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GEOMETRIC APPROACH TO DISCRETE SERIES OF UNIRREPS FOR VIR.

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ABSTRACT. – We want to realize the discrete series of unirreps for the Virasoro-Bott group Vir (= the central extension of $Diff_+(S^1)$) in the space of holomorphic functions on the infinite dimensional Kähler manifold $M = Diff_+(S^1)/S^1$. The explicit formulae are given for the action of Vir in the space of polynomial functions in the natural complex coordinates on M. © Elsevier, Paris

RÉSUMÉ. – Nous voulons réaliser la série discrète d'unirreps (= représentations unitaires irréductibles) pour le groupe de Virasoro-Bott Vir (l'extension centrale de $Diff_+(S^1)$) dans l'espace des fonctions holomorphes sur la variété kählerienne de dimension infinie $M = Diff_+(S^1)/S^1$. Les formules explicites sont données pour l'action de Vir dans l'espace des fonctions polynômiales dans les coordonnées naturelles complexes sur M. © Elsevier, Paris

1. We recall the complex-analytic realization of the manifold M, described in [1]. Let $D^+ := \{z \in \mathbb{C} \mid |z| \leq 1\}$ and let $\mathcal{F}$ denote the space of all holomorphic functions f on D^+, which are univalent [1], smooth up to boundary and normalized by the conditions $f(0) = 0, f'(0) = 1$.

So, if we write

$$f(z) = z \cdot \left(1 + \sum_{n \geq 1} c_n z^n\right),$$

we can consider $\{c_n\}$ as coordinates on $\mathcal{F}$ which provide an embedding of this infinite dimensional manifold into $\mathbb{C}^\infty$ [2].

In fact, a function $f \in \mathcal{F}$ is uniquely defined by the simple smooth contour $K = f(S^1)$. The set $\mathcal{K}$ of all contours thus obtained can be considered as a geometric realization of $\mathcal{F}$. It consists of all contours K surrounding the origin such that the conformal radius of K w.r.t. the origin is equal to 1 [3].

Now we shall construct the bijection $\mathcal{K} \to M$ as follows. Let $f_K \in \mathcal{F}$ be the unique function wich maps S^1 to K (hence, it maps conformally D^+ to $D_K^+ :=$ the domain

I would like to express my gratitude to Peter Michor and to all the staff of the Erwin Schrödinger Institute for the friendly and creative atmosphere which made my staying here very agreable.

([1]) Univalent means that $f(z_1) \neq f(z_2)$ for $z_1 \neq z_2$.

([2]) The famous De Branges' Theorem (formerly the Bieberbach Conjecture) shows that the image lies in fact in the bounded domain $|c_n| < n + 1$, for $n \geq 1$.

([3]) The last requirement is simply a reformulation of the second normalization condition above.

containing the origin and bounded by K) and g_K be a function which maps conformally $D^- := \{z \in P^1(\mathbb{C}) \mid |z| \geq 1\}$ to $D_K^- :=$ the exterior domain bounded by K.

We normalize g_K by the condition $g_K(\infty) = \infty$. This leaves one parameter free and it is easy to see that g_K is defined modulo the composition (from the right) with a rotation $z \to z \cdot e^{i\alpha}$. The space of all such functions g forms an infinite dimensional complex manifold $\mathcal{G}$. A generic element $g \in \mathcal{G}$ looks like

$$g(w) = a_1 w + a_0 + a_{-1} w^{-1} + \cdots$$

and $\{a_k\}$, $k \leq 1$, are natural coordinates on $\mathcal{G}$.

Let finally

$$\gamma_K = f_K^{-1} \circ g_K. \tag{1}$$

The right hand side of (1) is well defined only on S^1. Due to the ambiguity in the definition of g_K, it is an element of $M = Diff_+(S^1)/S^1$. This is the promised map from $\mathcal{K}$ to M:

$$K \mapsto \gamma_K \mod S^1. \tag{2}$$

The fact that (2) is a bijection is proved in [1], where it is shown, that any diffeomorphism $\gamma \in G = Diff_+(S^1)$ can be uniquely written in the form $\gamma = f_\gamma^{-1} \circ g_\gamma$ with $f_\gamma \in \mathcal{F}$, $g_\gamma \in \mathcal{G}$.

