Available online at www.sciencedirect.com
ScienceDirect
Discrete Mathematics 308 (2008) 5144–5148
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Transformation graph G^{-+-}

Lan Xu[a, b], Baoyindureng Wu[a, *, 1]

[a] College of Mathematics and System Science, Xinjiang University, Urumqi 830046, Xinjiang, PR China
[b] Department of Mathematics, Changji University, Changji 831100, Xinjiang, PR China

Received 31 March 2007; received in revised form 10 September 2007; accepted 17 September 2007
Available online 7 November 2007

Abstract

The transformation graph G^{-+-} of a graph G is the graph with vertex set $V(G) \cup E(G)$, in which two vertices u and v are joined by an edge if one of the following conditions holds: (i) $u, v \in V(G)$ and they are not adjacent in G, (ii) $u, v \in E(G)$ and they are adjacent in G, (iii) one of u and v is in $V(G)$ while the other is in $E(G)$, and they are not incident in G. In this paper, for any graph G, we determine the connectivity and the independence number of G^{-+-}. Furthermore, for a graph G of order $n \geqslant 4$, we show that G^{-+-} is hamiltonian if and only if G is not isomorphic to any graph in $\{2K_1+K_2, K_1+K_3\} \cup \{K_{1,n-1}, K_{1,n-1}+e, K_{1,n-2}+K_1\}$.

Keywords: Transformation graph; Complement; Hamilton cycle

1. Introduction

All graphs considered here are finite, undirected and simple. We refer to [1] for unexplained terminology and notations. Let $G = (V(G), E(G))$ be a graph. For a vertex v of G, if there is no confusion, the degree $d_G(v)$ is simply denoted by $d(v)$. The symbols $\Delta(G)$, $\delta(G)$, $\kappa(G)$, $\alpha(G)$, $\omega(G)$, comp(G) and $M(G)$ denote the maximum degree, the minimum degree, the connectivity, the independence number, the clique number, the number of components and the cardinality of a maximum matching of G, respectively. As usual, K_n is the complete graph of order n. For two positive integers r and s, $K_{r,s}$ is the complete bipartite graph with two partite sets containing r and s vertices.

In particular, $K_{1,s}$ is called a star. For $s \geqslant 2$, $K_{1,s} + e$ is the graph obtained from $K_{1,s}$ by adding a new edge which joins two vertices of degrees one. $K_{r,s} - e$ is the graph obtained from $K_{r,s}$ by deleting an edge. We say two graphs G and H are disjoint if they have no vertex in common, and denote their union by $G + H$; such a graph is called the disjoint union of G and H. The disjoint union of k copies of G is written as kG.

The *complement* of G, denoted by $\overline{G}$, is the graph with the same vertex set as G, but where two vertices are adjacent if and only if they are not adjacent in G. The total graph $T(G)$ of G is the graph whose vertex set is $V(G) \cup E(G)$, and in which two vertices are adjacent if and only if they are adjacent or incident in G. Wu and Meng [6] introduced some new graphical transformations which generalize the concept of total graph.

Let $G = (V(G), E(G))$ be a graph, and α, β be two elements of $V(G) \cup E(G)$. We define the associativity of α and β is $+$ if they are adjacent or incident, and $-$ otherwise. Let xyz be a 3-permutation of the set $\{+, -\}$. We say that α and

* Corresponding author.
E-mail address: baoyin@xju.edu.cn (B. Wu).
[1] Research supported by NSFC(No. 10601044) and XJEDU2006S05.

0012-365X/$ - see front matter
doi:10.1016/j.disc.2007.09.040

Table 1
Hamiltonicity of G^{xyz}

Transformation graph	References and results
G^{+++}	In [4], a necessary and sufficient condition
G^{++-}	In [8], a sufficient condition
G^{+-+}	
G^{+--}	In [9], a necessary and sufficient condition
G^{-++}	In [7], a necessary and sufficient condition
G^{-+-}	In this paper, a necessary and sufficient condition
G^{--+}	
G^{---}	In [5], a necessary and sufficient condition

β correspond to the first term x (resp. the second term y or the third term z) if both α and β are in $V(G)$ (resp. both α and β are in $E(G)$, or one of α and β is $V(G)$ and the other is in $E(G)$. The transformation graph G^{xyz} of G is defined on the vertex set $V(G) \cup E(G)$. Two vertices α and β of G^{xyz} are joined by an edge if and only if their associativity in G is consistent with the corresponding term of xyz.

