^2(]0, L[)$, as $n \to \infty$. Moreover, we have $u_n \to v$ in $L^1(]0, L[)$, as $n \to \infty$. Therefore $v \in \overline{D(\mathcal{B}_\beta)}^{L^1(]0,L[)}$ and the proof of the density of $D(\mathcal{B}_\beta)$ in $L^1(]0, L[)^+$ is complete.

To finish the proof of the theorem, we only need to show that the operator $\mathcal{B}_\beta$ is closed in $L^1(]0, L[) \times L^1(]0, L[)$. Given $(u_n, v_n) \in \mathcal{B}_\beta$ such that $u_n \to u$ and $v_n \to v$ in $L^1(]0, L[)$, we need to prove that $(u, v) \in \mathcal{B}_\beta$. Since $(u_n, v_n) \in \mathcal{B}_\beta$, we have that $u_n \in TBV^+(]0, L[)$ and $\mathbf{z}_n := \mathbf{a}(u_n, u_n') \in C([0, L])$ satisfy

$$v_n = -D\mathbf{z}_n \quad \text{in } \mathcal{D}'\big(]0, L[\big), \tag{4.50}$$

$$h\big(u_n, DT(u_n)\big) \leqslant \mathbf{z}_n DT(u_n) \quad \text{as measures } \forall T \in \mathcal{T}^+, \tag{4.51}$$

$$h_S\big(u_n, DT(u_n)\big) \leqslant \mathbf{z}_n D J_{T'S}(u_n) \quad \text{as measures } \forall S \in \mathcal{P}^+,\ T \in \mathcal{T}^+,$$

$$-\mathbf{z}_n(0) = \beta \quad \text{and} \quad \mathbf{z}_n(L) = -c u_n(L_-). \tag{4.52}$$

Let $T = T_{a,b} \in \mathcal{T}_r$. Multiplying (4.50) by $T(u_n)$ and applying integration by parts (Lemma 2.1), we get

$$\int_0^L v_n T(u_n) \, dx = \int_0^L \mathbf{z}_n DT(u_n) - \mathbf{z}_n(L) T\big(u_n(L_-)\big) - \beta T\big(u_n(0_+)\big),$$

from where it follows that

$$\int_0^L \mathbf{z}_n DT(u_n) \leqslant b\big(\beta + \|v\|_1\big) \leqslant C. \tag{4.53}$$

Here we used the boundary condition (4.52) to be able to disregard the term related to $\mathbf{z}_n(L)$, as it has the right sign.

On the other hand, by (4.51) and having in mind (2.9), we get

$$\int_0^L \mathbf{z}_n DT(u_n) \geqslant \frac{c}{2} \int_0^L \big|D\big(\big[T(u_n)\big]^2\big)\big| - \frac{c^2}{\nu} \int_0^L T(u_n)^2 \, dx. \tag{4.54}$$

By (4.53) and (4.54), we obtain that

$$\int_0^L \big|D\big(\big[T(u_n)\big]^2\big)\big| \leqslant \frac{2c}{\nu} \int_0^L T(u_n)^2 \, dx + \frac{2C}{c} \leqslant \frac{2cLb^2}{\nu} + \frac{2C}{c} = C. \tag{4.55}$$

Using the co-area formula as in the proof of Theorem 3.2, from (4.55) we deduce that

$$\int_0^L \big|DT(u_n)\big| \leqslant \frac{C}{2a} \quad \forall n \in \mathbb{N}.$$

Then, since the total variation is semi-continuous in $L^1(]0, L[)$, we have

$$\int_0^L \left|DT(u)\right| \leqslant \liminf_{n\to\infty} \int_0^L \left|DT(u_n)\right| \leqslant \frac{C}{2a}.$$

Hence, $T(u) \in BV(]0, L[)$, and consequently, $u \in TBV^+(]0, L[)$.

Since $\mathbf{z}_n = c|u_n|\mathbf{b}(u_n, u_n')$ with $|\mathbf{b}(u_n, u_n')| \leqslant 1$, for all measurable subsets $E \subset]0, L[$, we have

$$\int_E |\mathbf{z}_n|\, dx \leqslant c \int_E |u_n|\, dx.$$

Therefore, by Dunford–Pettis' Theorem, we can assume that

$$\mathbf{z}_n \rightharpoonup \mathbf{z} \quad \text{weakly in } L^1\big(]0, L[\big). \tag{4.56}$$

Moreover, since $|\mathbf{b}(u_n, u_n')| \leqslant 1$, we also can assume that

$$\mathbf{b}\big(u_n, u_n'\big) \rightharpoonup \mathbf{z}_b \quad \text{weakly}^* \text{ in } L^\infty\big(]0, L[\big). \tag{4.57}$$

As $u_n \to u$ in $L^1(]0, L[)$, from (4.56) and (4.57), we obtain that

$$\mathbf{z} = cu\mathbf{z}_b. \tag{4.58}$$

As $v_n \to v$ in $L^1(]0, L[)$, from (4.56) and (4.50), we easily deduce that

$$v = -D\mathbf{z} \quad \text{in } \mathcal{D}'\big(]0, L[\big), \tag{4.59}$$

and by (4.58) and (4.59), we have $\mathbf{z} \in W^{1,1}(]0, L[) \subset C([0, L])$.

Lemma 4.6. *The following equality is verified*

$$\mathbf{z}(x) = \mathbf{a}\big(u(x), u'(x)\big) \quad a.e.\ x \in]0, L[.$$

Proof. We use Minty–Browder's technique. Let $0 < a < b$, let $0 \leqslant \phi \in C_c^1(]0, L[)$ and let $g \in C^2([0, L])$. By (2.10), we have that

$$\int_0^L \phi\big[\mathbf{a}\big(u_n, u_n'\big) - \mathbf{a}\big(u_n, g'\big)\big] T_{a,b}'(u_n)(u_n - g)'\, dx \geqslant 0. \tag{4.60}$$

Let us denote

$$J_{\mathbf{a}}(x, r) := \int_0^r \mathbf{a}\big(s, g'(x)\big)\, ds,$$

$$J_{\mathbf{a}'}(x, r) := \int_0^r \partial_x\big[\mathbf{a}\big(s, g'(x)\big)\big]\, ds = \int_0^r \frac{\partial \mathbf{a}}{\partial \xi}\big(s, g'(x)\big) g''(x)\, ds$$

and observe that

$$-\mathbf{a}\big(T_{a,b}\big(u_n(x)\big), g'(x)\big)\big[T_{a,b}(u_n)\big]' = -D^{ac}\big[J_{\mathbf{a}}\big(x, T_{a,b}\big(u_n(x)\big)\big)\big] + J_{\mathbf{a}'}\big(x, T_{a,b}\big(u_n(x)\big)\big),$$

this we will substitute into (4.60). Note now that, using (4.51)

$$\int_0^L \phi\big[\mathbf{z}_n D^s T_{a,b}(u_n) - D^s J_{\mathbf{a}}\big(x, T_{a,b}(u_n)\big)\big] \geqslant \int_0^L \phi\big[h\big(u_n, DT_{a,b}(u_n)\big)^s - D^s J_{\mathbf{a}}\big(x, T_{a,b}(u_n)\big)\big] \geqslant 0,$$

where the last inequality is proved using the properties of the Lagrangian (see [4]). Then we can add this inequality to (4.60):

$$\begin{aligned} 0 \leqslant & \int_0^L \phi\big[\mathbf{z}_n DT(u_n) - D J_{\mathbf{a}}\big(x, T_{a,b}\big(u_n(x)\big)\big)\big] \\ & + \int_0^L \phi\big[J_{\mathbf{a}'}\big(x, T\big(u_n(x)\big)\big) - \mathbf{z}_n g' T'_{a,b}(u_n) + g' T'_{a,b}(u_n)\mathbf{a}\big(u_n, g'\big)\big]\,dx. \end{aligned}$$

Now, since

$$\begin{aligned} & \int_0^L \phi \mathbf{z}_n\big[DT_{a,b}(u_n) - g' T'_{a,b}(u_n)\big] \\ & = \int_0^L \phi \mathbf{z}_n D\big[T_{a,b}(u_n) - g\big] + \int_0^L \phi \mathbf{z}_n g'\big(1 - T'_{a,b}(u_n)\big)\,dx \\ & = -\int_0^L v_n \phi\big(T_{a,b}(u_n) - g\big)\,dx - \int_0^L \big(T_{a,b}(u_n) - g\big)\mathbf{a}\big(u_n, u_n'\big)\phi'\,dx + \int_0^L \phi \mathbf{z}_n g'\big(1 - T'_{a,b}(u_n)\big)\,dx \end{aligned}$$

we get

$$\lim_{n\to+\infty} \int_0^L \phi \mathbf{z}_n\big[DT_{a,b}(u_n) - g' T'_{a,b}(u_n)\big]\,dx \leqslant \big\langle \mathbf{z} D\big(T_{a,b}(u) - g\big), \phi\big\rangle + \|g'\|_\infty \int_0^L |\mathbf{z}|\phi\big(1 - T'_{a,b}(u)\big)\,dx.$$

On the other hand, the almost everywhere convergence of u_n implies that

$$J_{\mathbf{a}'}\big(x, T_{a,b}\big(u_n(x)\big)\big) \to J_{\mathbf{a}'}\big(x, T_{a,b}\big(u(x)\big)\big) \quad \text{a.e.}$$

and we also have (see [1, Proposition 3.13]) that

$$D\big[J_{\mathbf{a}}\big(x, T_{a,b}\big(u_n(x)\big)\big)\big] \rightharpoonup D\big[J_{\mathbf{a}}\big(x, T_{a,b}\big(u(x)\big)\big)\big] \quad \text{weakly as measures.}$$

As a consequence, we have

$$\lim_{n\to+\infty}\int_0^L \phi\big[J_{\mathbf{a}'}\big(x,T_{a,b}\big(u_n(x)\big)\big)-DJ_{\mathbf{a}}\big(x,T_{a,b}\big(u_n(x)\big)\big)+g'T'_{a,b}(u_n)\mathbf{a}\big(u_n,g'\big)\big]$$

$$=\big\langle J_{\mathbf{a}'}\big(x,T_{a,b}(u)\big)-DJ_{\mathbf{a}}\big(x,T(u)\big),\phi\big\rangle+\int_0^L \phi g'\mathbf{a}\big(u,g'\big)T'_{a,b}(u)\,dx.$$

Consequently we obtain

$$\big\langle \mathbf{z}D\big(T_{a,b}(u)-g\big),\phi\big\rangle+\|g'\|_\infty\int_0^L |\mathbf{z}|\phi\big(1-T'_{a,b}(u)\big)\,dx$$

$$+\int_0^L \phi\mathbf{a}\big(u,g'\big)g'T'_{a,b}(u)\,dx-\big\langle D\big[J_{\mathbf{a}}\big(x,T_{a,b}\big(u(x)\big)\big)\big]-J_{\mathbf{a}'}\big(x,T_{a,b}\big(u(x)\big)\big),\phi\big\rangle\geqslant 0$$

for all $0\leqslant\phi\in C_c^1(]0,L[)$. This means that, as measures,

$$\mathbf{z}D\big(T_{a,b}(u)-g\big)-D\big[J_a\big(x,T_{a,b}\big(u(x)\big)\big)\big]+J_{\mathbf{a}'}\big(x,T_{a,b}\big(u(x)\big)\big)$$
$$+\big\{\mathbf{a}\big(u,g'\big)g'T'_{a,b}(u)+|\mathbf{z}|\|g'\|_\infty\big(1-T'_{a,b}(u)\big)\big\}\mathcal{L}^1\geqslant 0,$$

and we obtain

$$\mathbf{z}\big(T_{a,b}(u)-g\big)'-\mathbf{a}\big(u,g'\big)\big(T_{a,b}(u)\big)'+\mathbf{a}\big(u,g'\big)g'T'_{a,b}(u)+|\mathbf{z}|\|g'\|_\infty\big(1-T'_{a,b}(u)\big)\geqslant 0.$$