Consider the map

$$Diff_+(S^1) \to \mathcal{F} \times \mathcal{G} : \quad \gamma \mapsto (f_\gamma, g_\gamma).$$

Let Γ be the image of this map and denote by p_1, p_2 its projections to the first and second factor respectively. Then p_1 is surjective, but not injective (it coincides essentially with the natural projection of $Diff_+(S^1)$ on $M \simeq \mathcal{F}$) while p_2 is injective, but not surjective (its image is a real hypersurface $\mathcal{G}_1 \in \mathcal{G}$ which consists of $g \in \mathcal{G}$ s.t. $g(S^1) \in \mathcal{K}$(4)).

2. Our next goal would be to describe the left action of the group G on $\mathcal{F}$ and the right action on $\mathcal{G}$. These actions are defined by

$$\gamma_1 \cdot f_\gamma := f_{\gamma_1\gamma}, \quad g_\gamma \cdot \gamma_2 := g_{\gamma\gamma_2}. \tag{3}$$

In fact, the second formula defines the action of G only on the hypersurface $\mathcal{G}_1$. But we extend this action to the whole $\mathcal{G}$ by real homogeneity which in this case implies complex homogeneity (since for any rotation r_α we have $e^{i\alpha} g_\gamma = r_\alpha \circ g_\gamma = g_{r_\alpha\gamma}$, the multiplication by $e^{i\alpha}$ commutes with the right action of G on $\mathcal{G}$).

(4) I do not know, how to express this property in terms of coordinates $\{a_k\}$.

Unfortunately, it is rather difficult to write these actions explicitly because the construction uses the Riemann uniformization theorem which gives no explicit formula [5].

But instead we can give the formulae for the corresponding infinitesimal action. Thus, for any $V \in Vect\ S^1$ we shall compute the corresponding $L_V \in Vect\ \mathcal{F}$ and $R_V \in Vect\ \mathcal{G}$. For this end we rewrite (1) in the form $f \circ \gamma = g$ and denote by $f_\epsilon, g_\epsilon, \gamma_\epsilon$ the small variation of the initial quantities which still satisfy

$$f_\epsilon \circ \gamma_\epsilon = g_\epsilon. \tag{4}$$

Let

$$f_\epsilon = f + \epsilon\phi + o(\epsilon),\ \ \phi \in H(D^+),$$

$$g_\epsilon = g + \epsilon\psi + o(\epsilon),\ \ \psi \in H(D^-),$$

$$\gamma_\epsilon = \gamma(1 + i\epsilon\chi) + o(\epsilon),\ \ \chi \in C^\infty(S^1).$$

Here $H(D^+)$ is the space of analytic functions on D^+, smooth up to boundary and normalized by $\phi(0) = \phi'(0) = 0$; $H(D^-)$ is the space of analytic functions on D^- which have at most a simple pole at ∞; $C^\infty(S^1)$ is the space of real-valued smooth functions on S^1.

It is clear that these three spaces are just tangent spaces to the corresponding manifolds $\mathcal{F}, \mathcal{G}, G$ (in fact, $H(D^-)$ coincides with $\mathcal{G}$).

Let us substitute the above expressions for $f_\epsilon, g_\epsilon, \gamma_\epsilon$ in (4) and compare the terms of the first order in ϵ. We get:

$$\phi \circ \gamma + f' \circ \gamma \cdot \gamma \cdot \chi = \psi.$$

Take now the right composition of both parts with g^{-1}. Using (1), we obtain

$$\phi \circ f^{-1} + f' \circ f^{-1} \cdot f^{-1} \cdot \chi \circ g^{-1} = \psi \circ g^{-1}. \tag{5}$$

We remark that the first term in the left hand side belongs to $H(D_K^+)$ and the right hand side belongs to $H(D_K^-)$ [6].