Therefore, one can obtain eight graphical transformations of graphs, since there are eight distinct 3-permutation of $\{+, -\}$. Note that G^{+++} is just the total graph $T(G)$ of G, and G^{---} is the complement of $T(G)$. Fleischner and Hobbs [4] showed that G^{+++} is hamiltonian if and only if G contains an EPS-subgraph, that is, a connected spanning subgraph S which is the edge-disjoint union of a (not necessarily connected) graph E, all of whose vertices have even degree, with a (possibly empty) forest P each of whose component is a path. Ma and Wu [5] showed that for a graph G of order $n \geqslant 3$, G^{---} is hamiltonian if and only if G is not isomorphic to any graph in $\{K_{1,n-1}, K_{1,n-1}+e, K_{1,n-2}+K_1\} \cup \{K_2+2K_1, K_3+K_1, K_3+2K_1, K_4\}$. Wu et al. [7] proved that for any graph G of order n, G^{-++} is hamiltonian if and only if $n \geqslant 3$. Chen [2] studied the super-connectivity of these transformation graphs. Table 1 summarizes the known results on hamiltonicity of G^{xyz}.

In this paper, we shall investigate the transformation graph G^{-+-} of a graph G, and determine its connectivity and independence number. Furthermore, we obtain a necessary and sufficient condition for G^{-+-} to be hamiltonian when the order of G is at least 4.

Theorem 1. *For a graph G of order $n \geqslant 4$, G^{-+-} is hamiltonian if and only if G is not isomorphic to any graph in* $\{K_{1,n-1}, K_{1,n-1}+e, K_{1,n-2}+K_1\} \cup \{2K_1+K_2, K_1+K_3\}$.

2. Preliminary

We start with some simple observations. Let G be a graph of order n and size m. Then the order of G^{-+-} is $n+m$, $d_{G^{-+-}}(x) = n+m-1-2d(x)$ for $x \in V(G)$ and $d_{G^{-+-}}(e) = n-4+d(u)+d(v)$ for any $e = uv \in E(G)$.

So $\delta(G^{-+-}) = \min\{n+m-1-2\Delta(G), n-4+\min_{uv\in E(G)}\{d(u)+d(v)\}\}$. Wu and Meng [6] proved that G^{-+-} is connected if and only if G is not a star, and that diam $(G^{-+-}) \leqslant 3$ if G is not a star.

Theorem 2. *For a graph G of order n and size m,* $\kappa(G^{-+-}) = \min\{\delta(G^{-+-}), n+\kappa(L(G))-1, m+\kappa(\overline{G})\}$ *or* $\min\{\delta(G^{-+-}), n+\kappa(L(G)), m+\kappa(\overline{G})\}$.

Proof. If G is a star then its center must be an isolated vertex in G^{-+-}, and thus $\kappa(G^{-+-}) = 0 = \delta(G^{-+-})$. Next we assume that G is not a star. It is easy to see that $\kappa(G^{-+-}) \leqslant \min\{\delta(G^{-+-}), n+\kappa(L(G)), m+\kappa(\overline{G})\}$. So it suffices to prove $\kappa(G^{-+-}) \geqslant \min\{\delta(G^{-+-}), n+\kappa(L(G))-1, m+\kappa(\overline{G})\}$. Let S be a minimum cut of G^{-+-} with $|S| < \delta(G^{-+-})$.

Thus each component of $G^{-+-}-S$ has at least two vertices. We say that a component H of $G^{-+-}-S$ is of type-1 (respectively, type-2, or type-3) if $V(H) \subseteq V(G)$ (respectively, $V(H) \subseteq E(G)$, or $V(H) \cap V(G) \neq \emptyset$ and $V(H) \cap E(G) \neq \emptyset$).

