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Full Length Article

Dynamic response of non-uniform beam subjected to moving load and resting on non-linear viscoelastic foundation

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ABSTRACT

In this paper, it was purposed to comprehend the dynamic response of non-uniform Euler–Bernoulli simply supported beam which is subjected to moving load. The beam is rested on a nonlinear viscoelastic foundation. Galerkin with Runge-Kutta methods are employed. The influences of variations of the traveling velocity and the effect of increase in the magnitude of the moving load on the dynamic response are studied. A MATLAB code is designed to compute and plot the deflection.

1. Introduction

All branches of transport have experienced great advances characterized by increasingly higher speeds and weights of vehicles. As a result, structures and media over in which the vehicles move have been subjected to vibrations and dynamic stresses far larger than ever before. Many problems of considerable practical importance can be related to the solution of beams on a nonlinear viscoelastic foundation. Reinforced-concrete pavements of highways and airport runways and foundation slabs of buildings are well-known direct applications. That is, an analogy exists between the governing differential equation of a beam on a nonlinear viscoelastic foundation.

Transverse vibration analysis of uniform and non-uniform Euler–Bernoulli beams was explained by Safa et al. (2011). They explained the theory and analytical techniques about lateral vibration of Euler–Bernoulli beams, and then described the methods used in the analysis. The dynamic effect of moving loads was not known until mid-nineteenth century. When the Stephenson's bridge across river Dee Chester in England in 1947 collapsed, it motivates the engineers for research of moving load problem. Moving loads have a great effect on the bodies or structures over which it travels. Fryba (1999) has presented fundamental studies in this area including most of the published articles before year 1999. The dynamic response of finite elastic structures (Rayleigh beams and plates) having arbitrary end supports and under an arbitrary number of moving masses was developed by Gbadeyan and Oni (1995). Serdar (2005) comprehend the dynamic response of beams and frames which are subjected to moving point load. Serdar employed finite element method and numerical time integration method (Newmark method) in the vibration analysis. The equation of motion in matrix form for an Euler beam subjected to a con-

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centrated mass moving at a steady speed was formulated by using the Green's function approach by Sudhansu (2012). He evaluated the solutions of the declared problems for the case of simply supported and cantilever beams by using dynamic Green's function.

It is clear that the main factor, which strongly affects the dynamic soil structure problem, is the validity of the foundation model. Free vibration of an Euler–Bernoulli beam resting on a variable Winkler foundation was studied by Alev et al. (2011). He solved the governing differential equations of the beam by using Differential Transform Method. The case of a nonlinear elastic foundation is of great theoretical and practical significance in railway engineering. In practice the foundation is highly nonlinear. Pellicano and Mastroddi (1997) studied nonlinear dynamics of beams resting on a nonlinear spring bed. Coskun and Engin (1999) presented the nonlinear vibration of a beam resting on nonlinear tensionless foundation subjected to a concentrated load at the center using the perturbation technique. Tsiatas (2010) presented the nonlinear problem of non-uniform beams. The nonlinear governing equations of beams rested on a nonlinear elastic foundation were solved using the Variational Iteration Method by Younesian et al. (2012). Cao and Zhong (2008) used the double Fourier transform and its inversion to solve the problem of dynamic response of an infinite beam placed on a Pasternak foundation when the system was subjected to a moving load. Oni and Awodola (2010) investigated the dynamic response under a concentrated moving mass of an elastically supported non-prismatic Bernoulli–Euler beam resting on an elastic foundation with stiffness variation. Ansari et al. (2010, 2011) studied vibration of a finite Euler–Bernoulli beam traversed by a moving load; the solution was obtained using the Galerkin method in conjunction with the Multiple Scales Method. Yang et al. (2010) presented the dynamical behavior of the vehicle–pavement–foundation coupled system using the Galerkin method and quick direct integral method. Contreras et al. (2010) described the experimental procedure followed for direct determination of dynamic modulus of asphalt mixtures by ultrasonic direct test at a specified temperature. Ding et al. (2012) introduce an investigation of the convergence of the Galerkin method for the dynamic response of a uniform beam resting on a nonlinear foundation with viscous damping subjected to a moving concentrated load.

In this paper, dynamic response of non-uniform Euler–Bernoulli simply supported beam will be obtained. The beam is subjected to moving load and rested on a nonlinear viscoelastic foundation. System parameters and magnitude of the moving load effects on the vertical deflections of non-uniform beam will be studied. Mahmoud et al. (2013) used the differential transformation method for analysis of the free vibration of beams with uniform and non-uniform cross sections.

2. Basic equations

Consider an Euler–Bernoulli beam with a non-uniform cross-section. This beam is resting on a nonlinear viscoelastic foundation and subjected to a moving load Fig. 1. The foundation is taken as a nonlinear Winkler's foundation with linear-plus-cubic stiffness and viscous damping with three parameters as follows Ding et al. (2012):

$$P=\bar{k}_1\bar{u}(\bar{x},\bar{t})+\bar{k}_3\bar{u}^3(\bar{x},\bar{t})+\bar{\mu}\frac{\partial\bar{u}(\bar{x},\bar{t})}{\partial\bar{t}} \tag{1}$$

where P is the foundation force induced per unit length of the beam, $\bar{u}(\bar{x},\bar{t})$ is displacement of the beam, $\bar{k}_1$, $\bar{k}_3$ and $\bar{\mu}$ are the linear, nonlinear and damping parameters of the foundation, respectively, $\bar{t}$ is the time.