The classical result from complex analysis claims that for any $F \in C^\infty(K)$ the functions $F^\pm$ defined by

$$F^\pm(\xi^\pm) = \frac{(\xi^\pm)^2}{2\pi i} \oint_K \frac{F(\zeta)d\zeta}{\zeta^2(\zeta - \xi^\pm)},$$

([5]) The only exception is the action of the rotation subgroup. As a direct corollary of the definition (3) we get

$$f_{r_\alpha \gamma r_\beta} = r_\alpha \circ f \circ r_\alpha^{-1}, \quad g_{r_\alpha \gamma r_\beta} = r_\alpha \circ g \circ r_\beta.$$

([6]) These two spaces are defined exactly as $H(D^\pm)$, to which they specialize in the case $K = S^1$.

for $\xi^{\pm} \in D^{\pm}_K$ satisfy [7]

$$F^{\pm} \in H(D^{\pm}_K) \text{ and } F = F^+ - F^- \text{ on } K.$$

Applying this result to the second term of the left hand side of (5), we get the equality

$$\phi \circ f^{-1}(\xi) = -\frac{\xi^2}{2\pi} \oint_K \frac{f' \circ f^{-1}(\zeta) \cdot f^{-1}(\zeta) \cdot \chi \circ g^{-1}(\zeta) \cdot d\zeta}{\zeta^2(\zeta - \xi)}. \tag{6}$$

In order to compute the field L_V we take for γ_ϵ the initial element $\gamma \in G$ shifted from the left by an element close to the identity of the form $\exp\{-\epsilon\, v(e^{i\tau})\frac{d}{d\tau} + o(\epsilon)\}$. Then χ will be equal to $-v \circ \gamma$. The corresponding infinitesimal shift on $\mathcal{F}$ will be $\phi \in T_f(\mathcal{F}) = L_V(f)$. We substitute $\chi = -v \circ \gamma$ in (6) and put $\xi = f(z)$, $\zeta = f(t)$. Then we get

$$L_V(f)(z) = \frac{f^2(z)}{2\pi} \oint_{S^1} \left(\frac{tf'(t)}{f(t)}\right)^2 \cdot \frac{v(t)}{(f(t) - f(z))} \cdot \frac{dt}{t}. \tag{7}$$

3. It is convenient to extend (7) by complex linearity to the Lie algebra homomorphism of $\mathbb{C}Vect\ (S^1)$ to $Vect\ \mathcal{F}$, given by the same formula.

In the complex Lie algebra $\mathbb{C}Vect\ (S^1)$ there is a subalgebra of polynomial vector fields with the natural basis consisting of the fields

$$V_k = t^{k+1}\frac{d}{dt} \quad \left(\text{corresponding to the function} \quad v_k(e^{i\tau}) = -ie^{ik\tau}\right)$$

which satisfy the commutation relations $[V_k, V_j] = (j - k) \cdot V_{k+j}$. We shall write simply L_k instead of L_{V_k}. It turns out, that L_k for $k \geq 1$ have a rather simple expression. Namely, the integrand in (7) for $v(t) = -it^k$, $k \geq 1$, has the only simple pole at $t = z$. Taking the residue in it, we get

$$L_k(f)(z) = z^{k+1}f'(z). \tag{7 - k}$$

It may appear that the vector field (7) on $\mathcal{F}$ for $k \geq 1$ is induced by the natural action of $V_k \in \mathbb{C}Vect\ (\mathbb{C})$ on complex plane. But in fact the flow induced by V_k do not preserve neither D^+ nor $\mathcal{F}$. Only the combinations $\frac{V_k - V_{-k}}{2}$ and $\frac{i(V_k + V_{-k})}{2}$ are real vector fields on S^1 and generate flows on $\mathcal{F}$. For instance (cf. (8-0) below and the footnote 5 above), to iV_0 there corresponds the flow $f \mapsto r_\alpha \circ f \circ r_\alpha^{-1}$.

The computation of L_k for $k \leq 0$ is more difficult because now one has also the pole of order $1 - k$ at $t = 0$. However, one can show easily that the additional term has always the form $\sum_{s=k+1}^{s=1} a_s f^s$ where the coefficients a_s are polynomials in the coordinates $\{c_n\}$ introduced in the section 1.

