Multicriteria fuzzy decision-making methods based on intuitionistic fuzzy sets

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Abstract
In this paper, a new method for handling multicriteria fuzzy decision-making problems based on intuitionistic fuzzy sets is presented. The proposed method allows the degrees of satisfiability and non-satisfiability of each alternative with respect to a set of criteria to be represented by intuitionistic fuzzy sets, respectively. Furthermore, the proposed method allows the decision-maker to assign the degrees of membership and non-membership of the criteria to the fuzzy concept “importance.” The method proposed here can provide a useful way to efficiently help the decision-maker to make his decision.

Keywords: Intuitionistic fuzzy sets; Multicriteria fuzzy decision-making; Linear programming model

1. Introduction

Since Zadeh presented the theory of fuzzy sets in 1965 [1], fuzzy sets theory has been used for handling fuzzy decision-making problems. Roughly speaking, a fuzzy set A of the universe of discourse \( U = \{u_1, u_2, \ldots, u_n\} \), is a set of ordered pairs \( \{(u_1, \mu_A(u_1)), \ldots, (u_n, \mu_A(u_n))\} \), where \( \mu_A \) is the membership function of the fuzzy set A, \( \mu_A: U \rightarrow [0, 1] \), and \( \mu_A(u_i) \) indicates the grade of membership of \( u_i \) in A. It is obvious that \( \forall u_i \in U \), the membership value \( \mu_A(u_i) \) is a single value between zero and one. W.L. Gau and D.J. Buehrer [2] pointed out that this single value combines the evidence for \( u_i \in U \) and the evidence against \( u_i \in U \), without indicating how much there is of each. They also pointed out that the single number tells us nothing about its accuracy. Thus they presented the concepts of vague sets. But in [3], H. Bustince and P. Burillo pointed out that the notion of vague set is the same as that of intuitionistic fuzzy set defined by Atanassov [4] practically ten years earlier. Shyi-Ming Chen, Jiann-Mean Tan [5] presented new techniques for handling multicriteria fuzzy decision-making problems based on vague set theory. And then Dug Hun Hong and Chang-Hwan Choi [6] provided another techniques for handling multicriteria fuzzy decision-making problems based on vague set theory, they provided new functions to measure the degree of...
accuracy in the grades of membership of each alternative with respect to a set of criteria. However, they assumed that the degree of importance to each criterion is constant. Deng-Feng Li [7] investigated multiattribute decision-making using intuitionistic fuzzy sets and constructed several linear programming models to generate optimal weights for criteria. But the method he put forward had to deal bigger calculation.

In this paper, we present a new method for handling multicriteria fuzzy decision-making problems based on intuitionistic fuzzy sets, where the characteristics of the alternatives are represented by intuitionistic fuzzy sets. The proposed method uses the truth-membership function and non-truth-membership function to indicate the degrees of satisfiability and non-satisfiability of each alternative with respect to a set of criteria, respectively, and it allows each criterion to have the degrees of membership and non-membership to the fuzzy concept “importance,” and it is only needed to establish one linear programming model, so the method is easier to apply to reality and it need smaller calculation and it can provide a useful way to efficiently help the decision-maker to make his decision.

This paper is organized as follows. The definition and properties of intuitionistic fuzzy sets are briefly introduced in Section 2. Multicriteria decision-making method based on intuitionistic fuzzy sets is then proposed, and the corresponding linear programming model is established in Section 3. A numerical example and a short conclusion are given in Sections 4 and 5, respectively.

2. Intuitionistic fuzzy sets

Let \( X \) be an ordinary finite non-empty set. An intuitionistic fuzzy set in \( X \) is an expression given by

\[
A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\},
\]

where \( \mu_A : X \to [0, 1], \nu_A : X \to [0, 1] \) with the condition \( 0 \leq \mu_A(x) + \nu_A(x) \leq 1 \), for all \( x \in X \).

The numbers \( \mu_A(x) \) and \( \nu_A(x) \) denote, respectively, the membership degree and the non-membership degree of the element \( x \) in \( A \).

For convenience of notation, we abbreviate “intuitionistic fuzzy set” to \( IFS \) and represent \( IFSs(X) \) as the set of all the \( IFSs \) in \( X \).

We call

\[
\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)
\]

the intuitionistic fuzzy index of the element \( x \) in the \( IFS \) \( A \). The value denotes a measure of non-determinacy.