According to Hamilton principle and the Euler–Bernoulli beam theory, the governing differential equation of motion for the non-uniform beam as

$$\rho A(\bar{x})\frac{\partial^2\bar{u}}{\partial\bar{t}^2}+\frac{\partial^2}{\partial\bar{x}^2}\left(EI(\bar{x})\frac{\partial^2\bar{u}}{\partial\bar{x}^2}\right)+\bar{k}_1\bar{u}+\bar{k}_3\bar{u}^3+\bar{\mu}\frac{\partial\bar{u}}{\partial\bar{t}}=\bar{Q}\delta(\bar{x}-\bar{v}\bar{t}) \tag{2}$$

Where ρ is the density of the beam material, $A(\bar{x})$ and $I(\bar{x})$ are the cross sectional area and the inertia of the beam at a point at distance $\bar{x}$, respectively, E is young's modulus of the beam, $\delta(\bar{x}-\bar{v}\bar{t})$ is the Dirac delta function used to deal with the moving concentrated load, $\bar{Q}$ and $\bar{v}$ are magnitude and speed of the load, respectively.

The boundary conditions of simply supported beam are:

$$\begin{aligned}\bar{u}(0,\bar{t})&=\frac{\partial^2\bar{u}(0,\bar{t})}{\partial\bar{x}^2}=0\\ \bar{u}(L,\bar{t})&=\frac{\partial^2\bar{u}(L,\bar{t})}{\partial\bar{x}^2}=0\end{aligned} \tag{3}$$

Introducing the dimensionless variables and parameters as follows:

$$u=\frac{\bar{u}}{L},\quad x=\frac{\bar{x}}{L},\quad h(x)=h_0(1-\beta x),\quad 0\le x\le 1,\quad 0\le\beta\le 1,$$
$$A(x)=A_0(1-\beta x),I(x)=I_0(1-\beta x)^3,S(x)=(1-\beta x)^3,$$
$$t=\frac{\bar{t}}{L}\sqrt{\frac{E}{\rho}}$$

Where x and L are the dimensionless spatial coordinate and the beam length, respectively, h_0, A_0 and I_0 are the height, the cross sectional area and the inertia of the beam at its beginning point at distance $x = 0$, respectively, and t is the dimensionless time. Eq. (2) can be transformed into the dimensionless equation as

$$(S(x))^{1/3}\frac{\partial^2u}{\partial t^2}+k_f^2\frac{\partial^2}{\partial x^2}\left(S(x)\frac{\partial^2u}{\partial x^2}\right)+k_1u+k_3u^3+\mu\frac{\partial u}{\partial t}=Q_Z\delta(x-vt) \tag{4}$$

where,

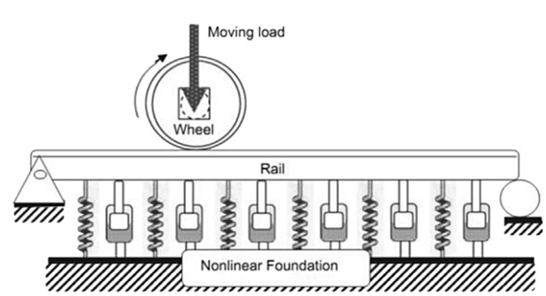

Fig. 1 – Beam on nonlinear viscoelastic foundation subjected to a moving load. Ansari et al. (2011).

$$Q_Z = \frac{\bar{Q}}{EA_0}, \upsilon = \bar{\upsilon}\sqrt{\frac{\rho}{E}}, k_f = \frac{1}{L}\sqrt{\frac{I_0}{A_0}}, k_1 = \frac{\bar{k}_1 L^2}{EA_0}, k_3 = \frac{\bar{k}_3 L^4}{EA_0}, \mu = \frac{\bar{\mu}}{A_0}\sqrt{\frac{L^2}{\rho E}}$$

3. Solution of the problem

3.1. *Galerkin approximation*

The Galerkin method is used to discretize the system and the series expansion form when $u(x,t)$ is assumed as Ding et al. (2012):

$$u(x,t) = \sum_{m=1}^{\infty} u_m(t)\varphi_m(x)$$

where $\varphi_m(x)$ are the trial functions, $u_m(t)$ are sets of generalized displacements of beam. For simple-simple at the ends, $\varphi_m(x) = sin(m\pi x)$. In this research, the first N terms of are considered in order to determine $u(x,t)$, then

$$u(x,t) = \sum_{m=1}^{N} u_m(t)\sin(m\pi x)$$

where N is Galerkin truncation term. For example, take $N = 3$, then the displacement is written as follow:

$$\begin{aligned} u(x,t) &= u_1(t)\sin(\pi x) + u_2(t)\sin(2\pi x) + u_3(t)\sin(3\pi x) \\ &= X(t)\sin(\pi x) + Y(t)\sin(2\pi x) + Z(t)\sin(3\pi x) \end{aligned} \tag{5}$$

Substituting Eq. (5) into Eq. (4), we get

$$\begin{aligned} &(S(x))^{1/3}\left(\ddot{X}(t)\sin(\pi x) + \ddot{Y}(t)\sin(2\pi x) + \ddot{Z}(t)\sin(3\pi x)\right) \\ &- k_f^2\pi^2[X(t)(S(x)\sin(\pi x))'' + 4Y(t)(S(x)\sin(2\pi x))'' \\ &+ 9Z(t)(S(x)\sin(3\pi x))''] + k_1[X(t)\sin(\pi x) + Y(t)\sin(2\pi x) \\ &+ Z(t)\sin(3\pi x)] + k_3[X(t)\sin(\pi x) + Y(t)\sin(2\pi x) + Z(t)\sin(3\pi x)]^3 \\ &+ \mu\left(\dot{X}(t)\sin(\pi x) + \dot{Y}(t)\sin(2\pi x) + \dot{Z}(t)\sin(3\pi x)\right) = Q_Z\delta(x - \upsilon t) \end{aligned} \tag{6}$$

