Periodic steady state response of large scale mechanical models with local nonlinearities

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Long term dynamics of a class of mechanical systems is investigated in a computationally efficient way. Due to geometric complexity, each structural component is first discretized by applying the finite element method. Frequently, this leads to models with a quite large number of degrees of freedom. In addition, the composite system may also possess nonlinear properties. The method applied overcomes these difficulties by imposing a multi-level substructuring procedure, based on the sparsity pattern of the stiffness matrix. This is necessary, since the number of the resulting equations of motion can be so high that the classical coordinate reduction methods become inefficient to apply. As a result, the original dimension of the complete system is substantially reduced. Subsequently, this allows the application of numerical methods which are efficient for predicting response of small scale systems. In particular, a systematic method is applied next, leading to direct determination of periodic steady state response of nonlinear models subjected to periodic excitation. An appropriate continuation scheme is also applied, leading to evaluation of complete branches of periodic solutions. In addition, the stability properties of the located motions are also determined. Finally, representative sets of numerical results are presented for an internal combustion car engine and a complete city bus model. Where possible, the accuracy and validity of the applied methodology is verified by comparison with results obtained for the original models. Moreover, emphasis is placed in comparing results obtained by employing the nonlinear or the corresponding linearized models.

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1. Introduction

There are many occasions in engineering practice, where there is a strong interest in determining and studying the long term behavior of a structural or mechanical system, which is subjected to periodic excitation. In such cases, the main interest lies mostly in locating and investigating periodic steady state response. A typical example is the production of response spectra, which are used in order to detect structural or acoustical resonances (Craig, 1981; Kropp and Heiserer, 2003) or to predict fatigue failure of critical structural parts (Lutes and Sarkani, 1997; Benasciutti and Tavo, 2005). When the dynamical system resulting after the modeling possesses linear characteristics, there are standard methods that can lead to this information (Rao, 1990). In cases where the dimension of the models considered becomes excessive, due to the many geometrical details that need to be included in order to cover critical design needs, the volume of the calculations can become prohibitive quite frequently. In such cases, application of appropriate dimension reduction methods in either the time or the frequency domain becomes necessary (Craig, 1981; Cuppens et al., 2000).

However, more difficulties arise when nonlinearities are present. In particular, for small scale nonlinear systems subjected to periodic external excitation there is a lot of analytical and numerical work referring to their long term response (Doedel, 1986; Nayfeh and Balachandran, 1995). Among other things, it is well known by now that these systems can exhibit many types of periodic motion as well as more complex behavior, including quasi-periodic and chaotic response. On the other hand, little is still available on capturing long term dynamics for large scale nonlinear systems (Fey et al., 1996; Verros and Natsiavas, 2002).

The main objective of the present work is to develop and apply a systematic methodology leading to a direct determination of periodic steady state response of periodically excited complex mechanical systems. Here, the term complex refers to two characteristic properties of the class of systems examined. The first level of complexity is related to the large number of the corresponding equations of motion. In fact, the detailed geometrical discretization of some of the structural substructures, based on the application of the finite element method (Zienkiewicz, 1986), leads to a large number of equations of motion. As a consequence, in many cases it may not be feasible even to carry over the evaluation of the dynamic quantities needed for the transformations leading to the classical dimension reduction methods (Fey et al., 1996; Chen...
et al., 1998). The second level of complexity is related to the nonlinearities associated with the system response. This poses severe restrictions in the applicability of most of the available and commonly employed methods. On the positive side, a good feature of the class of systems examined is that the nonlinear characteristics are associated with a relatively small number of degrees of freedom of the class of dynamical systems examined.

Based on the localized nature of the nonlinear action, the basic idea of this work is to first reduce the dimension of all the linear components of the structural system examined by applying an appropriate coordinate transformation. In particular, the reduction method applied is based on an automatic multi-level substructuring (Bennighof et al., 2000; Bennighof and Lehoucq, 2004). Apart from increasing the computational efficiency and speed, the reduction of the system dimensions makes amenable the subsequent application of numerical techniques for determining the dynamic response of complex systems, which are applicable and efficient for low order systems. For instance, this method has already been applied successfully to the solution of the real eigenproblem and the prediction of periodic response of large scale linear models with nonclassical damping (Kim and Bennighof, 2006; Papalukopoulos and Natsiavas, 2007), large order gyroscopic systems (Essel and Voss, 2006) and broadband vibro-acoustic simulations of vehicle models (Kropp and Heiserer, 2003). In addition, the same method has also facilitated determination of the transient response of large scale nonlinear models (Papalukopoulos and Natsiavas, 2007; Theodosiou and Natsiavas, 2009).