If $x\in[a<u<b]$, this reduces to

$$\big(\mathbf{z}-\mathbf{a}\big(u,g'\big)\big)(u-g)'\geqslant 0,$$

which holds for all $x\in\Omega\cap[a<u<b]$, where $\mathcal{L}^1(]0,L[\setminus\Omega)=0$, and all $g\in C^2([0,L])$. Being $x\in\Omega\cap[a<u<b]$ fixed and $\xi\in\mathbb{R}$ given, we find g as above such that $g'(x)=\xi$. Then

$$\big(\mathbf{z}(x)-\mathbf{a}\big(u(x),\xi\big)\big)\big(u'(x)-\xi\big)\geqslant 0,\quad \forall\xi\in\mathbb{R}.$$

By an application of Minty–Browder's method in $\mathbb{R}$, these inequalities imply that

$$\mathbf{z}(x)=\mathbf{a}\big(u(x),u'(x)\big)\quad\text{a.e. on }[a<u<b].$$

Since this holds for any $0<a<b$, we obtain the identification a.e. on the points of $]0,L[$ such that $u(x)\neq 0$. Now, by our assumptions on $\mathbf{a}$ and (4.58) we deduce that $\mathbf{z}(x)=\mathbf{a}(u(x),u'(x))=0$ a.e. on $[u=0]$. The lemma is proved. □

To finish the proof we only need to show that

$$\frac{c}{2}\big|D^s\big(T(u)^2\big)\big|\leqslant \mathbf{z}D^sT(u)\quad\text{as measures }\forall T\in\mathcal{T}^+,$$
$$\big|D^sJ_{S\theta}\big(T(u)\big)\big|\leqslant \mathbf{z}D^sJ_{T'S}(u)\quad\text{as measures, }\forall S\in\mathcal{P}^+,\ T\in\mathcal{T}^+,$$
$$-\mathbf{a}\big(u,u'\big)(0)=\beta\quad\text{and}\quad\mathbf{a}\big(u,u'\big)(L)=-cu(L_-).$$

These proofs are similar to those in the previous section. □

From Theorem 4.5, according to Crandall–Liggett's Theorem (cf., e.g., [12]), for any $0 \leqslant u_0 \in L^1(]0, L[)$ there exists a unique mild solution $u \in C([0, T]; L^1(]0, L[))$ of the abstract Cauchy problem

$$u'(t) + \mathcal{B}_\beta u(t) \ni 0, \qquad u(0) = u_0.$$

Moreover, $u(t) = T_\beta(t)u_0$ for all $t \geqslant 0$, where $(T_\beta(t))_{t \geqslant 0}$ is the semigroup in $L^1(]0, L[)^+$ generated by Crandall–Liggett's exponential formula, i.e.,

$$T_\beta(t)u_0 = \lim_{n \to \infty} \left(I + \frac{t}{n} \mathcal{B}_\beta \right)^{-n} u_0.$$

On the other hand, as the operator $\mathcal{B}_\beta$ is T-accretive we have that the comparison principle also holds for $T_\beta(t)$, i.e., if $u_0, \bar{u}_0 \in L^1(]0, L[)^+$, we have the estimate

$$\left\| \left(T_\beta(t)u_0 - T_\beta(t)\bar{u}_0 \right)^+ \right\|_1 \leqslant \left\| (u_0 - \bar{u}_0)^+ \right\|_1. \tag{4.61}$$

Obviously, by Crandall–Liggett's exponential formula, from (4.46), we get that for all $u_0 \in L^1(]0, L[)^+$,

$$T_\beta(t)(\mu u_0) = \mu T_{\frac{\beta}{\mu}}(t)(u_0) \quad \text{for all } t > 0. \tag{4.62}$$

As a consequence of (4.61) and (4.62), for $u \in L^\infty(]0, L[)^+$, we have

$$0 \leqslant T_\beta(t)(u) \leqslant \mu u_\beta, \quad \text{with } \mu = \max\left\{ \frac{c\|u\|_\infty}{\beta}, 1 \right\}, \ \forall t \geqslant 0.$$

5. Existence and uniqueness of solutions of the parabolic problem

This section deals with the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \left(\mathbf{a}(u, u_x) \right)_x & \text{in }]0, T[\times]0, L[, \\ -\mathbf{a}\left(u(t,0), u_x(t,0)\right) = \beta > 0 \quad \text{and} \quad u(t, L) = 0 & \text{on } t \in]0, T[, \\ u(0, x) = u_0(x) & \text{in } x \in]0, L[. \end{cases} \tag{5.63}$$

To make precise our notion of solution we need to recall the following definitions given in [3]. We set $Q_T =]0, T[\times]0, L[$.

It is well known (see for instance [28]) that the dual space $[L^1(0, T; BV(]0, L[))]^*$ is isometric to the space $L^\infty(0, T; BV(]0, L[)^*, BV(]0, L[))$ of all weakly* measurable functions $f : [0, T] \to BV(]0, L[)^*$, such that $v(f) \in L^\infty([0, T])$, where $v(f)$ denotes the supremum of the set $\{|\langle w, f \rangle| : \|w\|_{BV(]0,L[)} \leqslant 1\}$ in the vector lattice of measurable real functions. Moreover, the duality pairing is

$$\langle w, f \rangle = \int_0^T \left\langle w(t), f(t) \right\rangle dt,$$

for $w \in L^1(0, T; BV(]0, L[))$ and $f \in L^\infty(0, T; BV(]0, L[)^*, BV(]0, L[))$.

By $L^1_w(0, T, BV(]0, L[))$ we denote the space of weakly measurable functions $w : [0, T] \to BV(]0, L[)$ (i.e., $t \in [0, T] \to \langle w(t), \phi \rangle$ is measurable for every $\phi \in BV(]0, L[)^*$) such that $\int_0^T \|w(t)\| \, dt < \infty$. Observe that, since $BV(]0, L[)$ has a separable predual (see [1]), it follows easily that the map

$t \in [0,T] \to \|w(t)\|$ is measurable. By $L^1_{loc,w}(0,T,BV(]0,L[))$ we denote the space of weakly measurable functions $w:[0,T] \to BV(]0,L[$ such that the map $t \in [0,T] \to \|w(t)\|$ is in $L^1_{loc}(]0,T[)$.

Note that if $w \in L^1(0,T;BV(]0,L[)) \cap L^\infty(Q_T)$ and $\mathbf{z} \in L^1(Q_T)$ is such that there exists an element $\xi \in [L^1(0,T;BV(]0,L[))]^*$ with $D_x\mathbf{z} = \xi$ in $\mathcal{D}'(Q_T)$, we define, associated with $(\mathbf{z},\xi)$, the distribution $\mathbf{z}D_x w$ in Q_T by

$$\langle \mathbf{z}D_x w, \varphi\rangle = -\langle \xi, \varphi w\rangle - \int_0^T\int_0^L \mathbf{z}(t,x)w(t,x)\partial_x\varphi(t,x)\,dx\,dt \tag{5.64}$$

for all $\varphi \in \mathcal{D}(Q_T)$.

Our concept of solution for the problem (5.63) is the following.

Definition 5.1. A measurable function $u:]0,T[\times]0,L[\to \mathbb{R}^+$ is an *entropy solution* of (5.63) in $Q_T =]0,T[\times]0,L[$ if $u \in C([0,T];L^1(]0,L[))$, $T(u(\cdot)) \in L^1_{loc,w}(0,T,BV(]0,L[))$ for all $T \in \mathcal{T}_r$, and $\mathbf{z}(t) := \mathbf{a}(u(t),\partial_x u(t)) \in L^1(Q_T)$, such that:

(i) the time derivative u_t of u in $\mathcal{D}'(Q_T)$ belongs to $[L^1(0,T;BV(]0,L[))]^*$ and satisfies

$$\int_0^T \langle u_t(t),\psi(t)\rangle\,dt = -\int_0^T\int_0^L u(t,x)\Theta(t,x)\,dx\,dt \tag{5.65}$$

for all test functions $\psi \in L^1(0,T;BV(]0,L[))$ compactly supported in time such that $\psi(t) = \int_0^t \Theta(s)\,ds$ and $\Theta \in L^1_w(0,T;BV(]0,L[)) \cap L^\infty(Q_T)$;

(ii) $D_x\mathbf{z} = u_t$ in $\mathcal{D}'(Q_T)$, and for any $w \in L^1(0,T;BV(]0,L[))$, the distribution $\mathbf{z}D_x w$ defined by (5.64) is a Radon measure in Q_T and verifies, for all $w \in L^1(0,T;BV(]0,L[))$, the following integration by parts formula

$$\int_{Q_T} \mathbf{z}D_x w + \langle u_t, w\rangle = \beta\int_0^T w(t,0_+)\,dt - c\int_0^T u(t,L_-)w(t,L_-)\,dt; \tag{5.66}$$

(iii) the following inequality is satisfied

$$\int_{Q_T} \eta h_S\big(u,DT(u)\big)\,dt + \int_{Q_T} \eta h_T\big(u,DS(u)\big)\,dt \leqslant \int_{Q_T} J_{TS}(u)\partial_t\eta\,dx\,dt - \int_{Q_T} \mathbf{a}(u,\partial_x u)\cdot\partial_x\eta T(u)S(u)\,dx\,dt$$

for truncatures $S, T \in \mathcal{T}^+$ and any $\eta \in C^\infty(Q_T)$ of compact support.

In the following result we get a positive lower bound for $u(t,0_+)$.

Lemma 5.2. *If u is an entropy solution of* (5.63) *in $Q_T = (0,T)\times]0,L[$, then*

$$u(t,0_+) \geqslant \frac{\beta}{c} > 0, \quad \textit{for almost all } t \in]0,T[. \tag{5.67}$$

Proof. For any $n \in \mathbb{N}$, let v_n be the function defined by zero in $]1/n, L]$, 1 at $x = 0$, and a straight line joining both values at the rest of the points. Being $0 \leqslant \phi \in \mathcal{D}(]0,T[)$ fixed and taking w in (5.66) as $w_n(t) := \phi(t) v_n$, we get

$$\int_{Q_T} \mathbf{z} D w_n + \langle u_t, w_n \rangle = \beta \int_0^T \phi(t)\, dt. \tag{5.68}$$

By (5.65), we have

$$\langle u_t, w_n \rangle = -\int_0^T \phi'(t) \int_0^L u(t,x) v_n(x)\, dx\, dt,$$

so by the Dominate Convergence Theorem,

$$\lim_{n\to\infty} \langle u_t, w_n \rangle = 0. \tag{5.69}$$

On the other hand, given $\varphi \in \mathcal{D}(Q_T)$, we have

$$\langle \mathbf{z} D_x w_n, \varphi \rangle = \int_0^T \phi(t) \int_0^L \mathbf{z}(t,x) \varphi(t,x) v_n'(x)\, dx\, dt.$$

Hence,

$$\int_{Q_T} \mathbf{z}(t,x) D_x w_n(t,x) = -\int_0^T n\phi(t) \int_0^{\frac{1}{n}} \mathbf{z}(t,x)\, dx\, dt. \tag{5.70}$$

Now, by (5.68), (5.69) and (5.70), we get

$$\beta \int_0^T \phi(t)\, dt = -\lim_{n\to\infty} \int_0^T \phi(t) n \int_0^{\frac{1}{n}} \mathbf{z}(t,x)\, dx\, dt.$$