(7) The more familiar form of this result: $\Phi^{\pm}(\xi^{\pm}) = \frac{1}{2\pi i}\oint_K \frac{\Phi(\zeta)d\zeta}{(\zeta - \xi^{\pm})}$ for $\xi^{\pm} \in D^{\pm}_K$ satisfy

$$\Phi^{\pm} \in (\xi^{\pm})^{-2} \cdot H(D^{\pm}_K) \text{ and } \Phi = \Phi^+ - \Phi^- \text{ on } K.$$

In particular, we obtain:

$$L_0(f)(z) = zf'(z) - f(z), \tag{7-0}$$

$$L_{-1}(f)(z) = f'(z) - 1 - 2c_1 f(z), \tag{7-1}$$

$$L_{-2}(f)(z) = z^{-1}f'(z) - \frac{1}{f(z)} - 3c_1 + (c_1^2 - 4c_2)f(z). \tag{7-2}$$

In terms of coordinates our computations give:

$$L_k = \partial_k + \sum_{n\geq 1}(n+1)c_n\partial_{k+n} \quad \text{for } k \geq 1, \tag{7-k'}$$

$$L_0 = \sum_{n\geq 1} nc_n\partial_n, \tag{7-0'}$$

$$L_{-1} = \sum_{n\geq 1}((n+2)c_{n+1} - 2c_1c_n)\partial_n. \tag{7-1'}$$

4. We compute now the right action of the Lie algebra *Vect* S^1 on the manifold $\mathcal{G}$. This can be done by the same method with the only difference that now we are interested in the right hand side of (5) and consider the perturbation γ_ϵ of $\gamma \in G$ by a small right shift:

$$\gamma_\epsilon(t) = \gamma(t \cdot e^{i\epsilon\ v(t)}) = \gamma(t) + i\epsilon\gamma'(t)\cdot tv(t) + o(\epsilon).$$

Hence, the function χ has the form

$$\chi(t) = \frac{\gamma'(t)\cdot t\cdot v(t)}{\gamma(t)} = \frac{d\log\gamma}{d\log t}\cdot v(t)$$

and we get

$$R_V(g)(w) = -\frac{g^2(w)}{2\pi}\oint_{S^1}\left(\frac{sg'(s)}{g(s)}\right)^2\cdot\frac{v(s)}{(g(s)-g(w))}\cdot\frac{ds}{s}, \tag{8}$$

which shows that R_V is an analytic vector field in the coordinates $\{a_k\}$.

In particular,

$$R_k(g)(w) = w^{k+1}g'(w) \quad \text{for } k \leq 0, \tag{8-k}$$

$$R_1(g)(w) = w^2g'(w) - \frac{g^2}{a_1}. \tag{8-1.}$$

It is very interesting but very hard problem to express $\{a_k\}$ in terms of $\{c_k\}$ and vice versa. One special case will be studied below.

5. There is another interesting geometric structure on $\mathcal{F}$: the countable set of G-invariant foliations by complex discs. To construct them we again consider the small right shift of $\gamma \in G$ and compute the corresponding variation $\phi = \delta f$ of f_γ. The function χ has the form

$$\chi(t) = \frac{\gamma'(t) \cdot t \cdot v(t)}{\gamma(t)} = \frac{d \log \chi}{d \log t} \cdot v(t)$$

and we get

$$\delta f(z) = -\frac{f(z)^2}{2\pi} \oint_{S^1} \left(\frac{tg'(t)}{g(t)}\right)^2 \cdot \frac{v(t)}{(g(t) - f(z))} \cdot \frac{dt}{t}. \tag{9}$$

Assume now that $v(t) = t^n$. The complex vector field $v(t) \cdot t\frac{d}{dt}$ on S^1 is an eigenvector w.r.t. rotations.

Hence, the corresponding $\delta_n f \in T_f(\mathcal{F})$ generate a G-invariant 1-dimensional distribution on $\mathcal{F}$. In fact, $\delta_n f$ vanishes for $n \leq 0$ – the integrand in (9) has no singularities in D^-. So we get non-trivial distributions N_n only for $n \in \mathbb{N}$. For $n = 1$ the formula (9) gives

$$N_1(f) = \mathbb{C} \cdot f^2, \tag{10 – 1}$$

which has the simple geometric meaning. Namely, the functions f_1 and f_2 lie on the same leaf iff $\frac{1}{f_1} - \frac{1}{f_2} =$ const. In other words, the map $z \mapsto z^{-1}$ sends the contours K_1 and K_2, corresponding to functions from the same leaf, to contours which are obtained one from another by a parallel transport).