Claim 1. *If $G^{-+-}-S$ contains a component of type-1 then all components of $G^{-+-}-S$ are of type-1.*

Proof of Claim 1. To see this, let H_1 be a component of type-1 and we take two adjacent vertices x, y from H_1. Then they are not adjacent in G. If there is a component of type-2 or type-3 in $G^{-+-} - S$, we choose a vertex $e \in V(G^{-+-}) \cap E(G)$ from it. It is obvious that e is not adjacent to neither x nor y in G^{-+-}. So, e must be incident with both x and y in G by the definition of G^{-+-}. Namely, x and y are adjacent in G. It contradicts that x and y are not adjacent in G. The claim is true. □

Claim 2. *If $G^{-+-} - S$ has a component of type-3 then* $\text{comp}(G^{-+-} - S) = 2$.

Proof of Claim 2. By contradiction, suppose $\text{comp}(G^{-+-} - S) \geqslant 3$. By Claim 1, all components of $G^{-+-} - S$ are of type-2 or of type-3. We take a vertex v from a component of type-3 with $v \in V(G)$, and two vertices e_1 and e_2 from other two components with $e_1, e_2 \in E(G)$. By definition of G^{-+-}, v is the common end vertex of e_1 and e_2 in G while e_1 and e_2 are not adjacent in G, a contradiction. □

Claim 3. *All components of $G^{-+-} - S$ cannot be of type-3.*

Proof of Claim 3. By contradiction, suppose all components of $G^{-+-} - S$ are of type-3. By Claim 2, $\text{comp}(G^{-+-} - S) = 2$, and let H_1 and H_2 be the two components of $G^{-+-} - S$. By the adjacency relation between vertices of G^{-+-}, $|V(H_i) \cap V(G)| \leqslant 2$ for each $i = 1$ and 2, since otherwise one can find an edge of G from $V(H_i)$ which will have three end vertices coming from $V(H_j) \cap V(G)$, where $\{i, j\} = \{1, 2\}$, a contradiction. We consider two cases. Assume first that $|V(H_i) \cap V(G)| = 2$, and let $V(H_i) \cap V(G) = \{x_i, y_i\}$ for $i = 1, 2$. Again by the definition of G^{-+-}, the four vertices x_1, x_2, y_1, y_2 are pairwise adjacent in G, $V(H_1) \cap E(G) = \{x_2y_2\}$ and $V(H_2) \cap E(G) = \{x_1y_1\}$. Thus $\Delta(G) \geqslant d(x_1) \geqslant 3$ and $|S| = n + m - 6 > n + m - 1 - 2\Delta(G) \geqslant \delta(G^{-+-})$, a contradiction. So by symmetry, it remains to consider the case $|V(H_1) \cap V(G)| = 1$ and $|V(H_2) \cap V(G)| \leqslant 2$. Let $u_i \in V(H_i) \cap V(G)$ for $i = 1, 2$. Then $V(H_i) \cap E(G)$ is a set of edges incident with u_j in G, where $\{i, j\} = \{1, 2\}$, which gives $|V(H_i) \cap E(G)| \leqslant d(u_j)$. Therefore, $|S| \geqslant (n - 3) + m - ((d(u_1) - 1) + (d(u_2) - 1)) \geqslant n + m - 2\Delta(G) - 1$, a contradiction. So by Claim 1, 2 and 3, there are only three possibilities for the type of components of $G^{-+-} - S$: all components of $G^{-+-} - S$ are of type-1, all components of $G^{-+-} - S$ are of type-2, or $G^{-+-} - S$ consists of one component of type-2 and one of type-3. If all components of $G^{-+-} - S$ are of type-1 then $|S| \geqslant m + \kappa(\overline{G})$; if all components of $G^{-+-} - S$ are of type-2 then $|S| \geqslant + \kappa(L(G))$; in the last case, $|S| \geqslant n + \kappa(L(G)) - 1$. □

This completes the proof. □

Corollary 3. *For a graph G of order $n \geqslant 4$, the following statements are equivalent.*

(1) $\kappa(G^{-+-}) \geqslant 2$.
(2) $\delta(G^{-+-}) \geqslant 2$.
(3) $G \notin \{K_{1,n-1}, K_{1,n-1} + e, K_{1,n-2} + K_1\}$.

Proof. By Theorem 2 $\kappa(G^{-+-}) \geqslant \min\{\delta(G^{-+-}), n + \kappa(L(G)) - 1, m + \kappa(\overline{G})\}$, where $\delta(G^{-+-}) = \min\{n + m - 1 - 2\Delta(G), n - 4 + \min_{uv \in E(G)}(d(u) + d(v))\}$. First we claim that both $n + \kappa(L(G)) - 1 \geqslant 3$ and $m + \kappa(\overline{G}) \geqslant 3$.