Then, since $|\mathbf{z}(t,x)| \leqslant c u(t,x)$, by Fatou's Lemma we obtain that

$$\beta \int_0^T \phi(t)\, dt \leqslant c \int_0^T \phi(t) \left[\lim_{n\to\infty} n \int_0^{\frac{1}{n}} u(t,x)\, dx \right] dt = c \int_0^T \phi(t) u(t, 0_+)\, dt$$

from where it follows (5.67). □

Remark 5.3. Let u be a bounded entropy solution of (5.63) in Q_T. In the proof of the next result we need the following time regularization. For that, given $\phi \in \mathcal{D}(]0,T[)$ and $w \in L^1_{\mathrm{loc}}(0,T;BV(]0,L[))$, we define $(\phi w)^\tau$, as the Dunford integral (see [18])

$$(\phi w)^\tau(t) := \frac{1}{\tau}\int_{t-\tau}^{t} \phi(s)w(s)\,ds \in BV\big(]0,L[\big)^{**},$$

that is

$$\big\langle (\phi w)^\tau(t), \eta\big\rangle = \frac{1}{\tau}\int_{t-\tau}^{t} \big\langle \phi(s)w(s), \eta\big\rangle ds \quad \forall \eta \in BV\big(]0,L[\big)^{*}.$$

In [2] it is shown that $(\phi w)^\tau \in C([0,T];BV(]0,L[))$. If u is an entropy solution of (5.63) and $p \in \mathcal{T}^+$, it is easy to see that

$$\big|D_x\big(\phi p(u)\big)^\tau(t)\big|\big(]0,L[\big) \leqslant \frac{1}{\tau}\int_{t-\tau}^{t} \big|D_x\big(\phi(s)p\big(u(s)\big)\big)\big|\big(]0,L[\big)\,ds.$$

Then, by the lower semi-continuity of the total variation with respect to the L^1-convergence, we have

$$\begin{aligned}\big|D_x\big(\phi(t)p\big(u(t)\big)\big)\big|\big(]0,L[\big) &\leqslant \liminf_{\tau\to 0}\big|D_x\big(\phi p(u)\big)^\tau(t)\big|\big(]0,L[\big)\\ &\leqslant \limsup_{\tau\to 0}\frac{1}{\tau}\int_{t-\tau}^{t} \big|D_x\big(\phi(s)p\big(u(s)\big)\big)\big|\big(]0,L[\big)\,ds.\end{aligned}$$

Since the map $t \mapsto |D_x(\phi(t)p(u(t)))|(]0,L[)$ belongs to $L^1_{\mathrm{loc}}([0,T])$, we have that almost every $t \in [0,T]$ is a Lebesgue point of this map. So, for almost all $t \in [0,T]$, we have

$$\frac{1}{\tau}\int_{t-\tau}^{t} \big|D_x\big(\phi(s)p\big(u(s)\big)\big)\big|\big(]0,L[\big)\,ds \xrightarrow{\tau\to 0} \big|D_x\big(\phi(t)p\big(u(t)\big)\big)\big|\big(]0,L[\big),$$

and consequently,

$$\big|D_x\big(\phi p(u)\big)^\tau(t)\big|\big(]0,L[\big) \xrightarrow{\tau\to 0} \big|D_x\big(\phi(t)p\big(u(t)\big)\big)\big|\big(]0,L[\big) \quad \text{a.e. } t.$$

Respect to the existence and uniqueness of bounded entropy solutions we have the following result.

Theorem 5.4. *For any initial datum $0 \leqslant u_0 \in L^\infty(]0,L[)$ there exists a unique bounded entropy solution u of* (5.63) *in $Q_T =]0,T[\times]0,L[$ for every $T > 0$ such that $u(0) = u_0$. Moreover, if $u(t)$, $\bar{u}(t)$ are bounded entropy solutions of* (5.63) *in $Q_T =]0,T[\times]0,L[$ corresponding to initial data u_0, $\bar{u}_0 \in L^\infty(]0,L[)^+$, respectively, then*

$$\big\|\big(u(t)-\bar{u}(t)\big)^+\big\|_1 \leqslant \big\|(u_0-\bar{u}_0)^+\big\|_1 \quad \text{for all } t \geqslant 0.$$

In particular, we have uniqueness of bounded entropy solutions of (5.63)*.*

Proof. *The comparison principle.* Let $b > a > 2\epsilon > 0$, $T(r) := T_{a,b}(r) - a$. We need to consider truncature functions of the form $S_{\epsilon,l}(r) := T_\epsilon(r-l)^+ = T_{l,l+\epsilon}(r) - l \in \mathcal{T}^+$, and $S^l_\epsilon(r) := T_\epsilon(r-l)^- + \epsilon = T_{l-\epsilon,l}(r) + \epsilon - l \in \mathcal{T}^+$, where $l \geqslant 0$. Observe that $S^l_\epsilon(r) = -T_\epsilon(l-r)^+ + \epsilon$. Let us denote

$$J^+_{T,\epsilon,l}(r) = \int_0^r T(s)T_\epsilon(s-l)^+\,ds,$$

$$J^-_{T,\epsilon,l}(r) = \int_0^r T(s)T_\epsilon(s-l)^-\,ds = -\int_0^r T(s)T_\epsilon(l-s)^+\,ds.$$

Then, $J_{TS_{\epsilon,l}}(r) = J^+_{T,\epsilon,l}(r)$ and $J_{TS^l_\epsilon}(r) = J^-_{T,\epsilon,l}(r) + \epsilon J_T(r)$.

Let u, $\bar{u}$ be two entropy solutions of (5.63) corresponding to the initial conditions $u_0, \bar{u}_0 \in (L^1(]0,L[))^+$, respectively. Then, if $\mathbf{z}(t) := \mathbf{a}(u(t), \partial_x u(t))$, $\bar{\mathbf{z}}(t) := \mathbf{a}(\bar{u}(t), \partial_x \bar{u}(t))$, and $l_1, l_2 > \epsilon$, we have

$$\begin{aligned}
&-\int_0^T\int_0^L J^+_{T,\epsilon,l_1}\big(u(t)\big)\partial_t\eta(t)\,dx\,dt \\
&\quad + \int_0^T\int_0^L \eta(t)\big[h_T\big(u(t), D_x S_{\epsilon,l_1}\big(u(t)\big)\big) + h_{S_{\epsilon,l_1}}\big(u(t), D_x T\big(u(t)\big)\big)\big]\,dt \\
&\quad + \int_0^T\int_0^L \mathbf{z}(t)\partial_x\eta(t)T\big(u(t)\big)S_{\epsilon,l_1}\big(u(t)\big)\,dx\,dt \leqslant 0 \qquad (5.71)
\end{aligned}$$

and

$$\begin{aligned}
&-\int_0^T\int_0^L J^-_{T,\epsilon,l_2}\big(\bar{u}(t)\big)\partial_t\eta\,dx\,dt - \epsilon\int_0^T\int_0^L J_T\big(\bar{u}(t)\big)\partial_t\eta(t)\,dx\,dt \\
&\quad + \int_0^T\int_0^L \eta(t)\big[h_T\big(\bar{u}(t), D_x S^{l_2}_\epsilon\big(\bar{u}(t)\big)\big) + h_{S^{l_2}_\epsilon}\big(\bar{u}(t), D_x T\big(\bar{u}(t)\big)\big)\big]\,dt \\
&\quad + \int_0^T\int_0^L \bar{\mathbf{z}}(t)\partial_x\eta(t)T\big(\bar{u}(t)\big)S^{l_2}_\epsilon\big(\bar{u}(t)\big)\,dx\,dt \leqslant 0, \qquad (5.72)
\end{aligned}$$

for all $\eta \in C^\infty(Q_T)$, with $\eta \geqslant 0$, $\eta(t,x) = \phi(t)\rho(x)$, being $\phi \in \mathcal{D}(]0,T[)$, $\rho \in \mathcal{D}(]0,L[)$.

We choose two different pairs of variables (t,x), (s,y) and consider u, $\mathbf{z}$ as functions in (t,x), $\bar{u}$, $\bar{\mathbf{z}}$ in (s,y). Let $0 \leqslant \phi \in \mathcal{D}(]0,T[)$, $\psi \in \mathcal{D}(]0,L[)$, ρ_m and $\tilde{\rho}_n$ be sequences of mollifier in $\mathbb{R}$. Define

$$\eta_{m,n}(t,x,s,y) := \rho_m(x-y)\tilde{\rho}_n(t-s)\phi\left(\frac{t+s}{2}\right)\psi\left(\frac{x+y}{2}\right).$$

For (s,y) fixed, if we take in (5.71) $l_1 = \bar{u}(s,y)$, we get

$$
\begin{aligned}
&-\int_0^T\int_0^L J^+_{T,\epsilon,\bar{u}(s,y)}\big(u(t,x)\big)\partial_t\eta_{m,n}\,dx\,dt \\
&\quad+\int_0^T\int_0^L \eta_{m,n}\big[h_T\big(u(t,x),D_xS_{\epsilon,\bar{u}(s,y)}\big(u(t,x)\big)\big)+h_{S_{\epsilon,\bar{u}(s,y)}}\big(u(t,x),D_xT\big(u(t,x)\big)\big)\big]\,dt \\
&\quad+\int_0^T\int_0^L \mathbf{z}(t,x)\partial_x\eta_{m,n}T\big(u(t,x)\big)S_{\epsilon,\bar{u}(s,y)}\big(u(t,x)\big)\,dx\,dt\leqslant 0.
\end{aligned}
\tag{5.73}
$$

Similarly, for (t,x) fixed, if we take in (5.72) $l_2=u(t,x)$ we get

$$
\begin{aligned}
&-\int_0^T\int_0^L J^-_{T,\epsilon,u(t,x)}\big(\bar{u}(s,y)\big)\partial_s\eta_{m,n}\,dy\,ds-\epsilon\int_0^T\int_0^L J_T\big(\bar{u}(s,y)\big)\partial_s\eta_{m,n}\,dy\,ds \\
&\quad+\int_0^T\int_0^L \eta_{m,n}\big[h_T\big(\bar{u}(s,y),D_yS^{u(t,x)}_\epsilon\big(\bar{u}(s,y)\big)\big)+h_{S^{u(t,x)}_\epsilon}\big(\bar{u}(s,y),D_yT\big(\bar{u}(s,y)\big)\big)\big]\,ds \\
&\quad+\int_0^T\int_0^L \bar{\mathbf{z}}(s,y)\partial_y\eta_{m,n}T\big(\bar{u}(s,y)\big)S^{u(t,x)}_\epsilon\big(\bar{u}(s,y)\big)\,dy\,ds\leqslant 0.
\end{aligned}
\tag{5.74}
$$