Multiplying both sides of Eq. (6) by$sin(\pi x)$, $sin(2\pi x)$ and $sin(3\pi x)$ and integrating w.r.t x from 0 to 1, we obtain the following system of three simultaneously initial value problems:

$$\begin{aligned} &\beta_1\ddot{X} + \beta_2\ddot{Y} - k_f^2\pi^2(\beta_3 X + 4\beta_4 Y + 9\beta_{16} Z) + \frac{k_1}{2}X + \frac{3}{8}k_3 \\ &(X^3 - X^2Z + 2Y^2X + Y^2Z + 2Z^2X) + \frac{\mu}{2}\dot{X} = Q_Z\sin(\pi\upsilon t) \end{aligned} \tag{7-a}$$

$$\begin{aligned} &\beta_2\ddot{X} + \beta_1\ddot{Y} + \beta_{24}\ddot{Z} - k_f^2\pi^2(\beta_6 X + 4\beta_5 Y + 9\beta_{25} Z) + \frac{k_1}{2}Y \\ &+ \frac{3}{8}k_3(Y^3 + 2X^2Y + 2Z^2Y + 2XYZ) + \frac{\mu}{2}\dot{Y} = Q_Z\sin(2\pi\upsilon t) \end{aligned} \tag{7-b}$$

$$\begin{aligned} &\beta_1\ddot{Z} + \beta_{24}\ddot{Y} - k_f^2\pi^2(\beta_{36} X + 4\beta_{37} Y + 9\beta_{38} Z) + \frac{k_1}{2}Z + \frac{3}{8}k_3 \\ &\left(Z^3 - \frac{1}{3}X^3 + 2X^2Z + XY^2 + 2Y^2Z + 3Z^2Y\right) + \frac{\mu}{2}\dot{Z} = Q_Z\sin(3\pi\upsilon t) \end{aligned} \tag{7-c}$$

Solving Eq. (7) algebraically in $\ddot{X}$, $\ddot{Y}$ and $\ddot{Z}$, we obtain the following forms:

$$\begin{aligned} \ddot{X} = &\gamma_1 X + \gamma_2 Y + \gamma_3 Z + \gamma_4 X^3 + \gamma_5 Y^3 + \gamma_6 Z^3 + \gamma_7 X^2 Z + \gamma_8 X^2 Y \\ &+ \gamma_9 Y^2 X + \gamma_{10} Y^2 Z + \gamma_{11} Z^2 X + \gamma_{12} Z^2 Y + \gamma_{13} XYZ + \gamma_{141}\dot{X} \\ &+ \gamma_{142}\dot{Y} + \gamma_{143}\dot{Z} + \gamma_{15}\sin(\pi\upsilon t) + \gamma_{16}\sin(2\pi\upsilon t) \\ &+ \gamma_{17}\sin(3\pi\upsilon t) \end{aligned} \tag{8-a}$$

$$\begin{aligned} \ddot{Y} = &\gamma_{18} X + \gamma_{19} Y + \gamma_{20} Z + \gamma_{21} X^3 + \gamma_{22} Y^3 + \gamma_{23} Z^3 + \gamma_{24} X^2 Z \\ &+ \gamma_{25} X^2 Y + \gamma_{26} Y^2 X + \gamma_{27} Y^2 Z + \gamma_{28} Z^2 X + \gamma_{29} Z^2 Y + \gamma_{30} XYZ \\ &+ \gamma_{311}\dot{X} + \gamma_{312}\dot{Y} + \gamma_{313}\dot{Z} + \gamma_{32}\sin(\pi\upsilon t) + \gamma_{33}\sin(2\pi\upsilon t) \\ &+ \gamma_{34}\sin(3\pi\upsilon t) \end{aligned} \tag{8-b}$$

$$\begin{aligned} \ddot{Z} = &\gamma_{35} X + \gamma_{36} Y + \gamma_{37} Z + \gamma_{38} X^3 + \gamma_{39} Y^3 + \gamma_{40} Z^3 + \gamma_{41} X^2 Z \\ &+ \gamma_{42} X^2 Y + \gamma_{43} Y^2 X + \gamma_{44} Y^2 Z + \gamma_{45} Z^2 X + \gamma_{46} Z^2 Y + \gamma_{47} XYZ \\ &+ \gamma_{481}\dot{X} + \gamma_{482}\dot{Y} + \gamma_{483}\dot{Z} + \gamma_{49}\sin(\pi\upsilon t) + \gamma_{50}\sin(2\pi\upsilon t) \\ &+ \gamma_{51}\sin(3\pi\upsilon t) \end{aligned} \tag{8-c}$$

With initial conditions as:

$$\begin{aligned} X(0) &= \dot{X}(0) = 0 \\ Y(0) &= \dot{Y}(0) = 0 \\ Z(0) &= \dot{Z}(0) = 0 \end{aligned} \tag{9}$$

where, β_i and γ_i are resulted from definite integrals which are applied on Eq. (7) and they are functions of β. See Appendix [A, B].