Since $n \geqslant 4$, $n + \kappa(L(G)) - 1 \geqslant 4 - 1 = 3$. If $m \geqslant 3$, $m + \kappa(\overline{G}) \geqslant m \geqslant 3$. If $m = 2$, G is not connected since $n \geqslant 4$, and so $m + \kappa(\overline{G}) \geqslant 2 + 1 = 3$. For the case of $m = 1$, one can easily check that $\kappa(\overline{G}) \geqslant 2$, so we also obtain $m + \kappa(\overline{G}) \geqslant 3$. Thus the claim implies that (1) and (2) are equivalent. Moreover, one can easily check that $\delta(G^{-+-}) = 0$ if and only if $G \cong K_{1,n-1}$, $\delta(G^{-+-}) = 1$ if and only if $G \cong K_{1,n-1} + e$ or $K_{1,n-2} + K_1$. Thus (2) and (3) are equivalent. □

One can also note from the proof of Corollary 3 that:

Corollary 4. *For a graph G of order $n \geqslant 4$, $\kappa(G^{-+-}) = 2$ if and only if $\delta(G^{-+-}) = 2$.*

Theorem 5. *For any graph G, $\alpha(G^{-+-}) = 1$ if $\Delta(G) = 0$ and $\alpha(G^{-+-}) = \max\{\omega(G), M(G), 3\}$ otherwise.*

Proof. If $\Delta(G) = 0$ then G^{-+-} is a complete graph, and thus $\alpha(G^{-+-}) = 1$. Suppose $\Delta(G) > 0$ as follows. Since $\{u, v, e\}$ is an independent set of G^{-+-} for any $e = uv \in E(G)$, $\alpha(G^{-+-}) \geqslant 3$. Moreover, since all cliques and matchings of G are independent sets of G^{-+-}, $\alpha(G^{-+-}) \geqslant \omega(G)$ and $\alpha(G^{-+-}) \geqslant M(G)$. Hence $\alpha(G^{-+-}) \geqslant \max\{\omega(G), M(G), 3\}$.

To complete the proof, we will show that $\alpha(G^{-+-}) \leqslant \max\{\omega(G), M(G), 3\}$. Let S be a maximum independent set of G^{-+-} and $S = S_1 \cup S_2$, where $S_1 \subseteq V(G)$ and $S_2 \subseteq E(G)$. Note that $|S_1| \neq 1$. Otherwise, it implies that $|S_2| = 1$ since S_2 is a matching of G and each element of S_2 is incident with the vertex of S_1 in G. Thus $|S| = 2$, which contradicts $|S| \geqslant 3$. Therefore we consider the following three cases.

Case 1: $|S_1| \geqslant 3$.

Then $S_2 = \emptyset$, since otherwise each element of S_2 is an edge of G and has all vertices of S_1 as its end vertices in G. This is not possible because of $|S_1| \geqslant 3$. So $|S| = |S_1| \leqslant \omega(G) \leqslant \max\{\omega(G), M(G), 3\}$.

Case 2: $|S_1| = 2$.

Let $\{S_1\} = \{u, v\}$. By the same argument as in the proof of Case 1, $S_2 = \{uv\}$ since G is a simple graph. Hence $|S| = 3 \leqslant \max\{\omega(G), M(G), 3\}$.

Case 3: $|S_1| = 0$. Then $S = S_2$. Since S_2 is a matching of G, $|S_2| \leqslant M(G)$.

The proof is complete. □

We use the following classical theorem due to Chvátal and Erdös [3].

Theorem 6. *Let G be a graph of order at least three. If $\alpha(G) \leqslant \kappa(G)$, then G is hamiltonian.*