We integrate (5.73) in (s,y), (5.74) in (t,x), and add the two inequalities. Using that $a>2\epsilon$, and since

$$
\int_{Q_T\times Q_T}\eta_{m,n}h_{S_{\epsilon,\bar{u}(s,y)}}\big(u(t,x),D_xT\big(u(t,x)\big)\big)\,ds\,dt\,dy\geqslant 0
$$

and

$$
\int_{Q_T\times Q_T}\eta_{m,n}h_{S^{u(t,x)}_\epsilon}\big(\bar{u}(s,y),D_yT\big(\bar{u}(s,y)\big)\big)\,ds\,dt\,dx\geqslant 0,
$$

we get

$$
\begin{aligned}
&-\int_{Q_T\times Q_T}\big(J^+_{T,\epsilon,\bar{u}(s,y)}\big(u(t,x)\big)\partial_t\eta_{m,n}+J^-_{T,\epsilon,u(t,x)}\big(\bar{u}(s,y)\big)\partial_s\eta_{m,n}\big)\,ds\,dt\,dy\,dx \\
&\quad-\epsilon\int_{Q_T\times Q_T}J_T\big(\bar{u}(s,y)\big)\partial_s\eta_{m,n}\,ds\,dt\,dy\,dx+\int_{Q_T\times Q_T}\eta_{m,n}h_T\big(u(t,x),D_xS_{\epsilon,\bar{u}(s,y)}\big(u(t,x)\big)\big)\,ds\,dt\,dy \\
&\quad+\int_{Q_T\times Q_T}\eta_{m,n}h_T\big(\bar{u}(s,y),D_yS^{u(t,x)}_\epsilon\big(\bar{u}(s,y)\big)\big)\,ds\,dt\,dx \\
&\quad-\int_{Q_T\times Q_T}\bar{\mathbf{z}}(s,y)\partial_x\eta_{m,n}T\big(\bar{u}(s,y)\big)S^{u(t,x)}_\epsilon\big(\bar{u}(s,y)\big)\,ds\,dt\,dy\,dx
\end{aligned}
$$

$$
\begin{aligned}
&- \int_{Q_T\times Q_T} \mathbf{z}(t,x)\partial_y \eta_{m,n} T\big(u(t,x)\big) S_{\epsilon,\bar{u}(s,y)}\big(u(t,x)\big)\,ds\,dt\,dy\,dx \\
&+ \int_{Q_T\times Q_T} T_\epsilon^+\big(u(t,x)-\bar{u}(s,y)\big)\big[T\big(u(t,x)\big)\mathbf{z}(t,x) - T\big(\bar{u}(s,y)\big)\bar{\mathbf{z}}(s,y)\big] \\
&\times (\partial_x\eta_{m,n} + \partial_y\eta_{m,n})\,ds\,dt\,dy\,dx \\
&+ \epsilon \int_{Q_T\times Q_T} T\big(\bar{u}(s,y)\big)\bar{\mathbf{z}}(s,y)(\partial_x\eta_{m,n} + \partial_y\eta_{m,n})\,ds\,dt\,dy\,dx \leqslant 0.
\end{aligned} \tag{5.75}
$$

Let I_2 be the sum of the third up to the sixth terms of the above inequality. From now on, since u, $\mathbf{z}$ are always functions of (t,x), and $\bar{u}$, $\bar{\mathbf{z}}$ are always functions of (s,y), to make our expression shorter, we shall omit the arguments except when they appear as sub-index and in some additional cases where we find it useful to remind them. We also omit the differentials of the integrals.

Working as in the proof of uniqueness of Theorem 3 in [4], we obtain that $\frac{1}{\epsilon} I_2 \geqslant \|\phi\|_\infty \|\psi\|_\infty o(\epsilon)$. Hence, by (5.75), it follows that

$$
\begin{aligned}
&- \int_{Q_T\times Q_T} \big(J^+_{T,\epsilon,\bar{u}}(u)\partial_t\eta_{m,n} + J^-_{T,\epsilon,u}(\bar{u})\partial_s\eta_{m,n}\big) \\
&\quad + \int_{Q_T\times Q_T} T_\epsilon^+(u-\bar{u})\big[T(u)\mathbf{z} - T(\bar{u})\bar{\mathbf{z}}\big](\partial_x\eta_{m,n} + \partial_y\eta_{m,n}) + \epsilon \int_{Q_T\times Q_T} T(\bar{u})\bar{\mathbf{z}}(\partial_x\eta_{m,n} + \partial_y\eta_{m,n}) \\
&\quad \leqslant \epsilon o(\epsilon) + \epsilon \int_{Q_T\times Q_T} J_T(\bar{u})\partial_s\eta_{m,n}.
\end{aligned}
$$

Then, dividing by ϵ and letting $\epsilon \to 0$ we get

$$
\begin{aligned}
&- \int_{Q_T\times Q_T} \big(J^+_{T,\mathrm{sign},\bar{u}}(u)\partial_t\eta_{m,n} + J^-_{T,\mathrm{sign},u}(\bar{u})\partial_s\eta_{m,n}\big) \\
&\quad + \int_{Q_T\times Q_T} \mathrm{sign}_0^+(u-\bar{u})\big[T(u)\mathbf{z} - T(\bar{u})\bar{\mathbf{z}}\big](\partial_x\eta_{m,n} + \partial_y\eta_{m,n}) + \int_{Q_T\times Q_T} T(\bar{u})\bar{\mathbf{z}}(\partial_x\eta_{m,n} + \partial_y\eta_{m,n}) \\
&\quad \leqslant \int_{Q_T\times Q_T} J_T(\bar{u})\partial_s\eta_{m,n}
\end{aligned}
$$

where

$$
J^+_{T,\mathrm{sign},l}(r) = \int_0^r T\big(r'\big)\,\mathrm{sign}_0^+\big(r'-l\big)\,dr', \quad l \in \mathbb{R},\ r \geqslant 0,
$$

and

$$
J^-_{T,\mathrm{sign},l}(r) = \int_0^r T\big(r'\big)\,\mathrm{sign}_0^-\big(r'-l\big)\,dr', \quad l \in \mathbb{R},\ r \geqslant 0.
$$

Now, letting $m \to \infty$, we obtain

$$\begin{aligned}
&-\int_0^T\int_0^T\int_0^L \left(J^+_{T,\mathrm{sign},\bar u(s,x)}\big(u(t,x)\big)\partial_t\chi_n + J^-_{T,\mathrm{sign},u(t,x)}\big(\bar u(s,x)\big)\partial_s\chi_n\right)\\
&\quad+\int_0^T\int_0^T\int_0^L \mathrm{sign}_0^+\big(u(t,x)-\bar u(s,x)\big)\big[T\big(u(t,x)\big)\mathbf{z}(t,x)-T\big(\bar u(s,x)\big)\bar{\mathbf{z}}(s,x)\big]\partial_x\chi_n\\
&\quad+\int_0^T\int_0^T\int_0^L T\big(\bar u(s,x)\big)\bar{\mathbf{z}}(s,x)\partial_x\chi_n\\
&\quad\leqslant\int_0^T\int_0^T\int_0^L J_T\big(\bar u(s,x)\big)\partial_s\chi_n
\end{aligned}$$

where $\chi_n(t,s,x) := \tilde\rho_n(t-s)\phi(\frac{t+s}{2})\psi(x)$. We set $\psi = \psi_k \in \mathcal{D}(]0,L[) \uparrow \chi_{]0,L[}$ in the last expression and taking limit as $k \to +\infty$, we have

$$\begin{aligned}
&-\int_0^T\int_0^T\int_0^L \left(J^+_{T,\mathrm{sign},\bar u(s,x)}\big(u(t,x)\big)\partial_t\kappa_n(t,s) + J^-_{T,\mathrm{sign},u(t,x)}\big(\bar u(s,x)\big)\partial_s\kappa_n(t,s)\right)\\
&\quad+\lim_{k\to+\infty}\int_0^T\int_0^T\int_0^L \kappa_n(t,s)\,\mathrm{sign}_0^+\big(u(t,x)-\bar u(s,x)\big)T\big(u(t,x)\big)\mathbf{z}(t,x)\partial_x\psi_k(x)\\
&\quad-\lim_{k\to+\infty}\int_0^T\int_0^T\int_0^L \kappa_n(t,s)\,\mathrm{sign}_0^+\big(u(t,x)-\bar u(s,x)\big)T\big(\bar u(s,x)\big)\bar{\mathbf{z}}(s,x)\partial_x\psi_k(x)\\
&\quad+\lim_{k\to+\infty}\int_0^T\int_0^T\int_0^L \kappa_n(t,s)T\big(\bar u(s,x)\bar{\mathbf{z}}(s,x)\big)\partial_x\psi_k(x)\\
&\quad\leqslant\int_0^T\int_0^T\int_0^L J_T\big(\bar u(s,x)\big)\partial_s\kappa_n(t,s),
\end{aligned}\tag{5.76}$$

where $\kappa_n(t,s) := \tilde\rho_n(t-s)\phi(\frac{t+s}{2})$.

Let us study the second and the third term of the above expression. Let

$$\begin{aligned}
I_k &:= \int_0^T\int_0^T\int_0^L \kappa_n(t,s)\,\mathrm{sign}_0^+\big(u(t,x)-\bar u(s,x)\big)T\big(u(t,x)\big)\mathbf{z}(t,x)\partial_x\psi_k(x)\\
&= \int_0^T\int_0^T\int_0^L \kappa_n(t,s)\,\mathrm{sign}_0^+\big(u(t,x)-\bar u(s,x)\big)T\big(u(t,x)\big)\mathbf{z}(t,x)\partial_x\big(\psi_k(x)-1\big).
\end{aligned}$$

Let $H_n(s,r) := \kappa_n(r,s)\,\mathrm{sign}_0^+(u(r)-\bar{u}(s))T(u(r))$. For $\tau > 0$, we define the function $(\kappa_n(s))^\tau$, as the Dunford integral (see Remark 5.3)

$$\big(\kappa_n(s)\big)^\tau(t) := \frac{1}{\tau}\int_t^{t+\tau} H_n(s,r)\,dr.$$

Then,

$$\begin{aligned}
I_k &= \lim_{\tau\to 0}\int_0^T\int_0^T\int_0^L \big(\kappa_n(s)\big)^\tau(t)\mathbf{z}(t,x)\partial_x\big[\psi_k(x)-1\big]\,dx\,dt\,ds \\
&= -\lim_{\tau\to 0}\int_0^T\int_0^T\int_0^L \big[\psi_k(x)-1\big]\mathbf{z}(t,x)D_x\Big(\big(\kappa_n(s)\big)^\tau(t)\Big)\,ds\,dt \\
&\quad -\lim_{\tau\to 0}\int_0^T \Big\langle u_t, \big(\kappa_n(s)\big)^\tau\big(\psi_k(x)-1\big)\Big\rangle\,ds + c\lim_{\tau\to 0}\int_0^T\int_0^T u(t,L_-)\big(\kappa_n(s)\big)^\tau(t)(L_-)\,dt\,ds \\
&\quad -\beta\lim_{\tau\to 0}\int_0^T\int_0^T \big(\kappa_n(s)\big)^\tau(t)(0_+)\,dt\,ds = I_k^1+I_k^2+I_k^3+I_k^4.
\end{aligned}$$

Notice that

$$I_k^3 = c\int_0^T\int_0^T u(t,L_-)\kappa_n(t,s)\,\mathrm{sign}_0^+\big(u(t,L_-)-\bar{u}(s,L_-)\big)T\big(u(t,L_-)\big)\,dt\,ds$$

and

$$I_k^4 = -\beta\int_0^T\int_0^T \kappa_n(t,s)\,\mathrm{sign}_0^+\big(u(t,0_+)-\bar{u}(s,0_+)\big)T\big(u(t,0_+)\big)\,dt\,ds.$$

By Remark 5.3, we get

$$\Big|D_x\Big(\big(\kappa_n(s)\big)^\tau(t)\Big)\Big|\big(]0,L[\big) \xrightarrow{\tau\to 0} \Big|D_x\Big(\kappa_n(t,s)\,\mathrm{sign}_0^+\big(u(t)-\bar{u}(s)\big)T\big(u(t)\big)\Big)\Big|\big(]0,L[\big). \tag{5.77}$$

Using (5.77), we get

$$\big|I_k^1\big| \leqslant c\|u\|_{L^\infty(Q_T)}\int_0^T\int_0^T\int_0^L \big(1-\psi_k(x)\big)\Big|D_x\Big(\kappa_n(t,s)\,\mathrm{sign}_0^+\big(u(t)-\bar{u}(s)\big)T\big(u(t)\big)\Big)\Big|\,dt\,ds,$$

which implies $\lim_{k\to\infty} I_k^1 = 0$. Let us deal with I_k^2. We have

$$I_k^2 = \lim_{\tau\to 0}\int_0^T\int_0^T\int_0^L u(t,x)\frac{H_n(s,t+\tau)-H_n(s,t)}{\tau}\big(\psi_k(x)-1\big)\,dx\,dt\,ds.$$