For general n the distribution $N_n(f)$ has the form

$$N_n(f) = \mathbb{C} \cdot f^2 P_{n-1}(f), \tag{10 – n}$$

where P_k is a polynom of degree k in f whose coefficients are polynomials of weight 0 in $a_1, a_0, a_{-1}, \cdots$. E.g.,

$$N_2(f) = \mathbb{C} \cdot f^2(f - 3a_0). \tag{10 – 2}$$

6. Our next subject is the machinery of geometric quantization (see, e.g. [3]) for the group Vir, acting on $M \simeq \mathcal{F}$. We recall that the group $Vir = G \ltimes \mathbb{R}$ is the central extension of G, defined by the rule

$$(\gamma_1, \alpha_1) \cdot (\gamma_2, \alpha_2) = (\gamma_1 \circ \gamma_2,\ \alpha_1 + \alpha_2 + B(\gamma_1, \gamma_2)) \tag{11}$$

where B is the Bott cocycle:

$$B(\gamma_1, \gamma_2) = \frac{1}{2\pi} \int_{S^1} \log(\gamma_1 \circ \gamma_2)' d \log \gamma_2'. \tag{12}$$

The infinitesimal version of (11) is

$$[(v_1, a_1), (v_2, a_2)] = (v_1 v_2' - v_2 v_1', b(v_1, v_2)), \tag{11'}$$

where

$$b(v_1, v_2) = \frac{1}{\pi} \int_{S^1} v_1' dv_2' \tag{12'}$$

is the Gelfand-Fuchs-Virasoro cocycle. In fact, following the most physical papers, we shall use another cocycle $\tilde{b}$:

$$\tilde{b}(v_1, v_2) = \frac{1}{\pi} \int_{S^1} v_1' dv_2' - \frac{1}{\pi} \int_{S^1} v_1 dv_2. \tag{12''}$$

This cocycle $\tilde{b}$ is characterized by the property that it is the only cocycle from the same cohomology class with b satisfying: 1) $\tilde{b}$ is invariant under the adjoint action of the rotation subgroup; 2) $\tilde{b}$ vanishes on the subalgebra generated by L_1, L_0, L_{-1}.

Remark. – It is a good exercise to find the formula for the corresponding group cocycle $\tilde{B}$ which differs from B by a coboundary.

The action of Vir on $\mathcal{F}$ comes through the natural projection of Vir to G. This action being holomorphic, we can consider *the Kähler polarization on* $\mathcal{F}$ *and define* a representation of Vir in the space of analytic functions on $\mathcal{F}$ by:

$$(T(\gamma, \alpha)F)(f) = A((\gamma, \alpha), f)F(\gamma^{-1} \cdot f). \tag{13}$$

Here $A((\gamma, \alpha), f)$ is for each $(\gamma, \alpha) \in Vir$ a nowhere vanishing analytic function on $\mathcal{F}$, satisfying the cocycle equation

$$A((\gamma_1, \alpha_1) \cdot (\gamma_2, \alpha_2), f) = A((\gamma_1, \alpha_1), f)A((\gamma_2, \alpha_2), \gamma_1^{-1} \cdot f).$$

From this equation one can easily derive that

$$A((\gamma, \alpha),\ f) = e^{i\beta\alpha} a(\gamma,\ f)$$

for some constant β[8] and that the function $a(\gamma,\ f)$ satisfies

$$a((\gamma_1 \circ \gamma_2),\ f) = a(\gamma_1,\ f)a(\gamma_2,\ \gamma^{-1} \cdot f)e^{i\beta\tilde{B}(\gamma_1, \gamma_2)}. \tag{14}$$

The infinitesimal version of the representation T looks like

$$(T(v,\ a)F)(f) = \{[L_v(f) + \Phi(v,\ f)]F\}(f), \tag{15}$$

where $\Phi(v,\ f)$ is linear in $v \in C^\infty(S^1)$, analytic in $f \in \mathcal{F}$ and satisfies

$$L_{v_1}\Phi(v_2,\ f) - L_{v_1}\Phi(v_2,\ f) = \Phi(v_1 v_2' - v_2 v_1',\ f) + \beta\tilde{b}(v_1, v_2). \tag{16}$$

We shall assume now that we deal with the highest weight module $V_{c,\ h}$ over the Lie algebra vir which by definition possesses the vector x_0 with the properties:

$$L_k x_0 = 0 \text{ for } k > 0, \quad L_0 x_0 = h \cdot x_0, \quad C\ x_0 = c \cdot x_0. \tag{17}$$

([8]) The quantity $c = 12\beta$ is the so called central charge of the representation; it should be real for unitary representations.