3.2. *Runge-Kutta method*

One of the standard workhorses for solving ordinary differential equations (ODEs) is the Runge-Kutta method. Runge-Kutta method takes four steps, shooting across one quarter of the interval, estimating the derivative, then shooting to the midpoint, and so on. The precise manner in which the method propagates across a time step is done in the optimal way for the four steps. We will not provide a formal derivation of the Runge-Kutta algorithm; instead we will present the method and implement it. The general system of ODEs can be written as,

$$\begin{aligned} &\frac{\partial y}{\partial t} = f(y,t) \\ &k_1 = \Delta t\ f(t^N, y^N) \quad k_2 = \Delta t\ f(t^N + \Delta t/2, y^N + k_1/2) \\ &k_3 = \Delta t\ f(t^N + \Delta t/2, y^N + k_2/2) \quad k_4 = \Delta t\ f(t^N + \Delta t, y^N + k_3) \\ &y^{N+1} = y^N + \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6} \end{aligned}$$

The system of ODEs (8) and (9) are solved numerically by Runge-Kutta of fourth order to compute $X(t)$,$Y(t)$ and$Z(t)$. Thus the displacement of the beam is obtained at any time t and coordinate x.

4. Results and discussion

To discuss the accuracy and effects of system parameters, we will take a practical type of beam with foundation, represents the railway track on stiff soil foundation, running the train with the moving vehicle. The beam is assumed to be the UIC60 European high-speed rail (Contreras et al., 2010; Ding et al., 2012; Yang et al., 2010). The properties of the track, foundation and the moving load are listed in Table 1.

Table 1 – Properties of beam, foundation and load.

Item	Notation	Value	Dimensionless value
Asphalt mixture (UIC60)			
Young's modulus (steel)	E	210 GPa	–
Mass density	ρ	7850 kg/m^3	–
Cross-sectional area	A	7.69×10^{-3} m^2	–
Second moment of area	I	3.055×10^{-5} m^4	–
Modulus of elasticity	k_f	–	3.501×10^{-3}
Length	L	18 m	1
Foundation			
Mean stiffness	$\bar{k}_1$	3.5×10^7 N/m^2	7.0221
Nonlinear stiffness	$\bar{k}_3$	4×10^{14} N/m^4	2.6×10^{10}
Viscous damping	$\bar{\mu}$	1732.5×10^3 N S/m^2	99.879
Moving load			
Load	$\bar{Q}$	65 kN	4.025×10^{-5}
Speed	$\bar{v}$	10 m/s	0.001933

Fig. 2 shows the effects of the Galerkin truncation terms N on the vertical displacements of the beam's midpoint $u(1/2,1/2v)$. It shows the results of a few different numbers of Galerkin terms N as 4, 10, 20 and 30 terms. The results indicate that 4-term Galerkin method is not accurate enough for the dynamical response analysis of the railway track on stiff soil foundation running the train, and the 20-term Galerkin method yields rather accurate results. So,

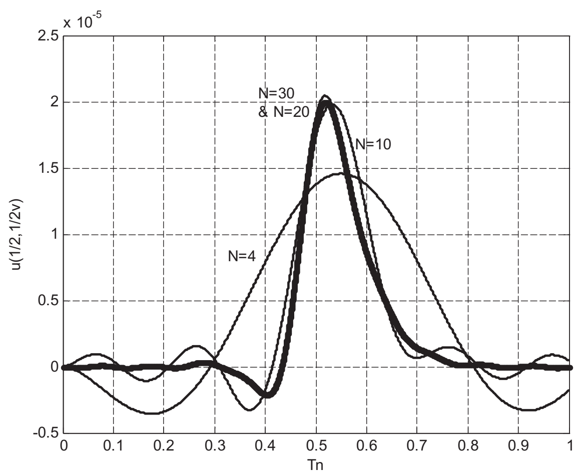

Fig. 2 – Effects of the Galerkin truncation terms N on the vertical displacements of the beam's midpoint.

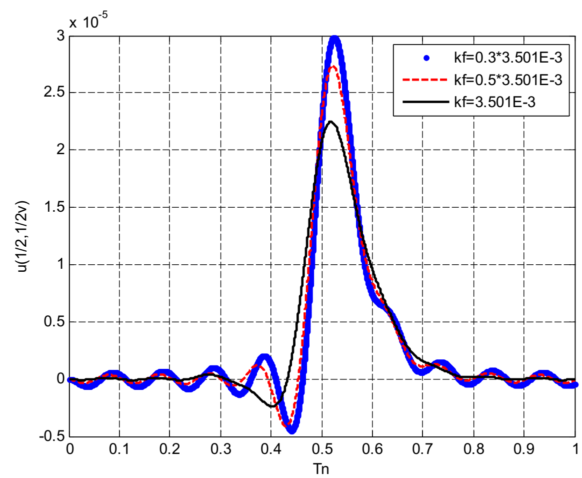

Fig. 4 – Effects of the dimensionless modulus of elasticity k_f on the vertical displacements of the beam's midpoint.

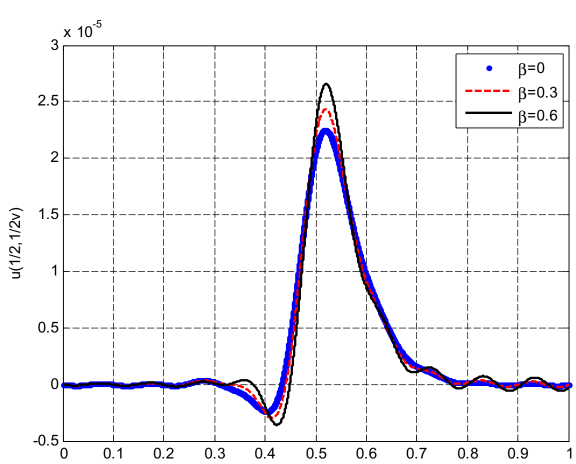

Fig. 3 – Effects of variability of beam's cross section β on the vertical displacements of the beam's midpoint.