In the present work, the same multi-level substructuring method is coupled with an appropriate numerical procedure in order to determine periodic steady state motions of the models examined, resulting in response to periodic excitation in a direct manner (Doedel, 1986). This coupling takes into account and exploits the characteristics of the general class of mechanical systems considered. Moreover, a suitable method is also applied, based on classical Floquet theory and leading to determination of the stability properties of the located periodic motions (Nayfeh and Balachandran, 1995). Finally, the methodology developed is complemented by a continuation method, yielding complete branches of periodic motions over a specified frequency interval (Ragon et al., 2002). The final outcome is expected to provide valuable information and insight to engineers dealing with the analysis and design of complex mechanical systems.

The validity and effectiveness of the methodology developed is illustrated and verified by presenting a selected set of numerical results. More specifically, some typical results are presented first for a detailed finite element model of a crankshaft, belonging to an internal combustion engine of a commercial car. Then, response spectra for a quite involved city bus model are also determined and presented. The results obtained are useful in assessing the dynamic response of the mechanical systems examined. In both cases, particular emphasis is placed on identifying and evaluating effects caused by the nonlinear action in the dynamics, in comparison with similar predictions of the linear theory.

The organisation of this paper is as follows. First, the characteristics of the class of mechanical models examined are presented in the following section. Then, the basic steps of the methodology developed, including both the application of the coordinate reduction and the direct determination of the periodic steady state response, are summarised in the third and fourth section, respectively. Next, the dynamic response of two example models subjected to periodic external excitation is investigated. Where possible, the accuracy and effectiveness of the present methodology is established by comparison of results obtained for the reduced and the corresponding complete dynamical models. The work is completed by summarizing the highlights in the last section.

2. Class of mechanical models examined

Accurate prediction of the dynamic response of mechanical systems requires frequently the development and examination of geometrically detailed dynamical models. On the one hand, this leads to a quite large set of equations of motion. The situation becomes more complicated when the systems are forced to operate in conditions involving activation of nonlinear characteristics. On the other hand, the information extracted from the systematic prediction of the dynamic response is essential and valuable in performing efficiently other useful studies, as well, related to fatigue, acoustics, identification, optimization and control of a system.

Typically, a complex mechanical system is composed of several structural components. Based on strict design requirements on accuracy, these components are usually discretized geometrically by a relatively large number of finite elements (Bathe, 1982; Hughes, 1987). In many cases, this gives rise to a dynamical system with an excessive number of degrees of freedom. For all practical purposes, the model of each mechanical substructure can be assumed to possess linear properties. However, the elements connecting the system substructures are typically characterized by strong nonlinear action. Taking all the above into account, the equations of motion of the general class of dynamical systems considered in this work can be put in the compact matrix form

\[
M \ddot{\mathbf{x}} + C \dot{\mathbf{x}} + K \mathbf{x} + \mathbf{p}(\mathbf{x}, \dot{\mathbf{x}}) = \mathbf{f}(t),
\]  

where all the unknown coordinates are included in the vector \( \mathbf{x}(t) = (x_1, x_2, \ldots, x_n)^T \), while \( M, C \) and \( K \) are the mass, damping and stiffness matrix of the system, respectively. These quantities include contribution from all the structural components of the system. Moreover, the elements of vector \( \mathbf{p}(\mathbf{x}, \dot{\mathbf{x}}) \) include the contribution of the nonlinear terms arising from the action of the coupled dynamical system, while vector \( \mathbf{f}(t) \) represents the action arising from the external forcing.

Besides the large number of degrees of freedom, the level of difficulty in determining the dynamic response of the class of systems examined increases considerably when nonlinearity effects become important. However, an attractive feature of these systems is that their nonlinearities usually appear mainly at a relatively small number of places, involving a small portion of the degrees of freedom. This makes possible the application of special techniques, which are appropriate for systems with local nonlinearities. Namely, for such systems it is possible to reduce significantly the number of the original degrees of freedom by applying suitable coordinate reduction methods (Craig, 1981; Fey et al., 1996; Verros and Natsiavas, 2002). Apart from increasing the computational efficiency and speed, the reduction of the system dimensions makes amenable the application of several numerical techniques, which are applicable and efficient for low order dynamical systems.

The dimension of the class of systems examined in the present work can be so high, that ordinary coordinate reduction methods may not be numerically efficient to apply. For this reason, a special coordinate reduction method is applied instead (Bennighof et al., 2000; Bennighof and Lehoucq, 2004), whose basic steps are presented in the following section. This reduction leads to a substantial acceleration of the subsequent calculations. More specifically, the emphasis of the present study is placed on locating periodic steady state motions, when the mechanical models examined are subjected to periodic external excitation.