3. The Proof of Theorem 1

If $G \in \{K_{1,n-1}, K_{1,n-1} + e, K_{1,n-2} + K_1\}$ then by Corollary 3, $\kappa(G^{-+-}) < 2$, and so G^{-+-} is not hamiltonian. It is easy to check that both $(2K_1 + K_2)^{-+-}$ and $(K_1 + K_3)^{-+-}$ are not hamiltonian. To show its sufficiency, assume $G \notin \{K_{1,n-1}, K_{1,n-1} + e, K_{1,n-2} + K_1\} \cup \{2K_1 + K_2, K_1 + K_3\}$. Then by Corollary 3 G^{-+-} is 2-connected. If $G \cong \overline{K_n}$ then $G^{-+-} \cong K_n$, and is hamiltonian. So assume G is not an empty graph. Recall that by Theorem 2 $\kappa(G^{-+-}) \geqslant \min\{\delta(G^{-+-}), n + \kappa(L(G)) - 1, m + \kappa(\overline{G})\}$, where $\delta(G^{-+-}) = \min\{n + m - 1 - 2\Delta(G), n - 4 + \min_{uv \in E(G)}(d(u) + d(v))\}$ and by Theorem 5 $\alpha(G^{-+-}) = \max\{\omega(G), M(G), 3\}$.

We consider three cases.

Case 1: $M(G) \geqslant \omega(G) \geqslant 3$.

Then by Theorem 5, $\alpha(G^{-+-}) = M(G)$. Since $\min\{m, \frac{n}{2}\} \geqslant M(G)$ and $n - 2 \geqslant \frac{n}{2}$ for $n \geqslant 4$, $m + \kappa(\overline{G}) \geqslant M(G)$ and $n + \kappa(L(G)) - 1 \geqslant M(G)$. Moreover, since G is not an empty graph, $\min_{uv \in E(G)}\{d(u) + d(v)\} \geqslant 2$ and $n - 4 + \min_{uv \in E(G)}(d(u) + d(v)) \geqslant n - 2 \geqslant M(G)$. Hence, if $n + m - 1 - 2\Delta(G) \geqslant M(G)$ then $\kappa(G^{-+-}) \geqslant M(G)$ and by Theorem 6 G^{-+-} is hamiltonian. Otherwise $n + m - 1 - 2\Delta(G) = M(G) - 1$ since $m \geqslant \Delta(G) + M(G) - 1$ and $n \geqslant \Delta(G) + 1$ hold for any graph G. In this case, $n = \Delta(G) + 1$ and $m = \Delta(G) + M(G) - 1$. Let $V(G) = \{v_0, v_1, \ldots, v_{n-1}\}$, $E(G) = \{e_1, e_2, \ldots, e_{n-1}, e'_1, e'_2, \ldots, e'_{M(G)-1}\}$, where $e_i = v_0 v_i$ for $i = 1, 2, \ldots, n - 1$ and $e'_j = v_{2j-1} v_{2j}$ for $j = 1, \ldots, M(G) - 1$. Then we can find a Hamilton cycle of G^{-+-}:

$$\begin{aligned} &v_0 e'_1 e_1 e_2 e_3 e_4 e_5 e'_3 e_6 e_7 e'_4 e_8 \cdots e'_{M(G)-1} e_{2(M(G)-1)} e_{2M(G)-1} \cdots e_{n-1} v_1 v_3 \\ &\quad \cdots v_{2M(G)-1} v_{2M(G)} \cdots v_{n-1} v_{2(M(G)-1)} v_{2(M(G)-2)} \cdots v_2 e'_2 v_0. \end{aligned}$$

Case 2: $\omega(G) > M(G) \geqslant 3$.

Then by Theorem 5 $\alpha(G^{-+-}) = \omega(G)$. We shall show that $\kappa(G^{-+-}) \geqslant \omega(G)$. Since $\omega(G) \geqslant 4$, $m + \kappa(\overline{G}) \geqslant m \geqslant \binom{\omega(G)}{2} \geqslant \omega(G)$. If G is connected then $\kappa(L(G)) \geqslant 1$, and $n - 1 \geqslant \omega(G)$, otherwise. Thus $n + \kappa(L(G)) - 1 \geqslant \omega(G)$.

It remains to show that $\delta(G^{-+-}) \geqslant \omega(G)$. Since $n - 4 + \min_{uv \in E(G)}\{d(u) + d(v)\} \geqslant n - 2$, if $\omega(G) \leqslant n - 2$ then $\delta(G^{-+-}) \geqslant \omega(G)$. For each case of $\omega(G) = n - 1$ and $\omega(G) = n$, one can easily check that $\min_{uv \in E(G)}\{d(u) + d(v)\} \geqslant \omega(G)$, which implies $n - 4 + \min_{uv \in E(G)}\{d(u) + d(v)\} \geqslant \omega(G)$.