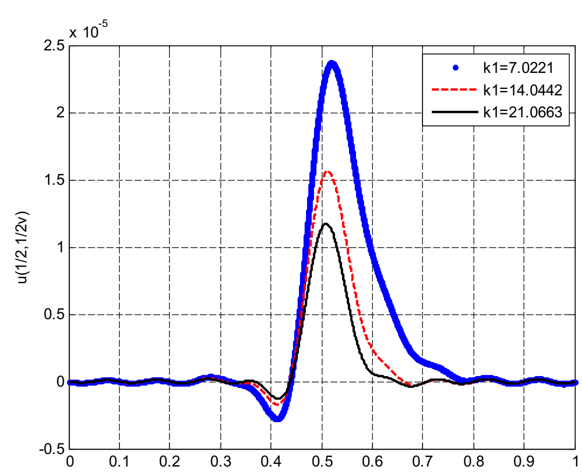

Fig. 5 – Effects of the dimensionless linear foundation parameters k_1 on the vertical displacements of the beam's midpoint.

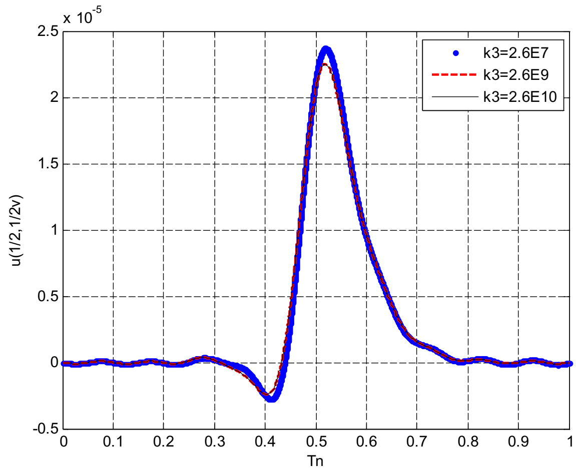

Fig. 6 – Effects of the dimensionless nonlinear foundation parameters k_3 on the vertical displacements of the beam's midpoint.

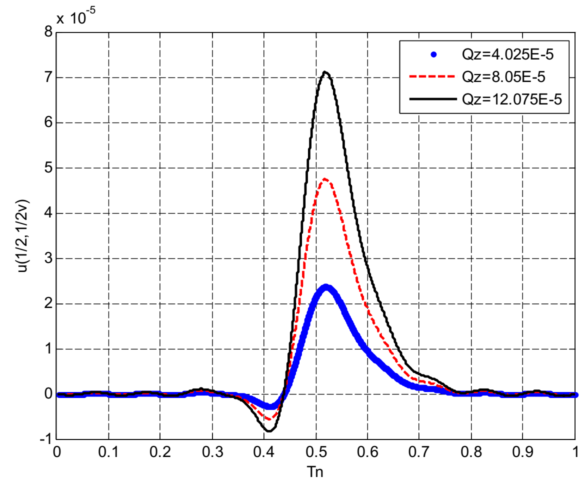

Fig. 8 – Effects of the moving load value or its velocity Q_Z on the vertical displacements of the beam's midpoint.

the beam's deflections at Figs. (3–9) *are drawn for 20 Galerkin terms*. Fig. 3 shows the effects of the variability of beam's cross section β on the vertical displacements of the beam's midpoint. Different values of β *are taken as* 0, 0.3 *and* 0.6. It can be concluded that here the increasing of the beam deflection according to cross sections decreasing is slower. Effect of the dimensionless modulus of elasticity k_f on the vertical displacements of the beam's midpoint is shown at Fig. 4. The figure shows that the deflection of the beam increases as its dimensions increases. The numerical results at Figs. (5–7) show that the vertical displacements decrease with growing of the linear and nonlinear foundation parameters k_1 and k_3, and the damping coefficient μ, respectively. The train with the moving vehicle has effects on the vertical displacements of the railway track. Figs. 8 and 9 show that the beam's dynamic deflections increase when the moving load value increases, but it decrease with increasing of the load velocity.

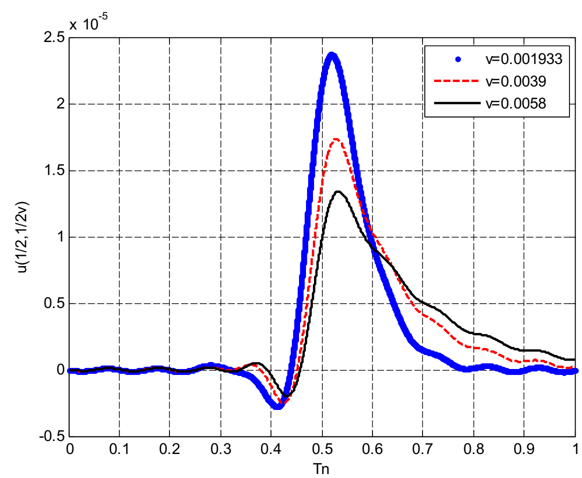

Fig. 9 – Effects of the moving load velocity v on the vertical displacements of the beam's midpoint.