In general, the long term response of a nonlinear dynamical system to periodic excitation can be either regular (periodic or quasi-periodic) or irregular (chaotic) (Nayfeh and Balachandran, 1995). Typically, such motions are determined by a direct integration of the equations of motion, starting from some selected set of initial
conditions (Craig, 1981; Hughes, 1987). However, application of such a method can possibly lead to determination of stable periodic solutions only. Difficulties arise when unstable periodic motions or many, periodic or nonperiodic, motions coexist for the same set of technical parameters. As a consequence, this brute force method cannot create a global picture of the system resonances and dynamics. In the present study, the emphasis is placed on developing a systematic method leading to a direct determination of complete branches of periodic steady state motion, including their stability properties. More details on this method are presented in the fourth section.

3. Substructuring method

Besides the classical methods, which are available and have been employed successfully in the past for reducing the original number of degrees of freedom of a complex mechanical system, a new class of coordinate reduction methods has also been developed recently, which presents certain computational advantages (Kropp and Heiser, 2003; Kim and Bennighof, 2006; Elssel and Voss, 2006; Papalukopoulos and Natsiavas, 2007). The basic steps of these methods are briefly illustrated in this section.

First, the damping and nonlinear forces are neglected temporarily from the equations of motion. Then, taking into account the sparsity pattern of the stiffness matrix, the original set of equations of motion of the system examined is reordered and split into a number of mathematical substructures (Karypis and Kumar, 1995). As a result, the equations of motion for the ith substructure alone appear in the following linear form:

$$M_i \ddot{q}_i + K_g q_i = f_i(t)$$

where the columns of submatrix $M_i$ are determined by solving the eigenvalue problem

$$K_i \Phi_i = M_i \Psi_i A_i,$$

and include the modes corresponding to the lowest natural frequencies of the component up to a specified level. The squares of these frequencies are placed at the diagonal of the diagonal matrix $A_i$. On the other hand, the columns of submatrix $\Psi_i'$ represent static correction modes and are determined by solving the following system of linear algebraic equations:

$$K_i \Psi_i' = -K_{ig},$$

while $I_{gg}$ is an identity matrix with appropriate dimension.

By applying an analogous treatment, a similar set of equations of motion is obtained for all the components of the system. In fact, the multi-level substructuring method applied is a generalization of the classical Craig–Bampton method. Its main advantage is that the order of the eigenvalue and linear problems, defined by Eqs. (5) and (6), is much smaller than the dimension of the original system. For large order systems, this makes feasible and efficient the calculation of the coordinate reduction transformation. In particular, after performing the necessary algebra associated with the transformation and synthesis stages, the linear undamped terms of the equations of motion can eventually be put in the form

$$\ddot{q} = M \ddot{q} + K q - f(t),$$

with

$$q = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \end{pmatrix}, ~ M = \begin{bmatrix} I & \mu_{1,2} & \cdots & \mu_{1,B} \\ I_2 & \ddots & \cdots & \cdots \\ \vdots & \ddots & \ddots & \cdots \\ M_{BB} & \cdots & \cdots & \ddots \end{bmatrix}$$

and

$$K = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ A_2 & \ddots & \cdots & \cdots \\ \vdots & \ddots & \ddots & \cdots \\ 0 & \cdots & \cdots & A_{BB} \end{bmatrix}.$$
action is present. In this way, the exact nonlinear characteristics of the system are preserved in the reduced system. This leads to substantial numerical improvements, which have already been demonstrated in previous work on direct integration of mechanical systems with smooth and nonsmooth characteristics (Papalukopoulos and Natsiavas, 2007; Theodosiou and Natsiavas, 2009). Furthermore, adding all these effects, the equations of motion can eventually be cast in the general form expressed by

$$\mathbf{M} \ddot{\mathbf{q}} + \mathbf{C} \dot{\mathbf{q}} + \mathbf{K} \mathbf{q} + \mathbf{p}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{f}(t),$$

(7)

As a consequence of the applied transformations, the order of the final set of the equations of motion is reduced substantially, while maintaining numerical accuracy up to a suitably selected forcing frequency level. Moreover, since the individual transformations are performed on many small dimensional systems instead of one larger dimensional system, a drastic reduction in the computation time is achieved. Besides, this approach leads to other important numerical benefits, since it is associated with a much smaller volume of data transfer and causes a tremendous reduction in the computer memory required for the execution of the overall computations. Finally, apart from increasing the computational efficiency and speed, the reduction of the system dimensions must lead eventually to satisfaction of the following system of algebraic equations:

$$\mathbf{g}(\mathbf{u}) = \mathbf{0},$$

(10)

This converts the original initial value mathematical problem examined to a two point boundary value problem with unknowns included in vector \(\mathbf{u}_0\). Clearly, the correct set of initial conditions must lead eventually to satisfaction of the following system of algebraic equations:

$$\mathbf{g}(\mathbf{u}) = \mathbf{0},$$

(11)