Let

$$q(\tau) := \operatorname{sign}_0^+\bigl(\tau - \bar{u}(s,x)\bigr)T(\tau), \qquad Q(r) := \int_0^r q(\tau)\,d\tau.$$

Since q is non-decreasing, $Q(r) - Q(\bar{r}) \leqslant q(r)(r - \bar{r})$. Then, changing variables, since $H_n(s,t) = q(u(t))\kappa_n(t,s)$,

$$\begin{aligned}
I_k^2 &= \lim_{\tau\to 0} \int_0^T\int_0^T\int_0^L \bigl(1-\psi_k(x)\bigr)\frac{u(t,x)-u(t-\tau,x)}{\tau} H_n(s,t)\,dx\,dt\,ds \\
&\geqslant \lim_{\tau\to 0} \int_0^T\int_0^T\int_0^L \bigl(1-\psi_k(x)\bigr)\kappa_n(t,s)\frac{Q(u(t,x)) - Q(u(t-\tau,x))}{\tau}\,dx\,dt\,ds \\
&= \lim_{\tau\to 0} \int_0^T\int_0^T\int_0^L \bigl(1-\psi_k(x)\bigr)Q\bigl(u(t,x)\bigr)\frac{\kappa_n(t,s)-\kappa_n(t+\tau,s)}{\tau}\,dx\,dt\,ds \\
&= -\int_0^T\int_0^T\int_0^L \bigl(1-\psi_k(x)\bigr)Q\bigl(u(t,x)\bigr)\partial_t\kappa_n(t,s)\,dx\,dt\,ds, \qquad (5.78)
\end{aligned}$$

from where it follows that $\lim_{k\to\infty} I_k^2 \geqslant 0$. Taking into account the above facts, we get

$$\begin{aligned}
\lim_{k\to\infty} I_k \geqslant{}& -\beta\int_0^T\int_0^T \kappa_n(t,s)\operatorname{sign}_0^+\bigl(u(t,0_+) - \bar{u}(s,0_+)\bigr)T\bigl(u(t,0_+)\bigr)\,dt\,ds \\
&+ c\int_0^T\int_0^T u(t,L_-)\kappa_n(t,s)\operatorname{sign}_0^+\bigl(u(t,L_-) - \bar{u}(s,L_-)\bigr)T\bigl(u(t,L_-)\bigr)\,dt\,ds. \qquad (5.79)
\end{aligned}$$

Working similarly, we obtain

$$\begin{aligned}
&-\lim_{k\to\infty}\int_0^T\int_0^T\int_0^L \kappa_n(t,s)\operatorname{sign}_0^+\bigl(u(t,x)-\bar{u}(s,x)\bigr)T\bigl(\bar{u}(s,x)\bigr)\bar{\mathbf{z}}(s,x)\partial_x\psi_k(x) \\
&\quad\geqslant \beta\int_0^T\int_0^T \kappa_n(t,s)\operatorname{sign}_0^+\bigl(u(t,0_+) - \bar{u}(s,0_+)\bigr)T\bigl(\bar{u}(s,0_+)\bigr)\,dt\,ds \\
&\qquad - c\int_0^T\int_0^T \bar{u}(s,L_-)\kappa_n(t,s)\operatorname{sign}_0^+\bigl(u(t,L_-) - \bar{u}(s,L_-)\bigr)T\bigl(\bar{u}(s,L_-)\bigr)\,dt\,ds. \qquad (5.80)
\end{aligned}$$

Analogously,

$$\lim_{k\to+\infty}\int_0^T\int_0^T\int_0^L \kappa_n(t,s)T\big(\bar{u}(s,x)\big)\bar{\mathbf{z}}(s,x)\partial_x\psi_k(x)$$
$$\geqslant c\int_0^T\int_0^T \bar{u}(s,L_-)\kappa_n(t,s)T\big(\bar{u}(s,L_-)\big)\,dt\,ds-\beta\int_0^T\int_0^T \kappa_n(t,s)T\big(\bar{u}(s,0_+)\big)\,dt\,ds. \tag{5.81}$$

From (5.76), by (5.79), (5.80) and (5.81), we have

$$-\int_0^T\int_0^T\int_0^L \big(J^+_{T,\mathrm{sign},\bar{u}(s,x)}\big(u(t,x)\big)\partial_t\kappa_n(t,s)+J^-_{T,\mathrm{sign},u(t,x)}\big(\bar{u}(s,x)\big)\partial_s\kappa_n(t,s)\big)\,dt\,ds\,dx$$
$$+c\int_0^T\int_0^T u(t,L_-)\kappa_n(t,s)\,\mathrm{sign}_0^+\big(u(t,L_-)-\bar{u}(s,L_-)\big)T\big(u(t,L_-)\big)\,dt\,ds$$
$$-c\int_0^T\int_0^T \bar{u}(s,L_-)\kappa_n(t,s)\,\mathrm{sign}_0^+\big(u(t,L_-)-\bar{u}(s,L_-)\big)T\big(\bar{u}(s,L_-)\big)\,dt\,ds$$
$$-\beta\int_0^T\int_0^T \kappa_n(t,s)\,\mathrm{sign}_0^+\big(u(t,0_+)-\bar{u}(s,0_+)\big)T\big(u(t,0_+)\big)\,dt\,ds$$
$$+\beta\int_0^T\int_0^T \kappa_n(t,s)\,\mathrm{sign}_0^+\big(u(t,0_+)-\bar{u}(s,0_+)\big)T\big(\bar{u}(s,0_+)\big)\,dt\,ds$$
$$+c\int_0^T\int_0^T \bar{u}(s,L_-)\kappa_n(t,s)T\big(\bar{u}(s,L_-)\big)\,dt\,ds-\beta\int_0^T\int_0^T \kappa_n(t,s)T\big(\bar{u}(s,0_+)\big)\,dt\,ds$$
$$\leqslant\int_0^T\int_0^T\int_0^L J_T\big(\bar{u}(s,x)\big)\partial_s\kappa_n(t,s)\,dt\,ds\,dx. \tag{5.82}$$

By Lemma 5.2, we have

$$u(t,0_+)\geqslant\frac{\beta}{c}>0,\qquad \bar{u}(s,0_+)\geqslant\frac{\beta}{c}>0\quad\text{for almost every } t,s>0. \tag{5.83}$$

Letting $a\to 0$, dividing by b and letting $b\to 0$ in (5.82), we obtain

$$-\int_0^T\int_0^T\int_0^L \big(u(t,x)-\bar{u}(s,x)\big)^+\big(\partial_t\kappa_n(t,s)+\partial_s\kappa_n(t,s)\big)\,dt\,ds\,dx$$
$$+c\int_0^T\int_0^T u(t,L_-)\kappa_n(t,s)\,\mathrm{sign}_0^+\big(u(t,L_-)-\bar{u}(s,L_-)\big)\,\mathrm{sign}_0^+\big(u(t,L_-)\big)\,dt\,ds$$

$$- c \int_0^T \int_0^T \bar{u}(s, L_-) \kappa_n(t,s) \operatorname{sign}_0^+ \big(u(t, L_-) - \bar{u}(s, L_-) \big) \operatorname{sign}_0^+ \big(\bar{u}(s, L_-) \big) \, dt \, ds$$

$$- \beta \int_0^T \int_0^T \kappa_n(t,s) \operatorname{sign}_0^+ \big(u(t, 0_+) - \bar{u}(s, 0_+) \big) \big[\operatorname{sign}_0 \big(u(t, 0_+) \big) - \operatorname{sign}_0 \big(\bar{u}(s, 0_+) \big) \big] \, dt \, ds$$

$$+ c \int_0^T \int_0^T \bar{u}(s, L_-) \kappa_n(t,s) \operatorname{sign}_0^+ \big(\bar{u}(s, L_-) \big) \, dt \, ds - \beta \int_0^T \int_0^T \kappa_n(t,s) \, dt \, ds$$

$$\leqslant \int_0^T \int_0^T \int_0^L \bar{u}(s,x) \partial_s \kappa_n(t,s) \, dt \, ds \, dx.$$

Having in mind (5.83), the fourth term of the above expression vanishes. Moreover, the sum of the second and third terms is non-negative. On the other hand, since $\bar{u}_s = D_x(\bar{\mathbf{z}})$ in the sense given in (ii) of Definition 5.1,

$$\int_0^T \int_0^T \int_0^L \bar{u}(s,x) \partial_s \kappa_n(t,s) \, dx \, dt \, ds = - \int_0^T \langle \bar{u}_s, \kappa_n(\cdot, t) \rangle \, dt$$

$$= c \int_0^T \int_0^T \bar{u}(s, L_-) \kappa_n(t,s) \, dt \, ds - \beta \int_0^T \int_0^T \kappa_n(t,s) \, dt \, ds.$$

Therefore,

$$- \int_0^T \int_0^T \int_0^L \big(u(t,x) - \bar{u}(s,x) \big)^+ \big(\partial_t \kappa_n(t,s) + \partial_s \kappa_n(t,s) \big) \, dt \, ds \, dx \leqslant 0.$$

Letting $n \to \infty$,

$$- \int_0^T \int_0^L \big(u(t,x) - \bar{u}(t,x) \big)^+ \phi'(t) \, dx \, dt \leqslant 0.$$

Since this is true for all $0 \leqslant \phi \in \mathcal{D}(]0, T[)$, we have

$$\frac{d}{dt} \int_0^L \big(u(t,x) - \bar{u}(t,x) \big)^+ \, dx \leqslant 0.$$

Hence

$$\int_0^L \big(u(t,x) - \bar{u}(t,x) \big)^+ \, dx \leqslant \int_0^L \big(u_0(x) - \bar{u}_0(x) \big)^+ \, dx \quad \text{for all } t \geqslant 0,$$

which finishes the uniqueness part.

Existence of bounded entropy solution. Given $0 \leqslant u_0 \in L^1(]0, L[)$, let $u(t) = T_\beta(t)u_0$, being $(T_\beta(t))_{t\geqslant 0}$ the semigroup in $L^1(]0, L[)^+$ generated by the accretive operator $\mathcal{B}_\beta$. Then, according to the general theory of nonlinear semigroups [12], we have that $u(t)$ is a mild solution of the abstract Cauchy problem

$$u'(t) + \mathcal{B}_\beta u(t) \ni 0, \qquad u(0) = u_0.$$

Let us prove that, assuming $0 \leqslant u_0 \in L^\infty(]0, L[)$, then u is a bounded entropy solution of (5.63) in Q_T. We divide the proof of existence in several steps.

Step 1. *Approximation with Crandall–Liggett's scheme.* Let $T > 0$, $K \geqslant 1$, $\Delta t = \frac{T}{K}$, $t_n = n\Delta t$, $n = 0, \ldots, K$. We define inductively u^{n+1}, $n = 0, \ldots, K-1$, to be the unique entropy solution of

$$\begin{cases} \dfrac{u^{n+1} - u^n}{\Delta t} - \big(\mathbf{a}\big(u^{n+1}, \big(u^{n+1}\big)'\big)\big)' = 0 & \text{in }]0, L[, \\ -\mathbf{a}\big(u^{n+1}(0), \big(u^{n+1}\big)'(0)\big) = \beta > 0 \quad \text{and} & u^{n+1}(L_-) = 0, \end{cases} \tag{5.84}$$

where $u^0 = u_0$.