We can also assume that in the realisation of $V_{c,\,h}$ in $H(\mathcal{F})$ the vector x_0 is represented by the functional $F_0(f) \equiv 1$ [9]. Then for $\Phi_k(f) := \Phi(v_k,\ f)$ we should have

$$\Phi_k(f) = \begin{cases} 0, & \text{for } k > 0 \\ P_n(c_1, \cdots, c_n), & \text{for } k = -n \le 0, \end{cases}$$

where P_n is a polynomial of weight n in the variables $\{c_k\}$ (the weight of c_k being equal to k) which depends on parameter c.

The equations (16) allows us to compute recurrently all these polynomials starting with $P_0 = const = h$. The result we formulate in terms of the generating function $\mathcal{P}$ for the series $\{P_n\}$.

THEOREM. – *We can write*

$$\mathcal{P}(z) = \sum_{n\ge 0} P_n(c_1, \cdots, c_n) z^n = h\left(\frac{zf'(z)}{f(z)}\right)^2 + \frac{cz^2}{12} S(f),$$

where $S(f) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$ *is the so called Schwarzian derivative of* f.

Proof. – From (16), taking into account $c = 12\beta$, we derive the system of linear differential equations for P_n:

$$(16') \qquad L_k P_n = (n + k) P_{n-k} + \delta_{k,n} \cdot \frac{k^3 - k}{12} c$$

which can be solved consequentively using the initial condition $P_0 = h$. E.g., we have: $P_0 = h$, $P_1 = 2hc_1$, $P_2 = (4h + \frac{c}{2})c_2 - (h + \frac{c}{2})c_1^2$, $P_3 = (6h + 2c)c_3 - (2h + 4c)c_1c_2 + 2cc_1^3$, $P_4 = (8h + 5)c_4 - (2h + 10c)c_1c_3 - 6cc_2^2 + (17c - 2h)c_1^2c_2 + (h - 6c)c_1^4$, $P_5 = (10h + 10c)c_5 - (2h + 20c)c_1c_4 + (2h - 26c)c_2c_3 + (36c - 4h)c_1^2c_3 + (8h - 58c)c_1^3c_2 + (42c - 6h)c_1c_2^2 + (16c - 2h)c_1^5$.

Since our system has the unique solution with given initial condition $P_0 = h$, it remains only to verify that the polynomials defined by the generating function in the Theorem satisfy (16′). We consider separately two cases: $h = 1,\ c = 0$ and $h = 0,\ c = 1$. In the first case the generating function has the form

$$\mathcal{P}_0(z) = \left(\frac{zf'(z)}{f(z)}\right)^2.$$

We start with the square root of this expression and write

$$\frac{zf'(z)}{f(z)} = \frac{d\log f(z)}{d\log z} = \sum_{n\ge 0} A_n z^n,$$

where A_n are again the polynomials of weight n in $\{c_k\}$.

LEMMA. – *The action of* L_k *on* A_n *looks like*

$$L_k A_n = n A_{n-k},$$

where we assume that $A_n = 0$ *for* $n < 0$.

[9] Otherwise one can pass to an equivalent representation.

Proof. – Consider the decomposition

$$\log f(z) = \log z + \sum_{n\geq 1} B_n z^n.$$

The effect of the action of L_k on the coefficients $\{B_n\}$ in this decomposition reduces to the infinitesimal shift $t \to z(1+\epsilon z^k)$. From that one derives

$$L_k B_n = \begin{cases} (n-k)B_{n-k} & \text{for } k < n \\ 1 & \text{for } k = n \\ 0 & \text{for } k > n. \end{cases}$$

Since $A_n = nB_n$, the Lemma follows.