This last system is nonlinear and an appropriate Newton–Raphson type method is needed for its numerical solution (Bathe, 1982; Pozrikidis, 1998). All these methods are based on an iterative process defined by the scheme

$$\mathbf{u}^{i+1} = \mathbf{u}^i + \Delta \mathbf{u}^i,$$

In particular, the correction \(\Delta \mathbf{u}^i\) imposed on the \(i\)th iteration is determined by solving the linear system of algebraic equations

$$\mathbf{J}(\mathbf{u}^i) \Delta \mathbf{u}^i = -\mathbf{g}(\mathbf{u}^i),$$

(12)

where

$$\mathbf{J}(\mathbf{u}) = \frac{\partial \mathbf{g}}{\partial \mathbf{u}}(\mathbf{u})$$

is the corresponding Jacobian matrix.

Numerically, the computation of vector \(\mathbf{g}(\mathbf{u}^i)\) in Eq. (12) is obtained by performing direct integration of Eq. (8) over a response period. In addition, the Jacobian matrix of the dynamical system examined is also needed in the calculations and computed by employing Eq. (10) in the form

$$\mathbf{J} = \frac{\partial \mathbf{u}}{\partial \mathbf{u}_0}(T) - \frac{\partial \mathbf{u}}{\partial \mathbf{u}_0}(0) = \Phi(T) - \mathbf{I},$$

(13)

where \(\mathbf{I}\) is an identity matrix with appropriate dimension, while

$$\Phi(T) = \frac{\partial \mathbf{u}}{\partial \mathbf{u}_0}(T)$$

(14)

is the so-called monodromy matrix of the dynamical system represented by Eq. (8). The evaluation of this matrix starts with the definition of the corresponding transfer matrix

$$\Phi(t) = \frac{\partial \mathbf{u}}{\partial \mathbf{u}_0}(t).$$

(15)

Then, employing this definition and the equations of motion in the first order form Eq. (8), it turns out that the elements of this matrix can be evaluated from

$$\Phi(t) = \frac{\partial \mathbf{u}}{\partial \mathbf{u}_0}(t) = \frac{\partial \mathbf{h}}{\partial \mathbf{u}_0}(\mathbf{u}, t) = \frac{\partial \mathbf{h}}{\partial \mathbf{u}}(\mathbf{u}, t) \frac{\partial \mathbf{u}}{\partial \mathbf{u}_0}(t),$$

or eventually by solving the system

$$\mathbf{u}(T) = \mathbf{u}_0.$$
\( \Phi(t) = A(t)\Phi(t), \)  
(16)

with

\[ A(t) = \frac{\partial H}{\partial l}(u(t), t). \]  
(17)

Therefore, the determination of the transfer matrix \( \Phi(t) \) and consequently of the monodromy matrix \( \Phi(T) \) and of the Jacobian matrix \( J \), defined by Eq. (13), is achieved by integrating the system of linear ordinary differential equations (16), subject to the initial conditions

\[ \Phi(0) = I. \]  
(18)

In order to avoid costly operations, related to determination and inversion of the Jacobian matrix, a suitable quasi-Newton method has been developed for solving Eq. (11) and consequently for locating periodic motions of Eq. (8), or equivalently of Eq. (7) (Broyden, 1965; Golub and van Loan, 1996). The numerical process ends when the vector \( u_0 \) is computed with sufficient accuracy.

Once a periodic motion is located, it is equally important to determine its stability properties, since only stable motions are observable in practice. As usual, the stability analysis of a located periodic motion, say \( u(T) \), starts by introducing a small perturbation into this motion. The idea is to examine how this perturbation evolves with time. More specifically, let

\[ u(t) = u_0(t) + \delta u(t), \]

with \( \delta u \ll 1 \). Substituting the last expression in Eq. (8), Taylor-expanding around \( u_0(t) \) and keeping only up to first order terms, it turns out that the perturbation introduced satisfies the following linear set of equations:

\[ \delta y(t) = A(t)\delta y(t), \]  
(19)

where the matrix \( A(t) \) is defined by Eq. (17). It is easy to verify that this matrix satisfies the periodicity condition

\[ A(t) = A(t + T). \]

Therefore, taking into account the last condition and applying classical Floquet theory in Eq. (19), it turns out that the stability properties of the periodic motion \( u(T) \) depend on the magnitude of the eigenvalues of the monodromy matrix \( \Phi(T) \) (Nayfeh and Mook, 1979). In particular, if all the eigenvalues of this matrix have magnitude less than one, the original perturbation dies out gradually and the motion examined is stable. However, if there exists at least one eigenvalue of the monodromy matrix with magnitude larger than one then the original perturbation grows with time and the located periodic motion is unstable. Finally, for parameter combinations leading to at least one eigenvalue of matrix \( \Phi(T) \) with magnitude equal to one bifurcations occur, signaling qualitative changes in the system response (Wiggins, 1990).