If we define

$$u^K(t) := u^0 \chi_{[0,t_1]}(t) + \sum_{n=1}^{K-1} u^n \chi_{]t_n,t_{n+1}]}(t),$$

by Crandall–Liggett's Theorem, we get that u^K converges uniformly to $u \in C([0, T], L^1(]0, L[))$, as $K \to \infty$.

We also define

$$\xi^K(t) := \sum_{n=0}^{K-1} \frac{u^{n+1} - u^n}{\Delta t} \chi_{]t_n,t_{n+1}]}(t)$$

and

$$\mathbf{z}^K(t) := \mathbf{a}\big(u^1, \big(u^1\big)'\big)\chi_{[0,t_1]}(t) + \sum_{n=1}^{K-1} \mathbf{a}\big(u^{n+1}, \big(u^{n+1}\big)'\big)\chi_{]t_n,t_{n+1}]}(t).$$

Since u^{n+1} is the entropy solution of (5.84), we have

$$\xi^K(t) = D_x \mathbf{z}^K(t) \quad \text{in } \mathcal{D}'\big(]0, L[\big),\ \forall t \in]0, T], \tag{5.85}$$

$$\mathbf{z}^K(t)(L) = -cu^K(t + \Delta t)(L_-), \quad \forall t \in]0, T - \Delta t], \qquad -\mathbf{z}^K(t)(0) = \beta, \quad \forall t \in [0, T] \tag{5.86}$$

and for all $S \in \mathcal{P}^+$, $T \in \mathcal{T}^+$, we have $\forall t \in]0, T - \Delta t]$

$$\begin{aligned} &h\big(u^K(t + \Delta t), D_x T\big(u^K(t + \Delta t)\big)\big) \leqslant \mathbf{z}^K(t) D_x T\big(u^K(t + \Delta t)\big) \quad \text{as measures}, \\ &h_S\big(u^K(t + \Delta t), D_x T\big(u^K(t + \Delta t)\big)\big) \leqslant \mathbf{z}^K(t) D_x J_{T'S}\big(u^K(t + \Delta t)\big) \quad \text{as measures}. \end{aligned} \tag{5.87}$$

Note that (5.87) is equivalent to

$$\frac{c}{2}\big|D_x^s\big(\big(T\big(u^K(t + \Delta t)\big)\big)^2\big)\big| \leqslant \mathbf{z}^K(t) D_x^s T\big(u^K(t + \Delta t)\big) \quad \text{as measures}.$$

Since $\mathbf{a}(u^{n+1}, (u^{n+1})')D_xT(u^{n+1}) \geqslant h(u^{n+1}, D_xT(u^{n+1}))$ as measures in $]0, L[$, using (2.9) we can write

$$h\big(u^{n+1}, D_xT\big(u^{n+1}\big)\big) = \mathbf{a}\big(u^{n+1}, \big(u^{n+1}\big)'\big)\big(T\big(u^{n+1}\big)\big)'\mathcal{L}^1 + \frac{c}{2}\big|D_x^s\big[\big(T\big(u^{n+1}\big)\big)^2\big]\big|$$
$$\geqslant \frac{c}{2}\big|\big(\big(T\big(u^{n+1}\big)\big)^2\big)'\big|\mathcal{L}^1 - \frac{c^2}{\nu}\big(T\big(u^{n+1}\big)\big)^2\mathcal{L}^1 + \frac{c}{2}\big|D_x^s\big[\big(T\big(u^{n+1}\big)\big)^2\big]\big|$$
$$= \frac{c}{2}\big|D_x\big[\big(T\big(u^{n+1}\big)\big)^2\big]\big| - \frac{c^2}{\nu}\big(T\big(u^{n+1}\big)\big)^2\mathcal{L}^1,$$

from where we get the following inequality as measures

$$\mathbf{z}^K(t)D_xT\big(u^K(t+\Delta t)\big) \geqslant \frac{c}{2}\big|D_x\big[\big(T\big(u^K(t+\Delta t)\big)\big)^2\big]\big| - \frac{c^2}{\nu}\big(T\big(u^K(t+\Delta t)\big)\big)^2. \tag{5.88}$$

Lemma 5.5. *There exists $M := M(\beta, c, \nu, L, \|u_0\|_\infty)$ such that*

$$\big\|u^K(t)\big\|_\infty \leqslant M \quad \forall K \in \mathbb{N} \text{ and } \forall t \in [0, T]. \tag{5.89}$$

Consequently, $\|u(t)\|_\infty \leqslant M$ $\forall t \in [0, T]$.

Proof. Since

$$(I + \Delta t\mathcal{B}_\beta)^{-1}\big(u^n\big) = u^{n+1}, \quad \text{for } n = 0, \ldots, K-1,$$

by Proposition 4.4, if $\mu := \max\{\frac{c\|u_0\|_\infty}{\beta}, 1\}$, we have

$$0 \leqslant u^1 = (I + \Delta t\mathcal{B}_\beta)^{-1}(u_0) \leqslant \mu u_\beta.$$

Then, repeating this process, we obtain

$$0 \leqslant u^{n+1} = (I + \Delta t\mathcal{B}_\beta)^{-1}\big(u^n\big) \leqslant (I + \Delta t\mathcal{B}_\beta)^{-1}(\mu u_\beta)$$
$$= \mu(I + \Delta t\mathcal{B}_{\frac{\beta}{\mu}})^{-1}(u_\beta) \leqslant \mu(I + \Delta t\mathcal{B}_\beta)^{-1}(u_\beta) = \mu u_\beta,$$

and the proof concludes. □

Step 2. By (5.89), $\|\mathbf{z}^K(t)\|_\infty \leqslant C$ for all $K \in \mathbb{N}$ and a.e. $t \in [0, T]$. Then we may assume that $\mathbf{z}^K \rightharpoonup \mathbf{z} \in L^\infty(Q_T)$ weakly*. Moreover, since $\mathbf{z}^K(t) = cu^K(t+\Delta t)\mathbf{b}(u^K(t+\Delta t), \partial_x u^K(t+\Delta t))$ $\forall t \in]0, T-\Delta t]$, with $\|\mathbf{b}(u^K(t+\Delta t), \partial_x u^K(t+\Delta t))\|_\infty \leqslant 1$ and u^K converges uniformly to u in $C([0, T], L^1(]0, L[))$, we may also assume that $\mathbf{b}(u^K(t+\Delta t), \partial_x u^K(t+\Delta t)) \rightharpoonup \mathbf{z_b}(t) \in L^\infty(Q_T)$ weakly* and

$$\mathbf{z}(t) = cu(t)\mathbf{z_b}(t) \quad \text{for almost all } t \in [0, T]. \tag{5.90}$$

Given $w \in BV(]0, L[)$, from (5.85) and (5.89), it follows that for each $t \in]0, T]$

$$\left|\int_0^L \xi^K(t,x)w(x)\,dx\right| = \left|-\int_0^L \mathbf{z}^K(t)Dw + \mathbf{z}^K(L)w(L) + \beta w(0_+)\right|$$
$$\leqslant C\|w\|_{BV(]0,L[)} + \left|\mathbf{z}^K(L)w(L)\right|$$
$$\leqslant \left(C + c\mu\|u_\beta\|_\infty\right)\|w\|_{BV(]0,L[)},$$

where the continuous injection of $BV(]0,L[)$ into $L^\infty(]0,L[)$ was used. Thus, $\|\xi^K(t)\|_{BV(]0,L[)^*} \leqslant C$, $\forall K \in \mathbb{N}$ and $t \in]0,T]$. Consequently, $\{\xi^K\}$ is a bounded sequence in $L^\infty(0,T;BV(]0,L[)^*)$. Now, since $L^\infty(0,T;BV(]0,L[)^*)$ is a vector subspace of the dual space $(L^1(0,T;BV(]0,L[)))^*$, we can find a subnet ξ^α of ξ^K such that

$$\xi^\alpha \rightharpoonup \xi \in \left(L^1\left(0,T;BV\left(]0,L[\right)\right)\right)^* \quad \text{weakly}^*.$$

Working as in the proof of Theorem 5.5 of [4], Step 2, we can prove that (5.65) holds and $u_t = D_x\mathbf{z}$ in $\mathcal{D}'(Q_T)$.

Step 3. Next, we prove that $u_t = D_x\mathbf{z}$ in the sense given in (ii) of Definition 5.1. To do this, let us first observe that we can prove, as in the proof of Theorem 5.5 of [4], Step 4, that the distribution $\mathbf{z}Dw$ in Q_T defined by (5.64) is a Radon measure in Q_T for all $w \in L^1(0,T;BV(]0,L[))$, and also that

$$\langle \mathbf{z}D_x w, \varphi\rangle = \lim_\alpha \int_0^T\int_0^L \mathbf{z}^\alpha(t,x)D_x w(t,x)\varphi_x(t,x)\,dx\,dt.$$

From where it follows, combining with (5.86) and integrating by parts,

$$\begin{aligned}
\int_{Q_T} \mathbf{z}D_x w &= \lim_\alpha \int_0^T\int_0^L \mathbf{z}^\alpha(t)D_x w(t)\,dt = -\lim_\alpha \int_0^T\int_0^L w(t,x)D_x\mathbf{z}^\alpha(t,x)\,dx\,dt \\
&\quad + \lim_\alpha\left[\int_0^T \mathbf{z}^\alpha(t,L)w(t,L_-)\,dt - \int_0^T \mathbf{z}^\alpha(0)w(t,0_+)\,dt\right] \\
&= \lim_\alpha\left[-\langle\xi^\alpha, w\rangle - c\int_0^T u^\alpha(t+\Delta t)(L_-)w(t,L_-)\,dt + \beta\int_0^T w(t,0_+)\,dt\right] \\
&= -\langle u_t, w\rangle - c\int_0^T u(t)(L_-)w(t,L_-)\,dt + \beta\int_0^T w(t,0_+)\,dt,
\end{aligned}$$

and (5.66) holds.

Step 4. Let $T = T_{a,b}$ be any cut-off function, let j be the primitive of T. Let $0 \leqslant \phi \in \mathcal{D}(]0,T[)$. Multiplying (5.84) by $T(u^{n+1})\phi(t)$, $t \in (t_n, t_{n+1}]$, integrating in $(t_n,t_{n+1}] \times]0,L[$ and adding from $n = 0$ to $n = K-1$, we have

$$\sum_{n=0}^{K-1}\int_{t_n}^{t_{n+1}} \phi(t)\int_0^L \frac{u^{n+1}-u^n}{\Delta t}T\left(u^{n+1}\right)dx\,dt + \int_0^T \phi(t)\int_0^L \mathbf{z}^K(t)D_x\left(T\left(u^K(t+\Delta t)\right)\right)dt$$

$$= \sum_{n=0}^{K-1} \int_{t_n}^{t_{n+1}} \left(\beta\phi(t)T\left(u^{n+1}(0_+)\right) - c\phi(t)u^{n+1}(L_-)T\left(u^{n+1}(L_-)\right)\right) dt. \tag{5.91}$$

Since ϕ has compact support in time in $(0, T)$, for K large enough, performing like in (5.78), we have

$$-\sum_{n=0}^{K-1} \int_{t_n}^{t_{n+1}} \phi(t) \int_0^L \frac{u^{n+1} - u^n}{\Delta t} T\left(u^{n+1}\right) dx\,dt \leqslant \int_0^T \int_0^L j\left(u^K(t)\right) \frac{\phi(t) - \phi(t - \Delta t)}{\Delta t}\, dx\,dt.$$

Hence, from (5.91) it follows that

$$\int_0^T \int_0^L \mathbf{z}^K(t)\phi(t) D_x T\left(u^K(t + \Delta t)\right) dt$$
$$\leqslant \int_0^T \int_0^L j\left(u^K(t)\right) \frac{\phi(t) - \phi(t - \Delta t)}{\Delta t}\, dx\,dt + \int_0^T \beta\phi(t)T\left(u^K(t + \Delta t, 0_+)\right) dt. \tag{5.92}$$

Given $\epsilon > 0$, if we take into (5.92) any test $0 \leqslant \phi \in \mathcal{D}(]0, T[)$ such that $\phi(t) = 1$ for $t \in]\epsilon, T - \epsilon[$, having in mind (5.89), we get

$$\int_\epsilon^{T-\epsilon} \int_0^L \mathbf{z}^K(t) D_x T\left(u^K(t + \Delta t)\right) dt$$
$$\leqslant \int_0^T \int_0^L j\left(u^K(t)\right) \frac{\phi(t) - \phi(t - \Delta t)}{\Delta t}\, dx\,dt + \int_0^T \beta T\left(u^K(t + \Delta t, 0_+)\right) dt \leqslant C.$$

This implies that $\{\mathbf{z}^K(t) D_x(T(u^K(t + \Delta t)))\}$ is a bounded sequence in $L^1_{\mathrm{loc},w}(0, T, \mathcal{M}(]0, L[))$, where $\mathcal{M}(]0, L[)$ denotes the space of bounded Radon measures in $]0, L[$.