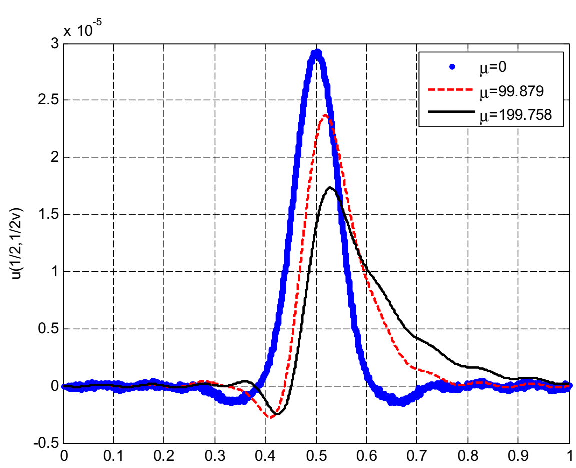

Fig. 7 – Effects of the dimensionless damping coefficients of the foundation μ on the vertical displacements of the beam's midpoint.

5. Conclusion

According to the presented results, it can be demonstrated that Galerkin with Runge-Kutta methods are efficient in calculating the dynamic response of non-uniform Euler–Bernoulli simply supported beam which is subjected to moving load and rested on a nonlinear viscoelastic foundation. Also, it clear that increasing of beam's dimensions causes an increasing of its dynamic deflection of the non-uniform beam. Foundation parameters have a significant effect on the beam deflection, while, increasing of foundation linear, nonlinear and damping parameters causes decreasing of the beam dynamic deflection. Moving load and its velocity have a clear effect on the beam dynamic deflection. It is also possible to extend this method to the use for beam with other cross section shapes and other boundary conditions.

Appendix A:

$$\beta_1=\int_0^1(1-\beta x)\sin^2(\pi x)dx, \qquad \beta_2=\int_0^1(1-\beta x)\sin(2\pi x)\sin(\pi x)dx,$$

$$\beta_3=\int_0^1\left[\left((1-\beta x)^3\sin(\pi x)\right)''\right]\sin(\pi x)dx, \qquad \beta_4=\int_0^1\left[\left((1-\beta x)^3\sin(2\pi x)\right)''\right]\sin(\pi x)dx,$$

$$\beta_5=\int_0^1\left[\left((1-\beta x)^3\sin(2\pi x)\right)''\right]\sin(2\pi x)dx, \qquad \beta_6=\int_0^1\left[\left((1-\beta x)^3\sin(\pi x)\right)''\right]\sin(2\pi x)dx,$$

$$\beta_{16}=\int_0^1\left[\left((1-\beta x)^3\sin(3\pi x)\right)''\right]\sin(\pi x)dx, \qquad \beta_{24}=\int_0^1(1-\beta x)\sin(2\pi x)\sin(3\pi x)dx,$$

$$\beta_{25}=\int_0^1\left[\left((1-\beta x)^3\sin(3\pi x)\right)''\right]\sin(2\pi x)dx, \qquad \beta_{36}=\int_0^1\left[\left((1-\beta x)^3\sin(\pi x)\right)''\right]\sin(3\pi x)dx,$$

$$\beta_{37}=\int_0^1\left[\left((1-\beta x)^3\sin(2\pi x)\right)''\right]\sin(3\pi x)dx \text{ and } \beta_{38}=\int_0^1\left[\left((1-\beta x)^3\sin(3\pi x)\right)''\right]\sin(3\pi x)dx$$

Appendix B:

$$\gamma_1=\frac{\left(-k_f^2\pi^2\left(\beta_2\beta_{24}\beta_{36}/\beta_1+\beta_2^2\beta_3/\beta_1-\beta_2\beta_6\right)\right)+\dfrac{\beta_2^2k_1}{2\beta_1}}{\left(\beta_2^2-\beta_1^2+\beta_{24}^2\right)}-\frac{\left\{k_f^2\pi^2\beta_3+\dfrac{k_1}{2}\right\}}{\beta_1},$$

$$\gamma_2=\frac{\left(-4k_f^2\pi^2\left(\beta_2\beta_{24}\beta_{37}/\beta_1+\beta_2^2\beta_4/\beta_1-\beta_2\beta_5\right)\right)-\dfrac{\beta_2k_1}{2}}{\left(\beta_2^2-\beta_1^2+\beta_{24}^2\right)}+\frac{4k_f^2\pi^2\beta_4}{\beta_1},$$

$$\gamma_3=\frac{\left(-9k_f^2\pi^2\left(\beta_2\beta_{24}\beta_{38}/\beta_1+\beta_2^2\beta_{16}/\beta_1-\beta_2\beta_{25}\right)\right)+\dfrac{\beta_2\beta_{24}k_1}{2\beta_1}}{\left(\beta_2^2-\beta_1^2+\beta_{24}^2\right)}+\frac{9k_f^2\pi^2\beta_{16}}{\beta_1},$$

$$\gamma_4=\frac{(3k_3/8)\left(\beta_2^2/\beta_1-\beta_2\beta_{24}/3\beta_1\right)}{\left(\beta_2^2-\beta_1^2+\beta_{24}^2\right)}-\frac{3k_3}{8\beta_1},$$

$$\gamma_5=\frac{(-3k_3\beta_2/8)}{\left(\beta_2^2-\beta_1^2+\beta_{24}^2\right)}, \quad \gamma_6=\frac{(3k_3/8)\left(\beta_2\beta_{24}/\beta_1\right)}{\left(\beta_2^2-\beta_1^2+\beta_{24}^2\right)},$$

$$\gamma_7=\frac{(3k_3/8)\left(2\beta_2\beta_{24}/\beta_1-\beta_2^2/\beta_1\right)}{\left(\beta_2^2-\beta_1^2+\beta_{24}^2\right)}+\frac{3k_3}{8\beta_1}$$