In many practical applications, it is frequently required to locate complete branches of periodic motions of a mechanical system, as an important parameter of the system is varied. For instance, in periodically excited dynamical systems, a typical such parameter is the fundamental forcing frequency. Finally, the inertia forces developed due to the pure rigid body rotation of the shaft were neglected (Morita and Okamura, 1995). More specifically, the rigid body rotation of

\[ u_{m+1} = u_m + u_{g.m}\Delta s_m \quad \text{and} \quad u_{m+1} = u_m + w_m\Delta s_m, \]

where the tangential vector \( u_{g.m} \) is evaluated from

\[ u_{g.m} = \frac{\partial H}{\partial l}(u_m, \mu_m), \]

while \( \Delta s_m \) and \( w_m \) are weighting scalars selected in an appropriate manner for each problem.

Introduction of parameter \( \mu \) in the set of unknowns necessitates the consideration of an additional algebraic condition. This extra equation is provided in the correction step. For instance, when the Riks–Wempner scheme is applied (Wempner, 1971; Riks, 1979), the corresponding new equation is expressed in the form

\[ u_{g.m}^T A\delta u(w) + w_m^T \Delta \mu_m = 0. \]

This reflects the fact that the correction imposed to the solution is perpendicular to the prediction. Addition of the last condition to the original system of Eq. (10) leads to a new system of algebraic equations, equal in number to the unknowns, including the elements of vector \( u_0 \) and parameter \( \mu \). In general, addition of this extra equation causes difficulties in the convergence of Newton’s method. Therefore, care should be exercised so that this is done only in the vicinity of some special points, like bifurcation points, of a frequency–response diagram.

### 5. Numerical results

In this section, some typical numerical results are presented for two selected mechanical models. Specifically, the first example focusses on periodic steady state response of the crankshaft of a car engine, subjected to periodic gas loading. Likewise, the second example investigates periodic response of the complete structure of a city bus subjected to harmonic excitation. In both cases, the main objective is to explore the accuracy and efficiency of the methodology developed. At the same time, the effect of some important technical parameters on the dynamics of the example systems is also investigated. For convenience in the presentation, these two examples are treated separately in the following subsections.

#### 5.1. Results for a car engine crankshaft

The first example model is the crankshaft of a four-cylinder inline internal combustion engine, belonging to a car. In fact, besides the shaft, the model examined includes the engine pulley at the left end and the flywheel at the right end, as shown in Fig. 1a. A finite element model of this system was obtained after discretizing its geometry by solid (mostly hexahedral) finite elements, leading to a mechanical model with more than 16,800 degrees of freedom. The crankshaft interacts with the engine block at five positions (indicated by Latin numbering in Fig. 1a) through oil journal bearings, possessing strongly nonlinear characteristics. In particular, the qualitative form of the restoring forces developed at these bearings is depicted in Fig. 1b. The corresponding damping forces exhibit similar characteristics also. In addition, the forcing is caused by the gas pressure developed within the engine cylinders and is a periodic function of time, as shown in Fig. 1c. In the last figure, the numbers in parentheses correspond to the position of the cylinders of the engine, which are assembled through the connecting rods with the crankshaft at the positions depicted in Fig. 1a. Therefore, they indicate the firing order of the cylinders. Moreover, in all the subsequent calculations the forces applied are assumed to have the same form for all the values of the fundamental forcing frequency. Finally, the inertia forces developed due to the pure rigid body rotation of the shaft were neglected (Morita and Okamura, 1995). More specifically, the rigid body rotation of...
the shaft was restrained elastically by placing an appropriate system of springs at its right end (on the flywheel side).

During the coordinate transformation phase, the highest response frequency of interest was set equal to 7000 Hz, due to the high frequency content of the gas forcing, while the cutoff frequency for the substructure eigenvalue problems was selected as 3. As a result, the original model was divided into 31 substructures, lying on four levels and the dimension of the substructures ranged from 567 to 1100. Eventually, this transformation led to a reduction in the size of the dynamical model from the original 16,822 degrees of freedom to only 208 degrees of freedom. Moreover, 54 of these were boundary degrees of freedom.

Since the forces applied to the mechanical model examined are periodic, it is natural to expect that the corresponding dynamical system will exhibit periodic steady state motions after a sufficiently long initial time interval. Indeed, Fig. 2 shows frequency–response diagrams obtained directly by applying the methodology developed, for a periodic forcing corresponding to an effective gas pressure equal to $p_{eff} = 1$ bar. In particular, the effective (root mean square) value of the periodic acceleration history signals recorded at two selected positions are displayed in Fig. 2, over a frequency range extending from 600 to 6000 rpm.