On the other hand, by (5.88)

$$\int_\epsilon^{T-\epsilon} \int_0^L \mathbf{z}^K(t) D_x T\left(u^K(t + \Delta t)\right) dt$$
$$\geqslant \frac{c}{2} \int_\epsilon^{T-\epsilon} \int_0^L \left|D_x\left[\left(T\left(u^K(t + \Delta t)\right)\right)^2\right]\right| dt - \int_\epsilon^{T-\epsilon} \int_0^L \frac{c^2}{\nu} \left(T\left(u^K(t + \Delta t)\right)\right)^2 dt.$$

Hence

$$\int_\epsilon^{T-\epsilon} \int_0^L \left|D_x\left[\left(T\left(u^K(t + \Delta t)\right)\right)^2\right]\right| dt \leqslant \frac{2C}{c} + \frac{2cLTb^2}{\nu} = C,$$

where by the co-area formula it follows that

$$\int_{\epsilon}^{T-\epsilon} \int_{0}^{L} \left| D_x T\left(u^K(t+\Delta t)\right)\right| dt \leqslant C. \tag{5.93}$$

Moreover, by Lemma 5 of [2], the map $t \mapsto \|T(u^K(t))\|_{BV(]0,L[)}$ is measurable, then by Fatou's Lemma and (5.93), it follows that

$$\int_{\epsilon}^{T-\epsilon} \liminf_{K\to\infty} \int_{0}^{L} \left| D_x T\left(u^K(t+\Delta t)\right)\right| dt \leqslant \liminf_{K\to\infty} \int_{\epsilon}^{T-\epsilon} \int_{0}^{L} \left| D_x T\left(u^K(t+\Delta t)\right)\right| dt \leqslant C. \tag{5.94}$$

Now, since the total variation is lower semi-continuous in $L^1(]0,L[)$, we have

$$\int_{0}^{L} \left| D_x T\left(u(t)\right)\right| \leqslant \liminf_{K\to\infty} \int_{0}^{L} \left| D_x T\left(u^K(t)\right)\right|,$$

thus we deduce that $T(u(t)) \in BV(]0,L[)$ for almost all $t \in (0,T)$ and consequently $u(t) \in TBV^+(]0,L[)$. Then, by (5.94), and applying again Lemma 5 of [2], we obtain that

$$T\left(u(\cdot)\right) \in L^1_{\mathrm{loc},w}\left(0,T,BV\left(]0,L[\right)\right). \tag{5.95}$$

Step 5. *Identification of the field.* Let us now prove that

$$\mathbf{z}(t) = \mathbf{a}\left(u(t), \partial_x u(t)\right) \quad \text{a.e. } t \in]0,T[. \tag{5.96}$$

Let $0 \leqslant \phi \in \mathcal{D}(Q_T)$ and $g \in C^2([0,L])$. Assume that $\phi = \eta(t)\rho(x)$ with $\eta \in \mathcal{D}(]0,T[)$ and $\rho \in \mathcal{D}(]0,L[)$. Let $0 < a < b$, and $T = T_{a,b}$. Let j denote the primitive of T. Recall that

$$J_{\mathbf{a}}(x,r) = \int_0^r \mathbf{a}\left(s, g'(x)\right) ds \quad \text{and} \quad J_{\mathbf{a}'}(x,r) = \int_0^r \partial_x\left[\mathbf{a}\left(s, g'(x)\right)\right] ds.$$

For simplicity, we write

$$D_2 J_{\mathbf{a}}\left(x, T\left(u^K(t+\Delta t)\right)\right) := D_x\left[J_{\mathbf{a}}\left(x, T\left(u^K(t+\Delta t)\right)\right)\right] - J_{\mathbf{a}'}\left(x, T\left(u^K(t+\Delta t)\right)\right).$$

Working as in the proof of Step 6 of Theorem 3 in [4] we find out that

$$\left[D_2 J_{\mathbf{a}}\left(x, T\left(u^K(t+\Delta t)\right)\right)\right]^{ac} = \mathbf{a}\left(u^K(t+\Delta t), g'\right)\partial_x\left[T\left(u^K(t+\Delta t)\right)\right]. \tag{5.97}$$

Using (5.97), (2.10) and (5.84), we obtain

$$\int_0^T\int_0^L \phi \mathbf{z}^K(t) D_x\big(T\big(u^K(t+\Delta t)\big) - g\big)\,dt$$

$$- \int_0^T\int_0^L \phi\big[D_2 J_{\mathbf{a}}\big(x, T\big(u^K(t+\Delta t)\big)\big) - \mathbf{a}\big(u^K(t+\Delta t), g'\big)g'\big]\,dt$$

$$= \int_0^T\int_0^L \phi\big[\mathbf{z}^K(t) D_x T\big(u^K(t+\Delta t)\big) - \mathbf{z}^K(t) g' + \mathbf{a}\big(u^K(t+\Delta t), g'\big)g'\big]\,dt$$

$$- \int_0^T\int_0^L \phi\big\{\big[D_2 J_{\mathbf{a}}\big(x, T\big(u^K(t+\Delta t)\big)\big)\big]^{ac} + \big[D_2 J_{\mathbf{a}}\big(x, T\big(u^K(t+\Delta t)\big)\big)\big]^{s}\big\}\,dt$$

$$= \int_0^T\int_0^L \phi\big(\mathbf{a}\big(u^K(t+\Delta t), g'\big) - \mathbf{z}^K(t)\big)\big(g' - \partial_x T\big(u^K(t+\Delta t)\big)\big)\,dt$$

$$+ \int_0^T\int_0^L \phi\big[\mathbf{z}^K(t) D_x^s T\big(u^K(t+\Delta t)\big) - \big[D_2 J_{\mathbf{a}}\big(x, T\big(u^K(t+\Delta t)\big)\big)\big]^{s}\big]\,dt$$

$$\geqslant \int_0^T\int_0^L \phi\big[\mathbf{z}^K(t) D_x^s T\big(u^K(t+\Delta t)\big) - \big[D_2 J_{\mathbf{a}}\big(x, T\big(u^K(t+\Delta t)\big)\big)\big]^{s}\big]\,dt$$

$$\geqslant \int_0^T\int_0^L \phi\left[\frac{c}{2}\big|D_x^s\big(T\big(u^K(t+\Delta t)\big)^2\big)\big| - \big[D_2 J_{\mathbf{a}}\big(x, T\big(u^K(t+\Delta t)\big)\big)\big]^{s}\right]dt.$$

Again working as in the proof of Step 6 of Theorem 3 in [4], we get

$$\int_0^T\int_0^L \phi\left[\frac{c}{2}\big|D_x^s\big(T\big(u^K(t+\Delta t)\big)^2\big)\big| - \big[D_2 J_{\mathbf{a}}\big(x, T\big(u^K(t+\Delta t)\big)\big)\big]^{s}\right]dt \geqslant 0.$$

Therefore, we obtain

$$\int_0^T\int_0^L \phi \mathbf{z}^K(t) D_x\big(T\big(u^K(t+\Delta t)\big) - g\big)\,dt$$

$$- \int_0^T\int_0^L \phi\big[D_2 J_{\mathbf{a}}\big(x, T\big(u^K(t+\Delta t)\big)\big) - \mathbf{a}\big(u^K(t+\Delta t), g'\big)g'\big]\,dt \geqslant 0. \tag{5.98}$$

Now we shall bound from above the first term. By (5.85) and for Δt small enough, performing like in (5.78), we get

$$\int_0^T \int_0^L \phi(t,x) T\big(u^K(t+\Delta t)\big) D_x \mathbf{z}^K(t)\,dt = \int_0^T \int_0^L \phi(t,x) T\big(u^K(t+\Delta t)\big)\xi^K(t)\,dx\,dt$$

$$\geqslant \int_0^T \int_0^L \frac{\phi(t-\Delta t,x)-\phi(t,x)}{\Delta t} j\big(u^K(t)\big)\,dt\,dx.$$

Then, integrating by parts, we have

$$\int_0^T \int_0^L \phi(t)\mathbf{z}^K(t) D_x\big(T\big(u^K(t+\Delta t)\big)-g\big)\,dt$$

$$\leqslant -\int_0^T \int_0^L \frac{\phi(t-\Delta t)-\phi(t)}{\Delta t} j\big(u^K(t)\big)\,dt\,dx$$

$$+\int_0^T \int_0^L \phi(t) g \xi^K(t)\,dt\,dx - \int_0^T \int_0^L \partial_x\phi(t)\mathbf{z}^K(t)\big[T\big(u^K(t+\Delta t)\big)-g\big]\,dt\,dx.$$

Thanks to this inequality we arrive from (5.98) to

$$-\int_0^T \int_0^L \frac{\phi(t-\Delta t)-\phi(t)}{\Delta t} j\big(u^K(t)\big)\,dt\,dx + \int_0^T \int_0^L \phi(t) g\xi^K(t)\,dt\,dx$$

$$-\int_0^T \int_0^L \partial_x\phi(t)\mathbf{z}^K(t)\big[T\big(u^K(t+\Delta t)\big)-g\big]\,dt\,dx$$

$$-\int_0^T \int_0^L \phi(t)\big[D_2 J_{\mathbf{a}}\big(x, T\big(u^K(t+\Delta t)\big)\big) - \mathbf{a}\big(u^K(t+\Delta t), g'\big)g'\big]\,dt \geqslant 0. \tag{5.99}$$

Letting $K \to \infty$ in (5.99) and having in mind that

$$D_2 J_{\mathbf{a}}\big(x, T\big(u^K(t+\Delta t)\big)\big) \rightharpoonup D_2 J_{\mathbf{a}}\big(x, T\big(u(t)\big)\big) \quad \text{weakly as measures}$$

we obtain

$$\int_0^T \int_0^L \partial_t\phi(t) j\big(u(t)\big)\,dt + \langle u_t, \phi g\rangle - \int_0^T \int_0^L \big[T\big(u(t)\big)-g\big]\mathbf{z}(t)\partial_x\phi(t)\,dx\,dt$$

$$+\int_0^T \int_0^L \phi(t)\big[-D_2 J_{\mathbf{a}}\big(x, T\big(u(t)\big)\big) + \mathbf{a}\big(u(t), g'\big)g'\big]\,dt \geqslant 0. \tag{5.100}$$