The operations of \( IFS \) are defined as follows, for every \( A, B \in IFSs(X) \):

- \( A \subseteq B \) if and only if \( \mu_A(x) \leq \mu_B(x) \) and \( \nu_A(x) \geq \nu_B(x) \) for all \( x \in X \).
- \( A = B \) if and only if \( A \subseteq B \) and \( B \subseteq A \).
- \( A \cap B = \{(x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))) \mid x \in X\} \).
- \( A \cup B = \{(x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))) \mid x \in X\} \).
- The complementary of an \( IFS \) \( A \) is \( A^c = \{(x, \nu_A(x), \mu_A(x)) \mid x \in X\} \).

3. Multicriteria fuzzy decision-making based on intuitionistic fuzzy sets

This section presents a new method for handling multicriteria fuzzy decision-making problems, where the characteristics of the alternatives are represented by intuitionistic fuzzy sets. Suppose there exists an alternative set \( A = \{A_1, A_2, \ldots, A_m\} \) which consists of \( m \) non-inferior decision-making alternatives from which a most preferred alternative is to be selected. Each alternative is assessed on \( n \) criteria. Denote the set of all criteria \( C = \{C_1, C_2, \ldots, C_n\} \). Assume that \( \mu_{ij} \) and \( \nu_{ij} \) are the degrees of membership and non-membership of the alternative \( A_i \) to \( C_j \) satisfies the criterion \( C_j \in C \), respectively, where \( 0 \leq \mu_{ij} \leq 1 \), \( 0 \leq \nu_{ij} \leq 1 \) and \( 0 \leq \mu_{ij} + \nu_{ij} \leq 1 \). In other words, the evaluation of the alternative \( A_i \) with respect to the criterion \( C_j \) is an intuitionistic fuzzy set. The intuitionistic indices \( \pi_{ij} = 1 - \mu_{ij} - \nu_{ij} \) are such that the larger \( \pi_{ij} \) the higher a hesitation margin of the decision maker of the alternative \( A_i \) with respect to the criterion \( C_j \) whose intensity is given by \( \mu_{ij} \). Intuitionistic indices allow us to calculate the best final result (and the worst one) we can expect in a process leading to a final optimal decision. During the process the decision-maker can change his evaluations in the following way. He can increase his evaluation by adding the value of
the intuitionistic index. So in fact his evaluation lies in the closed interval \([\mu_i^l, \mu_i^u]\) = \([\mu_{ij}, 1 - v_{ij}]\), where \(\mu_i^l = \mu_{ij}\), and \(\mu_i^u = 1 - v_{ij}\). Obviously, \(0 \leq \mu_i^l + \mu_i^u \leq 2\) for all \(A_i \in A\) and \(C_j \in C\).

So the characteristics of the alternative \(A_i\) are presented by the intuitionistic fuzzy set shown as follows:

\[
A_i = \{(C_1, \mu_{i1}, v_{i1}), (C_2, \mu_{i2}, v_{i2}), \ldots, (C_n, \mu_{in}, v_{in})\},
\]

where \(1 \leq i \leq m\). We can present \(A_i\) by another form as follows for the sake of performing the decision-maker’s evaluation more directly,

\[
A_i = \{(C_1, [\mu_i^l, \mu_i^u]), (C_2, [\mu_i^l, \mu_i^u]), \ldots, (C_n, [\mu_i^l, \mu_i^u])\}.
\]

Assume that there is a decision-maker who wants to choose an alternative which satisfies the criteria \(C_j, C_k, \ldots, C_p\) or which satisfies the criteria \(C_s\). This decision-maker’s requirement is represented by the following expression:

\[
C_j \text{ AND } C_k \text{ AND } \ldots \text{ AND } C_p \text{ OR } C_s.
\]

In this case, the degrees to which the decision-maker’s requirement can be measured by the evaluation function \(E\),

\[
E(A_i) = ([\mu_i^l, \mu_i^u] \land [\mu_i^l, \mu_i^u] \land \cdots \land [\mu_i^l, \mu_i^u]) \lor [\mu_i^l, \mu_i^u] = \left[\min\{\mu_i^l, \mu_i^u, \ldots, \mu_i^l, \mu_i^u\}, \min\{\mu_i^l, \mu_i^u, \ldots, \mu_i^u\}\right] \lor [\mu_i^l, \mu_i^u] = \left[\max\{\min\{\mu_i^l, \mu_i^l, \ldots, \mu_i^l, \mu_i^l\}, \mu_i^l\}, \max\{\min\{\mu_i^u, \mu_i^u, \ldots, \mu_i^u, \mu_i^u\}, \mu_i^u\}\right] = [\mu_{A_i}^l, \mu_{A_i}^u],
\]

where \(\land\) and \(\lor\) denote the minimum operator and the maximum operator of the IFS, respectively; \(1 \leq i \leq m\).