Now the proof of the Theorem in the first case we deduce from the obvious relation

$$P_n = \sum_{i+j=n} A_k A_j.$$

Indeed,

$$\begin{aligned} L_k P_n &= \sum_{i+j=n} (iA_{i-k}A_j + jA_iA_{j-k}) \\ &= \sum_{p+q=n-k} (p+q+2k)A_pA_q = (n+k)P_{n-k}. \end{aligned}$$

In the second case we have to consider the sequence of polynomials $\{P_n\}$ defined by the generating function

$$\mathcal{P}_1(z) = z^2 S(f)(z).$$

We use the following characteristic property of $S(f)$ (see, e.g. [1]):

$$S(f\circ g) = S(f)\circ g \cdot (g')^2 + S(g). \tag{18}$$

Let $M_t^{(k)}$ be the flow [10] on $\mathbb{C}$ defined by

$$M_t^{(k)}(z) = \frac{z}{(1-ktz^k)^{1/k}}.$$

One easily checks that the generator of $M_t^{(k)}$ is the vector field $V_k = z^{k+1}\frac{d}{dz}$. So we get:

$$\sum_n L_k P_n z^n = \frac{d}{dt} S(f\circ M_t^{(k)})|_{t=0}.$$

Our statement now follows from (18) and

$$\frac{d}{dt} S(M_t^{(k)})|_{t=0} = \frac{k^3-k}{z^2}, \tag{19}$$

which can be established by the direct computation. The Theorem is proved.

([10]) In fact, this is only formal flow, which defines for every $t \in \mathbb{C}$ an analytic map of some neighborhood of the origin (depending on t).

7. We recall some notions from the Kähler geometry. Let (M, w) be a complex manifold with an Hermitian form w which in local coordinates looks like $w_{\alpha,\bar{\beta}}dz^{\alpha}d\bar{z}^{\beta}$. The pair (M, w) is called a Kähler manifold if two real forms $g = \Re\, w$, $\omega = \Im\, w$ define on M respectively the Riemannian metric and the symplectic structure. In this case there exists locally a real-valued function K s.t. $w_{\alpha,\bar{\beta}} = \partial_{\alpha}\bar{\partial}_{\beta}K$. This function is called the Kähler potential of w and is defined modulo the addition of the real part of an analytic function.

Example. – 1. $M = \mathbb{C}$, $w = c(dz)^2$, $K = c|z|^2$.

2. $M = D^+$, $w = \frac{c(dz)^2}{(1-|z|^2)^2}$, $K = -c\log(1-|z|^2)$.

3. $M = \mathbb{C} \subset P^1(\mathbb{C})$, $w = \frac{c(dz)^2}{(1+|z|^2)^2}$, $K = c\log(1+|z|^2)$.

PROPOSITION. – *If in the local coordinate system the potential has the form* $K(z) = K(0) + |z|^2 - \frac{1}{2}R_{\alpha,\bar{\beta},\gamma,\bar{\delta}}z^{\alpha}\bar{z}^{\beta}z^{\gamma}\bar{z}^{\delta} + o(|z|^4)$, *then* $R_{\alpha,\bar{\beta},\gamma,\bar{\delta}}$ *are components of the curvature tensor.*

E.g., in the examples above one can check that the curvature is 0, $-c$ and $+c$ respectively.

If some group G acts on M preserving w and if the Kähler potential K is defined globally [11], then we have

$$K(g^{-1}\cdot z) = K(z) - 2\Re\, B(g, z),$$

where $B(g, z)$ is for any $g \in G$ an analytic function on M and the following cocycle equation holds:

$$\Re\, B(g_1g_2, z) = \Re\, B(g_1, z) + \Re\, B(g_2, g_1^{-1}\cdot z).$$

We conclude, that

$$B(g_1g_2, z) - B(g_1, z) - B(g_2, g_1^{-1}\cdot z) = iC(g_1, g_2), \tag{20}$$

where $C(g_1, g_2)$ is a real valued 2-cocycle on the group G.

Indeed, the left hand side of (20) is an analytic function in z with vanishing real part, hence a pure imaginary constant.