First, in Fig. 2a are shown acceleration spectra obtained at the position where the first connecting rod, based on the numbering of Fig. 1a, is connected to the shaft. The continuous lines were obtained from the nonlinear model, while the broken lines represent similar results, obtained by considering a linearized model, instead. Specifically, since there is no substantial static loading in the case examined, the linearized model was defined by considering the potential energy of the equivalent bearing springs at the given forcing level, so that the maximum displacements at the bearing locations of the fully nonlinear and the linearized model are about the same at 3000 rpm. Direct comparison confirms that significant differences are observed between the results of the
linearized and the nonlinear model. Moreover, these differences were found to be amplified considerably at other positions, especially at points close to the bearing nonlinearities. For instance, in Fig. 2b are shown similar acceleration spectra, obtained at the position where the crankshaft is connected to the first journal bearing. These results demonstrate that the acceleration levels are about one order of magnitude smaller than those recorded in the previous position. Also, the nonlinear effects can have a substantial effect on the system response and may be completely different than those predicted by applying linear analyses, something that is done in practice quite frequently (Morita and Okamura, 1995).

The lowest natural frequency of the linearized model examined was computed to be 9876 rpm. Obviously, this frequency is well above the maximum fundamental forcing frequency examined. This implies that the peaks observed in Fig. 2 correspond to superharmonic resonances induced by either the periodic nature of the forcing or the system nonlinearities (Nayfeh and Mook, 1979). In order to isolate the effects due to the action of the nonlinear forces, in Fig. 3 are shown results obtained by imposing a uniform increase in the original forcing amplitude by a factor of two and three, respectively. For instance, Fig. 3a shows the displacement spectra obtained at the position where the first connecting rod is attached to the crankshaft. Likewise, Fig. 3b presents a similar comparison for the acceleration amplitude values obtained at the same point. In both cases, the increase in the response amplitude is nonuniform and a frequency shifting occurs, which is in contrast to predictions obtained based on linear models (Craig, 1981; Rao, 1990). Moreover, more resonances become apparent as the forcing is increased.

Again, the situation becomes more intensified at points close to the nonlinear bearings. For instance, in Fig. 4 are shown similar results with those presented in Fig. 3, obtained at the position where the first journal bearing is connected to the crankshaft. Specifically, Fig. 4a shows the displacement spectra, while Fig. 4b presents the corresponding acceleration spectra. Once again, the effect of the bearing nonlinearities are quite pronounced.

Finally, similar differences were also observed in both the form and the amplitude of the periodic histories captured for the model response quantities. For instance, in Fig. 5 is presented the history obtained for the acceleration recorded over a response period at the same two points examined above, at steady state conditions. More specifically, these results were obtained at 3000 rpm and for the largest forcing level examined, corresponding to $p_{ef} = 3$ bar. First, direct comparison with the results obtained by the corresponding linearized model (represented by the broken curves) demonstrates that an acceptable agreement level is reached for the acceleration recorded at the position where the first connecting rod is connected to the crankshaft (Fig. 5a). On the other hand, substantial differences are observed in both the form and the maximum values for the same signals, recorded at the position of the first oil journal bearing (Fig. 5b). Among other things, it is again apparent that much smaller response amplitudes are developed in the latter case.
The following set of results is included in an effort to illustrate the numerical accuracy and efficiency of the methodology developed. First, in Fig. 6 are presented selected results, assessing the numerical accuracy of the coordinate reduction method applied. Specifically, in Fig. 6a is shown the periodic steady state displacement history obtained over a response period, developed at the connection position of the crankshaft with the first connecting rod. This response was caused by a periodic forcing corresponding to an effective gas pressure equal to $p_{\text{eff}} = 1$ bar, with a fundamental forcing frequency of 3000 rpm. The continuous line represents results obtained from the original model, while the broken curve corresponds to results obtained from the reduced model. Since the two curves are virtually indistinguishable, Fig. 6b is also included, illustrating the numerical error defined by

$$
\varepsilon_n = \frac{|x_{\text{ori}}^n - x_{\text{red}}^n|}{x_{\text{max}}^n}.
$$

In the last expression, $x_{\text{ori}}^n$ and $x_{\text{red}}^n$ represent the displacement value obtained from the original and the reduced model, respectively, at the time instant $t_n$, while $x_{\text{max}}$ corresponds to the maximum value of the signal. Among other things, these results verify the accuracy of the coordinate reduction method and especially the exact incorporation of the nonlinear terms in the reduced model.