Let \(A = [\mu_A, 1 - v_A]\), where \(\mu_A \in [0, 1], v_A \in [0, 1], \mu_A + v_A \leq 1\). The score of \(A\) can be evaluated by the score function \(S\) shown as \(S(A) = \mu_A - v_A\), where \(S(A) \in [-1, 1]\).

Next, we define an accuracy function \(H\) to evaluate the degree of accuracy of IFS as follows:

\[
H(A) = \mu_A + v_A,
\]

where \(H(A) \in [0, 1]\).

From the definition of accuracy function \(H\), it can be also expressed as

\[
H(A) = \mu_A + v_A = 1 - \pi_A.
\]

We know that the value of \(\pi\) denotes a measure of non-determinacy. The larger it the higher a hesitation margin of the decision-maker. So the larger the value of \(H(A)\), the more the degree of accuracy of the IFS \(A\). Now we want to make use of the two functions \(S\) and \(H\) to establish a function, which can measure the degree of alternatives satisfy the decision-maker’s requirement. But if we add them up directly, the value of \(v_A\) will be deleted. So based on the score function \(S\) and the accuracy function \(H\), the degree of suitability to which the alternative \(A_i\) satisfies the decision-maker’s requirement can be measured as follows:

\[
W(E(A_i)) = S(E(A_i)) + \frac{1 - H(E(A_i))}{2} = \frac{1}{2} \mu_{A_i}^l + \frac{3}{2} \mu_{A_i}^u - 1,
\]

where \(W(E(A_i)) \in [-1, 1]\). The larger the value of \(W(E(A_i))\), the more the suitability to which the alternative \(A_i\) satisfies the decision-maker’s requirement, where \(1 \leq i \leq m\).

Previously, we assumed that all criteria have the same degree of importance. However, if we can allow each criterion to have a different degree of importance, then there is room for more flexibility. In [5] and in [6], the authors presented a weighted technique for handling multicriteria fuzzy decision-making problems, but they assumed that the degree of importance of the criteria entered by the decision-maker are constant, it is hard to do in reality. So in this paper we assume that \(\rho_j\) and \(\tau_j\) are the degrees of membership and non-membership of the criteria \(C_j \in C\) to the fuzzy concept “importance,” respectively, where \(0 \leq \rho_j \leq 1, 0 \leq \tau_j \leq 1\) and \(0 \leq \rho_j + \tau_j \leq 1\). The intuitionistic indices \(\xi_j = 1 - \rho_j - \tau_j\) are such that the larger \(\xi_j\) the higher a hesitation margin of decision-maker as to the “importance” of the criteria \(C_j\) whose intensity is given by \(\rho_j\). Intuitionistic indices allow us to calculate the biggest weight (and the smallest one) we can expect in a process leading to a final decision. During the process the decision-maker can
change his evaluating weights in the following way. He can increase his evaluating weights by adding the value of the intuitionistic index. So in fact his weight lies in the closed interval $[\omega_j^l, \omega_j^u] = [\rho_j + \xi_j, \rho_j + \xi_j + 1 - \tau_j]$. Obviously, $0 \leq \omega_j^l \leq \omega_j^u \leq 1$ for each criterion $C_j \in C$. In addition, in this paper assume that $\sum_{j=1}^{n} \omega_j^l \leq 1$ and $\sum_{j=1}^{n} \omega_j^u \geq 1$ in order to find optimal weights satisfying $\omega_j^l \leq \omega_j \leq \omega_j^u$ and $\sum_{j=1}^{n} \omega_j = 1$.