By (5.66),

$$\langle u_t, \phi g\rangle = -\int_0^T\int_0^L \mathbf{z}(t) g \partial_x\phi(t)\,dt\,dx - \int_0^T\int_0^L \mathbf{z}(t) g'\phi(t)\,dt\,dx$$

and we can rearrange (5.100) in the following way

$$\int_0^T\int_0^L \partial_t\phi(t) j\big(u(t)\big)\,dt\,dx - \int_0^T\int_0^L \mathbf{z}(t) g'\phi(t)\,dt\,dx - \int_0^T\int_0^L T\big(u(t)\big)\mathbf{z}(t)\partial_x\phi(t)\,dx\,dt$$
$$+ \int_0^T\int_0^L \phi(t)\big[-D_2 J_{\mathbf{a}}\big(x, T\big(u(t)\big)\big) + \mathbf{a}\big(u(t), g'\big)g'\big]\,dt \geqslant 0. \tag{5.101}$$

Now, for τ small enough and using again the trick in (5.78), we have

$$\int_0^T\int_0^L \partial_t\phi(t,x) j\big(u(t,x)\big)\,dx\,dt = \lim_{\tau\to 0}\int_0^T\int_0^L \frac{\eta(t-\tau)-\eta(t)}{-\tau} j\big(u(t,x)\big)\rho(x)\,dx\,dt$$
$$\leqslant \lim_{\tau\to 0}\int_0^T\int_0^L u(t,x)\rho(x)\frac{d}{dt}\big(\eta T(u)\big)^\tau(t,x)\,dx\,dt,$$

where we used again the notion of Dunford integral (see Remark 5.3). Using (5.65), we have

$$\int_0^T\int_0^L u(t)\rho\frac{d}{dt}\big(\eta T(u)\big)^\tau(t)\,dx\,dt = -\big\langle u_t, \rho\big(\eta T(u)\big)^\tau(\cdot)\big\rangle$$
$$= -\lim_\alpha\big\langle \xi^\alpha, \rho\big(\eta T(u)\big)^\tau(\cdot)\big\rangle = -\lim_\alpha\int_0^T\left\langle D_x\mathbf{z}^\alpha(t), \rho\frac{1}{\tau}\int_{t-\tau}^t \eta(s)T\big(u(s)\big)\,ds\right\rangle dt$$
$$= \lim_\alpha\int_0^T\int_0^L \mathbf{z}^\alpha(t) D_x\left(\rho\frac{1}{\tau}\int_{t-\tau}^t \eta(s)T\big(u(s)\big)\,ds\right)dt = \lim_\alpha\int_0^T\int_0^L \partial_x\rho\mathbf{z}^\alpha(t)\int_{t-\tau}^t \frac{1}{\tau}\eta(s)T\big(u(s)\big)\,ds\,dx\,dt$$
$$+ \lim_\alpha\int_0^T\int_0^L \rho\mathbf{z}^\alpha(t) D_x\big[\big(\eta T(u)\big)^\tau(t)\big]\,dt = \int_0^T\frac{1}{\tau}\int_{t-\tau}^t \eta(s)\int_0^L T\big(u(s)\big)\mathbf{z}(t)\partial_x\rho\,dx\,ds\,dt$$
$$+ \lim_\alpha\int_0^T\int_0^L \rho\mathbf{z}^\alpha(t)\partial_x\big[\big(\eta T(u)\big)^\tau(t)\big]\,dx\,dt + \lim_\alpha\int_0^T\int_0^L \rho\mathbf{z}^\alpha(t) D_x^s\big[\big(\eta T(u)\big)^\tau(t)\big]\,dt$$

$$\leqslant \int_0^T \frac{1}{\tau} \int_{t-\tau}^t \eta(s) \int_0^L T\big(u(s)\big)\mathbf{z}(t)\partial_x\rho \, dx\, ds\, dt + \int_0^T \frac{1}{\tau} \int_{t-\tau}^t \eta(s) \int_0^L \rho\mathbf{z}(t)\partial_x\big(T\big(u(s)\big)\big)\, dx\, ds\, dt$$
$$+ \int_0^T \frac{1}{\tau} \int_{t-\tau}^t \eta(s) \int_0^L cM\rho\big|D_x^s\big[T\big(u(s)\big)\big]\big|\, ds\, dt.$$

Taking limits when $\tau \to 0$, having in mind (5.89), we obtain

$$\int_0^T\int_0^L \partial_t\phi(t)\, j\big(u(t)\big)\, dx\, dt \leqslant \int_0^T \eta(t) \int_0^L T\big(u(t)\big)\mathbf{z}(t)\partial_x\rho\, dx\, dt + \int_0^T \eta(t) \int_0^L \rho\mathbf{z}(t)\partial_x T\big(u(t)\big)\, dx\, dt$$
$$+ cM \int_0^T \eta(t) \int_0^L \rho\big|D_x^s\big[T\big(u(t)\big)\big]\big|\, dt.$$

From (5.101), all gathered together reads

$$0 \leqslant -\int_0^T\int_0^L \phi(t)\mathbf{z}(t)g'\, dx\, dt + \int_0^T \eta(t) \int_0^L \rho\mathbf{z}(t)\partial_x\big(T\big(u(t)\big)\big)\, dx\, dt + cM\int_0^T \eta(t) \int_0^L \rho\big|D_x^s T\big(u(t)\big)\big|\, dt$$
$$+ \int_0^T\int_0^L \phi\big[-D_2 J_{\mathbf{a}}\big(x, T\big(u(t)\big)\big) + \mathbf{a}\big(u(t), g'\big)g'\big]\, dt.$$

Using that $D_2 J_{\mathbf{a}}(x, T(u(t))) = \mathbf{a}(u(t), g')\partial_x(T(u(t))) + [D_2 J_{\mathbf{a}}(x, T(u(t)))]^s$, this is written as

$$0 \leqslant cM \int_0^T \eta(t) \int_0^L \rho\big|D_x^s T\big(u(t)\big)\big|\, dt - \int_0^T\int_0^L \phi\big[D_2 J_{\mathbf{a}}\big(x, T\big(u(t)\big)\big)\big]^s\, dt$$
$$+ \int_0^T\int_0^L \big[g' - \partial_x\big(T\big(u(t)\big)\big)\big]\big[\mathbf{a}\big(u(t), g'\big) - \mathbf{z}(t)\big]\phi\, dx\, dt.$$

As measures,

$$cM\big|D_x^s T\big(u(t)\big)\big| - \big[D_2 J_{\mathbf{a}}\big(x, T\big(u(t)\big)\big)\big]^s + \big[g' - \partial_x\big(T\big(u(t)\big)\big)\big]\big[\mathbf{a}\big(u(t), g'\big) - \mathbf{z}(t)\big]\mathcal{L}^2 \geqslant 0.$$

Taking the absolutely continuous part and particularizing to points $x \in [a < u(t) < b]$, this reduces to

$$\big[g' - \partial_x u(t)\big]\big[\mathbf{a}\big(u(t), g'\big) - \mathbf{z}(t)\big] \geqslant 0,$$

an inequality which holds for all $(t, x) \in S \cap [a < u < b]$, where $S \subseteq]0, T[\times]0, L[$ is such that $\mathcal{L}^2(]0, T[\times]0, L[\setminus S) = 0$, and all $g \in C^2([0, L])$. Being $(t, x) \in S \cap [a < u < b]$ fixed and $\xi \in \mathbb{R}$ given, we can find a function g as above such that $g'(x) = \xi$. Then

$$\big(\mathbf{z}(t, x) - \mathbf{a}\big(u(t), \xi\big)\big)\big(\partial_x u(t, x) - \xi\big) \geqslant 0, \quad \forall \xi \in \mathbb{R} \text{ and } \forall (t, x) \in S \cap [a < u < b].$$

By an application of Minty–Browder's method in $\mathbb{R}$, these inequalities imply that

$$\mathbf{z}(x) = \mathbf{a}\big(u(t,x), \partial_x u(t,x)\big) \quad \text{a.e. on } Q_T \cap [a < u < b].$$

Since this holds for any $0 < a < b$, we obtain (5.96) a.e. on the points of Q_T such that $u(t,x) \neq 0$. Now, by our assumptions on $\mathbf{a}$ and (5.90) we deduce that $\mathbf{z}(x) = \mathbf{a}(u(x), u'(x)) = 0$ a.e. on $[u = 0]$. We have proved (5.96).

Step 6. *The entropy inequality.* Given $S \in \mathcal{P}^+, T \in \mathcal{T}^+$ and $\phi \in \mathcal{D}(Q_T)$, working as in the proof of (5.92) we can get

$$\begin{aligned} &\int_0^T\int_0^L \phi \mathbf{z}^K(t) D_x\big(T\big(u^K(t+\Delta t)\big) S\big(u^K(t+\Delta t)\big)\big)\,dt \\ &\quad \leqslant \int_0^T\int_0^L J_{TS}\big(u^K(t)\big)\frac{\phi(t)-\phi(t-\Delta t)}{\Delta t}\,dx\,dt \\ &\qquad - \int_0^T\int_0^L \mathbf{z}^K(t)\partial_x\phi T\big(u^K(t+\Delta t)\big)S\big(u^K(t+\Delta t)\big)\,dx\,dt \end{aligned} \tag{5.102}$$

and the fact that $\{\mathbf{z}^K(t) D_x(T(u^K(t+\Delta t))S(u^K(t+\Delta t)))\}$ is a bounded sequence in $L^1_{\mathrm{loc}}(0,T;\mathcal{M}(]0,L[))$. From here, as in the proof of Theorem 4.5 of [6], we can get that the sequences $\{\mathbf{z}^K(t) D_x J_{T'S}(u^K(t+\Delta t))\}$ and $\{\mathbf{z}^K(t) D_x J_{S'T}(u^K(t+\Delta t))\}$ are bounded in $L^1_{\mathrm{loc}}(0,T;\mathcal{M}(]0,L[))$. This allows us to define, up to subsequence, the objects $\mu_T^S, \mu_S^T \in \mathcal{M}(Q_T)$ by means of

$$\langle \phi, \mu_S^T\rangle = \lim_K \int_0^T\int_0^L \phi \mathbf{z}^K(t) D_x J_{T'S}\big(u^K(t+\Delta t)\big)\,dt, \quad \forall \phi \in C_c(Q_T),$$

$$\langle \phi, \mu_T^S\rangle = \lim_K \int_0^T\int_0^L \phi \mathbf{z}^K(t) D_x J_{S'T}\big(u^K(t+\Delta t)\big)\,dt, \quad \forall \phi \in C_c(Q_T).$$

Then, passing to the limit in (5.102), we obtain

$$\begin{aligned} &\langle \phi, \mu_S^T\rangle + \langle \phi, \mu_T^S\rangle \\ &\quad \leqslant \int_0^T\int_0^L J_{TS}\big(u(t)\big)\partial_t\phi(t)\,dx\,dt - \int_0^T\int_0^L \mathbf{z}(t)\partial_x\phi T\big(u(t)\big)S\big(u(t)\big)\,dx\,dt, \quad \forall \phi \in \mathcal{D}(Q_T). \end{aligned} \tag{5.103}$$

Working as in proof of Lemma 4.11 in [6], we can get the following result.

Lemma 5.6. *For $S \in \mathcal{P}^+$, $T \in \mathcal{T}^+$, we have that $\mu_S^T \geqslant h_S(u, DT(u))$.*

By the above lemma and (5.103) we obtain the entropy inequality

$$\int_0^T\int_0^L \phi h_S\big(u, DT(u)\big)\,dt + \int_0^T\int_0^L \phi h_T\big(u, DS(u)\big)\,dt$$
$$\leqslant \int_0^T\int_0^L J_{TS}(u)\phi'\,dx\,dt - \int_0^T\int_0^L \mathbf{a}(u, \partial_x u)\cdot \partial_x \phi T(u)S(u)\,dx\,dt$$

for truncatures $S \in \mathcal{P}^+$, $T \in \mathcal{T}^+$ and any smooth function ϕ of compact support. □

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