Likewise, the results of Table 1 are presented in an effort to assess the numerical efficiency of the methodology developed. More specifically, these results refer to calculation of the same periodic steady state motion, obtained at 3000 rpm for $p_{\text{eff}} = 1$ bar. Due to the relatively small size of the problem examined, in addition to running the calculations after application of the multi-level dynamic substructuring method (MLDS), it was also possible to run similar calculations after application of the classical single-level component mode synthesis method (CMS) as well as the original model, without prior application of any coordinate reduction method. All the numerical results of this work were obtained on a workstation (CPU: Intel Pentium 4, 3.2 GHz, RAM: 2 GB, OS: GNU Linux 2.6 i586) and the results of Table 1 refer to the CPU time, the elapsed time and the volume of the input/output calculations performed for the particular case examined. Obviously, the results obtained after application of either of the coordinate reduction methods are much better than those obtained by employing the original model.

The last set of results referring to the crankshaft example are shown in Table 2. The results presented are similar to those of Table 1, with the emphasis placed now on the comparison of the performance of the multi-level substructuring method and the single-level component mode synthesis method. The results demonstrate that the performance of the multi-level substructuring...
method is superior. Moreover, this is amplified as the order of the system examined becomes larger. In addition, after a critical size of the original model, it may not be feasible to apply the single-level method at all, as verified by results presented in the following example.

5.2. Results for a city bus model

The numerical methodology developed is applicable to large models, with dimension in the order of millions of degrees of freedom. For instance, a mechanical system with a much bigger original dimension is considered next. More specifically, in Fig. 7a is shown the mechanical model of a city bus. Besides the bus upper body structure (superstructure) and chassis frame, the front and the rear suspensions are included among the other important structural subsystems. The main parts of the bus superstructure were geometrically discretized by a relatively large number of shell finite elements, so that the models examined can also be used at a later stage in order to produce results for vibro-acoustics studies as well (Kropp and Heiserer, 2003). As a result, the finite element discretization of the vehicle superstructure led to a model possessing 955,866 degrees of freedom. On the other hand, the bus chassis frame was also discretized by shell finite elements, leading to a model with 337,260 degrees of freedom. In addition, the driver and the passengers as well as the tire subsystems were modeled by appropriate sets of discrete masses, springs and dampers. Finally, special added mass elements were also employed in modeling systems like the air-condition unit, the fuel tanks, the bus floor including the passenger seats and the baggage store compartment. As a result, the final model possesses a total of 1,372,699 degrees of freedom.

For all practical purposes, the model of the bus body can be assumed to possess linear properties. However, the (bushing) elements connecting the system substructures are typically characterized by strong nonlinear action. Furthermore, strongly nonlinear characteristics are also encountered in the action of the shock absorber and spring units. In particular, the restoring force developed at the shock absorber unit of a typical bus, connecting the body with the wheels, can be represented by a combination of a linear and two nonlinear and asymmetric springs with hardening characteristics (Verros and Natsiavas, 2002). Likewise, the damping force developed in the suspension dampers can be represented by piecewise linear characteristics. For instance, Fig. 7b presents the damping force developed at the shock absorbers of all the suspension units of the vehicle examined. Clearly, all the main suspension dampers exhibit different behavior in tension than in compression, which is also typical in automotive engineering practice (Gillespie, 1992). Moreover, the equivalent damping coefficient (slope) is reduced as the relative speed increases.

The above description makes clear that the model examined is an ideal example for the methodology developed. The first set of numerical results, depicted in Fig. 8, were obtained for the linearized model obtained around the static equilibrium position, resulting by applying the weight of the structure, for three reduced models of the bus. More specifically, during the coordinate transformation phase, the dimensions of the reduced dynamical models were selected to include linear modes up to 20, 50 and 100 Hz, respectively, while the cutoff frequency for the substructure eigenvalue problems was selected equal to 3. As a result, the original model was divided into 842 substructures, lying on nine different levels and the dimension of the substructures varied from 2118 to 3600. This transformation led eventually to a reduction in the size of the dynamical model to only 201, 308 and 523 degrees of freedom, from which 177, 249 and 277 were boundary degrees of freedom, respectively.

<table>
<thead>
<tr>
<th>Method used</th>
<th>CPU time (m:s)</th>
<th>Elapsed time (m:s)</th>
<th>I/O (MB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>MLDS</td>
<td>00:12</td>
<td>00:14</td>
<td>206</td>
</tr>
<tr>
<td>CMS</td>
<td>01:01</td>
<td>01:04</td>
<td>223</td>
</tr>
</tbody>
</table>

Table 2
Comparison of numerical efficiency between MLDS and CMS.

Fig. 8. Comparison of natural frequencies, obtained for three reduced linearized bus models.

Fig. 7. (a) Finite element model of the bus structure. (b) Force developed in the main dampers of the front and the rear suspension shock absorber and spring units.
In particular, the horizontal axis of Fig. 8 presents the index, while the vertical axis presents the value of the natural frequencies predicted by the aforementioned models. These results demonstrate that the accuracy of the natural frequencies of each reduced model depends strongly on the pre-selected frequency range. In addition, a rapid and strong deviation is observed to occur in the results outside the pre-specified range, which is in accordance with similar observations in an earlier study (Papalukopoulos and Natsiavas, 2007). Finally, it is worth mentioning that in terms of numerical performance, application of the CMS method was not possible for any of the cases considered here, due mainly to the large dimension of the eigenvalue problem, defined by Eq. (5), that needs to be solved.