On the other hand, note that $m \geqslant \binom{\omega(G)}{2} + (\Delta(G) - (\omega(G) - 1))$. If $\omega(G) \geqslant 5$ then $\binom{\omega(G)}{2} \geqslant 2\omega(G) - 1$ and $n + m - 1 - 2\Delta(G) \geqslant (\Delta(G) + 1) + (\Delta(G) + \omega(G)) - 1 - 2\Delta(G) = \omega(G)$. So, there is only one case $\omega(G) = 4$ and $M(G) = 3$ to consider. Since $m \geqslant \Delta(G) + M(G) - 1 = \Delta(G) + 2$, $n + m - 1 - 2\Delta(G) \geqslant n + 1 - \Delta(G)$.

If $n + 1 - \Delta(G) \geqslant 4$ then by $\omega(G) = 4$, $n + 1 - \Delta(G) \geqslant \omega(G)$. So we treat the cases $n + 1 - \Delta(G) = 2$ and 3. If $n + 1 - \Delta(G) = 2$ then $\Delta(G) = n - 1$, and thus $m \geqslant n + 3$ by $\omega(G) = 4$ and $M(G) = 3$. Therefore $n + m - 1 - 2\Delta(G) \geqslant n + (n + 3) - 1 - 2(n - 1) = 4 = \omega(G)$.

If $\Delta(G)=n-2$ then by $\omega(G)=4$ and $M(G)=3$, $m\geqslant n+2$. Hence $n+m-1-2\Delta(G)\geqslant n+(n+2)-1-2(n-2)=5\geqslant\omega(G)$.

Thus $\alpha(G^{-+-})=\omega(G)\leqslant\kappa(G^{-+-})$, by Theorem 6 G^{-+-} is hamiltonian.

Case 3: $\max\{\omega(G), M(G), 3\}=3$.

Then $\alpha(G^{-+-})=3$. If $\kappa(G^{-+-})\geqslant 3$ we are done. So we assume $\kappa(G^{-+-})=2$ as follows. By Corollary 4 $\delta(G^{-+-})=2$. Since $\delta(G^{-+-})=\min\{n+m-1-2\Delta(G), n-4+\min_{uv\in E(G)}\{d(u)+d(v)\}\}$, we distinguish two cases.

First suppose $n-4+\min_{uv\in E(G)}\{d(u)+d(v)\}=2$. Since $n\geqslant 4$ and $\min_{uv\in E(G)}\{d(u)+d(v)\}\geqslant 2$, G must be isomorphic to $2K_2$ or $2K_1+K_2$. But by the hypothesis that $G\ncong 2K_1+K_2$, $G\cong 2K_2$. One can see that $(2K_2)^{-+-}\cong K_{3,3}-e$ is hamiltonian.

Now we consider the case $m+n-1-2\Delta(G)=2$. It follows from $n\geqslant\Delta(G)+1$ and $m\geqslant\Delta(G)$ that $(n,m)\in\{(\Delta(G)+1, \Delta(G)+2), (\Delta(G)+2, \Delta(G)+1), (\Delta(G)+3, \Delta(G))\}$. If $(n,m)=(\Delta(G)+3, \Delta(G))$ then $G\cong K_{1,n-3}+2K_1$. Let $V(G)=\{v_0, v_1, \ldots, v_{n-1}\}$ and $E(G)=\{e_1, e_2, \ldots, e_{n-3}\}$, where $e_i=v_0v_i$ for $i=1,2,\ldots,n-3$. Then we can find a Hamilton cycle of G^{-+-}: $v_0v_{n-1}e_1e_2\cdots e_{n-3}v_1v_2\cdots v_{n-3}v_{n-2}v_0$.