$$\gamma_8=\frac{(-6k_3\beta_2/8)}{\left(\beta_2^2-\beta_1^2+\beta_{24}^2\right)}, \qquad \gamma_9=\frac{(3k_3/8)\left(\beta_2\beta_{24}/\beta_1+2\beta_2^2/\beta_1\right)}{\left(\beta_2^2-\beta_1^2+\beta_{24}^2\right)}-\frac{6k_3}{8\beta_1},$$

$$\gamma_{10}=\frac{(3k_3/8)\left(2\beta_2\beta_{24}/\beta_1+\beta_2^2/\beta_1\right)}{\left(\beta_2^2-\beta_1^2+\beta_{24}^2\right)}-\frac{3k_3}{8\beta_1}, \qquad \gamma_{11}=\frac{(3k_3/8)\left(2\beta_2^2/\beta_1\right)}{\left(\beta_2^2-\beta_1^2+\beta_{24}^2\right)}-\frac{6k_3}{8\beta_1},$$

$$\gamma_{12}=\frac{(3k_3/8)\left(3\beta_2\beta_{24}/\beta_1-2\beta_2\right)}{\left(\beta_2^2-\beta_1^2+\beta_{24}^2\right)}, \qquad \gamma_{13}=\frac{(-6k_3\beta_2/8)}{\left(\beta_2^2-\beta_1^2+\beta_{24}^2\right)},$$

$$\gamma_{141}=\frac{\left(\beta_1-\beta_{24}^2/\beta_1\right)}{\left(\beta_2^2-\beta_1^2+\beta_{24}^2\right)}\frac{\mu}{2}, \qquad \gamma_{142}=\frac{-\beta_2}{\left(\beta_2^2-\beta_1^2+\beta_{24}^2\right)}\frac{\mu}{2},$$

$$\gamma_{143}=\frac{\beta_2\beta_{24}/\beta_1}{\left(\beta_2^2-\beta_1^2+\beta_{24}^2\right)}\frac{\mu}{2}, \qquad \gamma_{15}=\frac{-\beta_2^2Q_Z}{\beta_1\left(\beta_2^2-\beta_1^2+\beta_{24}^2\right)}+\frac{Q_Z}{\beta_1},$$

$$\gamma_{16}=\frac{\beta_2Q_Z}{\left(\beta_2^2-\beta_1^2+\beta_{24}^2\right)}, \qquad \gamma_{17}=\frac{-\beta_2\beta_{24}Q_Z}{\beta_1\left(\beta_2^2-\beta_1^2+\beta_{24}^2\right)},$$

$$\gamma_{18} = \frac{\left(-k_f^2\pi^2(\beta_{24}\beta_{36} + \beta_2\beta_3 - \beta_1\beta_6)\right) + \dfrac{\beta_2 k_1}{2}}{-(\beta_2^2 - \beta_1^2 + \beta_{24}^2)},$$

$$\gamma_{19} = \frac{\left(-4k_f^2\pi^2(\beta_{24}\beta_{37} + \beta_2\beta_4 - \beta_1\beta_5)\right) - \dfrac{\beta_1 k_1}{2}}{-(\beta_2^2 - \beta_1^2 + \beta_{24}^2)},$$

$$\gamma_{20} = \frac{\left(-9k_f^2\pi^2(\beta_{24}\beta_{38} + \beta_2\beta_{16} - \beta_1\beta_{25})\right) + \dfrac{\beta_{24} k_1}{2}}{-(\beta_2^2\beta_1^2 + \beta_{24}^2)},$$

$$\gamma_{21} = \frac{-(3k_3/8)(\beta_2 - \beta_{24}/3)}{(\beta_2^2 - \beta_1^2 + \beta_{24}^2)}, \quad \gamma_{22} = \frac{(3k_3\beta_1/8)}{(\beta_2^2 - \beta_1^2 + \beta_{24}^2)},$$

$$\gamma_{23} = \frac{-(3k_3\beta_{24}/8)}{(\beta_2^2 + \beta_1^2 + \beta_{24}^2)}, \quad \gamma_{24} = \frac{-(3k_3/8)(2\beta_{24} - \beta_2)}{(\beta_2^2 - \beta_1^2 + \beta_{24}^2)},$$

$$\gamma_{25} = \frac{(6k_3\beta_1/8)}{(\beta_2^2 - \beta_1^2 + \beta_{24}^2)}, \quad \gamma_{26} = \frac{-(3k_3/8)(\beta_{24} - 2\beta_2)}{(\beta_2^2 - \beta_1^2 + \beta_{24}^2)},$$

$$\gamma_{27} = \frac{-(3k_3/8)(2\beta_{24} + \beta_2)}{(\beta_2^2 - \beta_1^2 + \beta_{24}^2)}, \quad \gamma_{28} = \frac{-(3k_3/8)(2\beta_2)}{(\beta_2^2 - \beta_1^2 + \beta_{24}^2)},$$

$$\gamma_{29} = \frac{-(3k_3/8)(3\beta_{24} - 2\beta_1)}{(\beta_2^2 - \beta_2^2 + \beta_{24}^2)}, \quad \gamma_{30} = \frac{(6k_3\beta_1/8)}{(\beta_2^2 - \beta_1^2 + \beta_{24}^2)},$$

$$\gamma_{311} = \frac{-\beta_2}{(\beta_2^2 - \beta_1^2 + \beta_{24}^2)}\frac{\mu}{2}, \quad \gamma_{312} = \frac{\beta_1}{(\beta_2^2 - \beta_1^2 + \beta_{24}^2)}\frac{\mu}{2},$$