Moving along the same direction, Fig. 9 presents the spectra of the acceleration obtained at two specific points of the linearized bus models considered. The first set of them is shown in Fig. 9a and refers to the point where the driver seat is mounted to the bus frame, while the second is a selected point at the bus roof. The response was caused by applying a vertical harmonic excitation at the front left wheel of the bus, which is close to the driver position. The effective (root mean square) value of the acceleration history is presented within the forcing frequency interval 0–30 Hz, which is typical for ride studies referring to ground vehicles (Ellis, 1969; Gillespie, 1992). The results demonstrate that the accuracy level obtained is sufficient within the frequency range examined, at least for the last two reduced models. In fact, considering models with even more degrees of freedom causes virtually no change in the results obtained within the selected frequency range.

Next, Fig. 10 displays frequency–response diagrams obtained for the acceleration at the same two specific points of the bus considered. In all the results reported from here on, the reduced model employed in the calculations is sufficiently accurate within the range from 0 to 50 Hz. For comparison purposes, the broken curves represent similar results, obtained by running the model resulted by linearizing the equations of motion around the static equilibrium position of the bus. Clearly, there appear significant deviations between the predictions of the fully nonlinear and the linearized model employed, at least within certain frequency ranges.

Among other things, the results of the last figure demonstrate the fact that the methodology developed extends the classical frequency response analysis (FRA) from the linear to the nonlinear domain. In this respect, the information extracted from Fig. 10 is useful in assessing the forcing frequency ranges where the response quantity examined exhibits high level vibrations. In general, the deviations observed between the predictions of the nonlinear and the corresponding linearized models are amplified as the forcing amplitude is increased. Similar diagrams are also useful in predicting the effect of the important system parameters, like the horizontal velocity or the equivalent stiffness and damping parameters of the suspension, on the system response. This provides the basic information needed in selecting these parameters in an optimum manner. In addition, such information is also useful in health monitoring studies (Metallidis et al., 2008). For instance, Fig. 11 presents similar frequency–response diagrams, obtained after introducing a damage in one of the main suspension springs. Specifically, according to the damage scenario adopted, the stiff-
ness coefficient of the front left suspension unit was reduced to half of its original value. As confirmed by the results of Fig. 11, the results are affected significantly, especially around the range of 10 Hz, which is the range affected mostly by the wheel action.

6. Synopsis and conclusions

A complete methodology was presented for determining periodic steady state response of a class of periodically driven mechanical models in a direct and computationally efficient way. Specifically, the models examined involve a relatively large number of degrees of freedom and may possess nonlinear characteristics. However, the nonlinear action is confined to a relatively small number of degrees of freedom. Therefore, the basic idea was to first apply a multi-level dynamic substructuring method, in order to condensate the original dimension of the system significantly, so that the reduced model is sufficiently accurate up to a pre-specified level of forcing frequencies. This was achieved by employing an appropriate sequence of coordinate transformations, based on the sparsity pattern of the stiffness matrix. The analysis was then completed by a systematic method leading to a direct determination of steady state response of nonlinear systems subjected to periodic external excitation by exploiting the characteristics of the class of systems examined. A method for determining the stability properties of the located periodic motions was also developed, in conjunction with a continuation technique, yielding complete branches of periodic motions over a specified frequency interval.

The accuracy and effectiveness of the methodology developed was illustrated by presenting numerical results for an involved model of the crankshaft of a car engine under periodic excitation as well as for a quite complex finite element model of a city bus under harmonic base excitation. First, frequency spectra of several response quantities related to the crankshaft dynamics were constructed for steady state motions resulting from periodic gas excitation. Special emphasis was put on examining the deviations arising between predictions of the nonlinear and the corresponding linearized models. The effect of increasing the forcing amplitude on changing the qualitative form of the response spectra was also investigated. Then, the attention was shifted to examining ride dynamic performance of a detailed finite element model of a city bus. Initially, a comparison of response diagrams obtained by employing reduced models with a different number of linear normal modes and consequently with a different level of accuracy was performed. For the sufficiently accurate reduced model selected, the effect of the system nonlinearities on the response was investigated. Also, it was illustrated that the analysis developed can be useful in many areas, including optimal selection of critical parameters. In closing, it is worth mentioning that a great advantage of the present methodology may be realized by extending it to other important areas, such as identification, diagnostics and vehicle control.

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References


Fig. 11. Frequency–response diagrams: (a) at the driver seat position and (b) at a selected point of the bus roof.