If $(n,m)=(\Delta(G)+2, \Delta(G)+1)$ then G is isomorphic either $(K_{1,n-2}+e)+K_1$ or the tree obtained from joining a new vertex to a vertex with degree one in $K_{1,n-2}$. If $G\cong(K_{1,n-2}+e)+K_1$, let $V(G)=\{v_0, v_1, \ldots, v_{n-1}\}$ and $E(G)=\{e_1, e_2, \cdots e_{n-1}\}$, where $e_i=v_0v_i$ for $i=1,\cdots,n-2$ and $e_{n-1}=e_1e_2$. Note that for $n=4$, $(K_{1,n-2}+e)+K_1\cong K_3+K_1$, but $G\ncong K_1+K_3$ by the assumption, we have $n\geqslant 5$. Hence, one can find a Hamilton cycle of G^{-+-} as follows: $v_0v_{n-1}v_1e_2e_3\cdots e_{n-2}v_2v_3\cdots v_{n-2}e_1e_{n-1}v_0$. For the latter case, let $V(G)=\{v_0, v_1, \ldots, v_{n-1}\}$ and $E(G)=\{e_1, e_2, \ldots, e_{n-2}, e_{n-1}\}$, where $e_i=v_0v_i$ for $i=1,2,\ldots,n-2$ and $e_{n-1}=v_{n-2}v_{n-1}$. Then we can find a Hamilton cycle of G^{-+-}: $v_0e_{n-1}v_1v_2\cdots v_{n-2}e_1e_2\cdots e_{n-2}v_{n-1}v_0$.

If $(n,m)=(\Delta(G)+1, \Delta(G)+2)$ then G is isomorphic to a graph obtained from $K_{1,n-1}$ by adding two edges (there are two possibilities: the two new edges may be adjacent or not in G). If the two new edges are not adjacent in G, let $V(G)=\{v_0, v_1, \ldots, v_{n-1}\}$ and $E(G)=\{e_1, e_2, \ldots, e_{n-1}, e_n, e_{n+1}\}$, where $e_i=v_0v_i$ for $i=1,2,\ldots,n-1$, $e_n=v_1v_2$ and $e_{n+1}=v_3v_4$. Then $v_0e_nv_3v_5\cdots v_{n-1}v_1v_4v_2e_1e_2e_3e_5\cdots e_{n-1}e_4e_{n+1}v_0$ is a Hamilton cycle of G^{-+-}. If the two new edges are adjacent in G, let $V(G)=\{v_0, v_1, \ldots, v_{n-1}\}$ and $E(G)=\{e_1, e_2, \ldots, e_{n-1}, e_n, e_{n+1}\}$, where $e_i=v_0v_i$ for $i=1,2,\ldots,n-1$, $e_n=v_1v_2$ and $e_{n+1}=v_2v_3$. Then $v_0e_nv_3v_4\cdots v_{n-1}e_1v_2e_3e_4\cdots e_{n-1}v_1e_2e_{n+1}v_0$ is a Hamilton cycle of G^{-+-}.

The proof is complete.

4. Concluding remarks

In this note, we prove that for a graph G of order $n\geqslant 4$, G^{-+-} is hamiltonian if and only if G is not isomorphic to any graph in $\{K_{1,n-1}, K_{1,n-1}+e, K_{1,n-2}+K_1\}\cup\{2K_1+K_2\}$. Corollary 3 implies that if G is a graph of order $n\geqslant 4$ and not isomorphic to $2K_1+K_2$ then G^{-+-} is hamiltonian if and only if $\delta(G^{-+-})\geqslant 2$.

Acknowledgements

The authors are grateful to the referees for their critical comments.

References

[1] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, American Elsevier, Macmillan, New York, London, 1976.
[2] J. Chen, Super edge-connectivity of two classes of transformation graphs, Doctoral Thesis, Xinjiang University, 2006.
[3] V. Chvátal, P. Erdös, A note on hamiltonian circuits, Discrete Math. 2 (1972) 111–113.
[4] H. Fleischner, A.M. Hobbs, Hamiltonian total graphs, Math. Notes 68 (1975) 59–82.
[5] G. Ma, B. Wu, Hamiltonicity of complements of total graphs, Discrete Geometry, Combinatorics and Graph Theory, 7th China-Japan Conference, Lecture Notes in Computer Science, vol. 4381, 2007, Springer, pp. 109–119.
[6] B. Wu, J. Meng, Basic properties of total transformation graphs, J. Math. Study 34 (2) (2001) 109–116.
[7] B. Wu, L. Zhang, Z. Zhang, The transformation graph G^{xyz} when $xyz=-++$, Discrete Math. 296 (2005) 263–270.
[8] L. Yi, B. Wu, The transformation graph G^{++-}, Submitted for publication.
[9] L. Zhen, B. Wu, Hamiltonicity of transformation graph G^{+--}, Submitted for publication.