It follows that the formula

$$(T(g)f)(z) = e^{B(g,\, z)}f(g^{-1}\cdot z) \tag{21}$$

defines a projective representation of G in the space $H(M)$ of holomorphic functions on M. Of course, we can consider it as an ordinary representation of the central extension $\tilde{G} = G \ltimes \mathbb{R}$, defined by the cocycle $C(g_1, g_2)$. Moreover, if μ denotes a G-invariant measure on M, this representation is unitary w.r.t. the inner product

$$(f_1, f_2) = \int_M e^{-K(z)}f_1(z)\overline{f_2(z)}d\mu(z). \tag{22}$$

One can check that in the first of above examples one obtains the projective representation of the Abelian group $\mathbb{R}^2$ of translations (which is equivalent to the ordinary representation

([11]) Otherwise we have to consider instead of functions on M the sections of some holomorphic line bundle $\mathcal{L}$ over M.

of the Heisenberg group); in the second example one obtains the projective representation of the group $SU(1,1) \simeq PSL(2,\mathbb{R})$; in the third we get the representation of $SU(2)$, but only for $c \in \mathbb{N}$ [12].

The analogy between (21) and (13) is evident as well as the analogy between (20) and (14).

The manifold $\mathcal{F}$ possesses the two-parametric family of G-invariant Kähler structures. In [1] these structures were computed at the initial point $f_0 \in \mathcal{F},\ f_0(z) \equiv z$. The result in the standard coordinates $\{c_k\}$ looks like

$$w_{\alpha,\beta}(0) = \sum_{k\geq 1} (\alpha k + \beta k^3) dc_k d\bar{c}_k. \tag{23}$$

This form is positive non-degenerate for $\alpha > 0,\ \beta > 0$. Being G-invariant, $w_{\alpha,\ \beta}$ is uniquely defined by its value (23) at f_0, but the explicit formula is known only for $\beta = 0$.

In physical papers usually another parameters are chosen:

$$h = \frac{\alpha+\beta}{2}, \quad c = 12\beta.$$

The Taylor decomposition at f_0 of the corresponding Kähler potential $K_{c,\ h}$ was computed in [2] up to the terms of fourth order. This is equivalent to the calculation of the Riemann curvature tensor due to the Proposition above.

8. Recall now some known fact from the representation theory of the algebra vir (see [4], [5], [6]).

Let $V_{c,\ h}$ be the irreducible highest weight module over vir. It is called unitarizable if one can define an inner product on it s.t. $L_n^* = L_{-n}$ and the vacuum vector x_0 has the length 1. Such a product is unique and exists in the following cases:

a) $c \geq 1,\ h \geq 0$;

b) $c = 1 - \frac{6}{m(m+1)},\quad h = h_{p,q,m} = \frac{(mp+p-mq)^2-1}{4m(m+1)}$,

where $m = 2,3,4,\cdots,\quad 0 < q \leq p < m$.

In the first case $L_{c,\ h}$ coincides with the Verma module $V_{c,\ h}$ while in the second case it is the quotient of $V_{c,\ h}$ by a non-trivial submodule consisting of vectors of zero length. This submodule is generated by the so-called *singular vectors* annihilated by all $L_k,\ k > 0$, and different from the vacuum vector. The first singular vector is on the pq-th level.

The simplest (and trivial) example is $m = 2$ where $p = q = 1,\ c = h = 0$ and $L_{0,0}$ is a trivial one dimensional module. It is realized in the one dimensional space of constant functions on $\mathcal{F}$.

Consider in more details the first non-trivial case $m = 3$. Here $c = \frac{1}{2}$, $h = 0,\ \frac{1}{2}$, or $\frac{1}{16}$ and the singular vectors are respectively on the first, second and forth levels.

For $h = 0$ the singular vector is just $L_{-1}x_0$. Moreover, the basis in $V_{\frac{1}{2},0}$ consists of vectors of the form

$$L_{-k_1} L_{-k_2} \cdots L_{-k_r} x_0,\ k_i \geq 2. \tag{24}$$

([12]) This is the condition for the existing of the line bundle $\mathcal{L}$ over $P^1(\mathbb{C})$, see e.g. [3].

The geometric realization of this space has the simple description: it consists exactly of functions on $\mathcal{F}$ which are constant along the leaves of the first G-invariant foliation (see section 5).

The very interesting open problem is to find the analogous description for the remaining representations of the discrete series.

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