$$\gamma_{313} = \frac{-\beta_{24}}{(\beta_2^2 - \beta_1^2 + \beta_{24}^2)}\frac{\mu}{2}, \quad \gamma_{32} = \frac{\beta_2 Qz}{(\beta_2^2 - \beta_1^2 + \beta_{24}^2)},$$

$$\gamma_{33} = \frac{-\beta_1 Qz}{(\beta_2^2 - \beta_1^2 + \beta_{24}^2)}, \quad \gamma_{34} = \frac{\beta_{24} Qz}{(\beta_2^2 - \beta_1^2 + \beta_{24}^2)},$$

$$\gamma_{35} = \frac{\left(-k_f^2\pi^2(\beta_{24}^2\beta_{36}/\beta_1 + \beta_2\beta_{24}\beta_3/\beta_1 - \beta_{24}\beta_6)\right) + \dfrac{\beta_2\beta_{24}k_1}{2\beta_1}}{(\beta_2^2 - \beta_1^2 + \beta_{24}^2)} + \frac{k_f^2\pi^2\beta_{36}}{\beta_1},$$

$$\gamma_{36} = \frac{\left(-4k_f^2\pi^2(\beta_{24}^2\beta_{37}/\beta_1 + \beta_2\beta_{24}\beta_4/\beta_1 - \beta_{24}\beta_5)\right) + \dfrac{\beta_{24}k_1}{2}}{(\beta_2^2 - \beta_1^2 + \beta_{24}^2)} + \frac{4k_f^2\pi^2\beta_{37}}{\beta_1},$$

$$\gamma_{37} = \frac{\left(-9k_f^2\pi^2(\beta_{24}^2\beta_{38}/\beta_1 + \beta_2\beta_{24}\beta_{16}/\beta_1 - \beta_{24}\beta_{25})\right) + \dfrac{\beta_1 k_1}{2} - \dfrac{\beta_2^2 k_1}{2\beta_1}}{(\beta_2^2 - \beta_1^2 + \beta_{24}^2)} + \frac{9k_f^2\pi^2\beta_{38}}{\beta_1},$$

$$\gamma_{38} = \frac{(3k_3/8)(\beta_2\beta_{24}/\beta_1 - \beta_{24}^2/3\beta_1)}{(\beta_2^2 - \beta_1^2 + \beta_{24}^2)} + \frac{k_3}{8\beta_1}, \quad \gamma_{39} = \frac{(-3k_3\beta_{24}/8)}{(\beta_2^2 - \beta_1^2 + \beta_{24}^2)},$$

$$\gamma_{40} = \frac{(3k_3/8)(\beta_1^2 - \beta_2^2)}{(\beta_2^2 - \beta_1^2 + \beta_{24}^2)}, \quad \gamma_{41} = \frac{(3k_3/8)(2\beta_{24}^2/\beta_1 - \beta_2\beta_{24}/\beta_1)}{(\beta_2^2 - \beta_1^2 + \beta_{24}^2)} - \frac{6k_3}{8\beta_1},$$

$$\gamma_{42} = \frac{-(6k_3\beta_{24}/8)}{(\beta_2^2 - \beta_1^2 + \beta_{24}^2)}, \quad \gamma_{43} = \frac{(3k_3/8)(\beta_{24}^2 + 2\beta_2\beta_{24}/\beta_1)}{(\beta_2^2 - \beta_1^2 + \beta_{24}^2)} - \frac{3k_3}{8\beta_1},$$

$$\gamma_{44} = \frac{(3k_3/8)(2\beta_{24}^2/\beta_1 + \beta_2\beta_{24}/\beta_1)}{(\beta_2^2 - \beta_1^2 + \beta_{24}^2)} - \frac{6k_3}{8\beta_1}, \quad \gamma_{45} = \frac{(3k_3/8)(2\beta_2\beta_{24}/\beta_1)}{(\beta_2^2 - \beta_1^2 + \beta_{24}^2)},$$

$$\gamma_{46} = \frac{(3k_3/8)(3\beta_{24}^2/\beta_1 - 2\beta_{24})}{(\beta_2^2 - \beta_1^2 + \beta_{24}^2)} - \frac{9k_3}{8\beta_1}, \quad \gamma_{47} = \frac{-(6k_3\beta_{24}/8)}{(\beta_2^2 - \beta_1^2 + \beta_{24}^2)},$$

$$\gamma_{481} = \frac{\beta_2\beta_{24}/\beta_1}{(\beta_2^2 - \beta_1^2 + \beta_{24}^2)}\frac{\mu}{2}, \quad \gamma_{482} = \frac{-\beta_{24}}{(\beta_2^2 - \beta_1^2 + \beta_{24}^2)}\frac{\mu}{2},$$

$$\gamma_{483} = \frac{(\beta_2 - \beta_{24}^2/\beta_1)}{(\beta_2^2 - \beta_1^2 + \beta_{24}^2)}\frac{\mu}{2}, \quad \gamma_{49} = \frac{-(\beta_2\beta_{24}/\beta_1)Qz}{(\beta_2^2 - \beta_1^2 + \beta_{24}^2)},$$

$$\gamma_{50} = \frac{\beta_{24}Qz}{(\beta_2^2 - \beta_1^2 + \beta_{24}^2)} \text{ and } \quad \gamma_{51} = \frac{-(\beta_{24}^2/\beta_1)Qz}{(\beta_2^2 - \beta_1^2 + \beta_{24}^2)} + \frac{Qz}{\beta_1}$$

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