Assume that there is a decision-maker who wants to choose an alternative which satisfies the criteria $C_j, C_k, \ldots, C_p$ or which satisfies the criteria $C_s$. This decision-maker’s requirement can be represented by (1). The degree of importance of the criteria $C_j, C_k, \ldots, C_p$ entered by the decision-maker are $\omega_j, \omega_k, \ldots, \omega_p$, respectively, where $\omega_j \leq \omega_j^l, \omega_k \leq \omega_k^l \leq \omega_k^u, \ldots, \omega_p \leq \omega_p^l \leq \omega_p^u$ and $\omega_j + \omega_k + \cdots + \omega_p = 1$. Let

$$T(A_i) = H\left(\left[\mu_{ij}^l, \mu_{ij}^u\right]\right) * \omega_j + H\left(\left[\mu_{ik}^l, \mu_{ik}^u\right]\right) * \omega_k + \cdots + H\left(\left[\mu_{ip}^l, \mu_{ip}^u\right]\right) * \omega_p,$$

$$W(A_i) = S\left(\left[\mu_{is}^l, \mu_{is}^u\right]\right) * \omega_j + S\left(\left[\mu_{ik}^l, \mu_{ik}^u\right]\right) * \omega_k + \cdots + S\left(\left[\mu_{ip}^l, \mu_{ip}^u\right]\right) * \omega_p,$$

where $T(A_i) \in [0, 1], W(A_i) \in [-1, 1]$ and $1 \leq i \leq m$. Then the degree of suitability that the alternative $A_i$ satisfies the decision-maker’s requirement can be measured by the following function:

$$R(A_i) = \max \left\{ W(A_i) + \frac{1 - T(A_i)}{2}, S\left(\left[\mu_{is}^l, \mu_{is}^u\right]\right) + \frac{1 - H\left(\left[\mu_{is}^l, \mu_{is}^u\right]\right)}{2} \right\},$$

where $R(A_i) \in [-1, 1], 0 \leq i \leq m$. The larger the value of $R(A_i)$, the more the suitability to which the alternative $A_i$ satisfies the decision-maker’s requirement.

In Eq. (4), we know that the value of $R(A_i)$ bases on the value of $W(A_i) + \frac{1 - T(A_i)}{2}$. So, next, we will point out how to obtain the optimal weights $\omega_j, \omega_k, \ldots, \omega_p$ for criteria $C_j, C_k, \ldots, C_p$ so that we can obtain the maximum value of above formula.

The optimal weights value can be computed via the following programming:

$$\max \sum_{i=1}^{m} \left(\frac{1}{2} \mu_{ij}^l + \frac{3}{2} \mu_{ij}^u - 1\right) * \omega_j + \left(\frac{1}{2} \mu_{ik}^l + \frac{3}{2} \mu_{ik}^u - 1\right) * \omega_k + \cdots + \left(\frac{1}{2} \mu_{ip}^l + \frac{3}{2} \mu_{ip}^u - 1\right) * \omega_p,$$

s.t. $\omega_j^l \leq \omega_j \leq \omega_j^u,$

$$\vdots$$

$\omega_p^l \leq \omega_p \leq \omega_p^u,$

$\omega_j + \omega_k + \cdots + \omega_p = 1.$

We can solve Eq. (5) by Simplex method.

4. A numerical example

We take the example shown in [7] so that we can compare the process of calculation of the two methods. Consider an air-condition system selection problem. Suppose there exist three air-condition system set $A = \{a_1, a_2, a_3\}$. Suppose three criteria $c_1$ (economical), $c_2$ (function), and $c_3$ (being operative) are taken into consideration in the selection problem. Denote the set of all criteria by $C = \{c_1, c_2, c_3\}$. Using statistical methods, the degrees $\mu_{ij}$ of membership and the degrees $\nu_{ij}$ of non-membership for the alternative $a_i \in A$ satisfies the criterion $c_j \in C$ can be obtained, respectively. Namely,

$$((\mu_{ij}, \nu_{ij}))_{3 \times 3} = \begin{pmatrix} c_1 & (0.75, 0.10) & (0.80, 0.15) & (0.40, 0.45) \\ c_2 & (0.60, 0.25) & (0.68, 0.20) & (0.75, 0.05) \\ c_3 & (0.80, 0.20) & (0.45, 0.50) & (0.60, 0.30) \end{pmatrix},$$

$$([\mu_{ij}^l, \mu_{ij}^u])_{3 \times 3} = \begin{pmatrix} c_1 & [0.75, 0.90] & [0.80, 0.85] & [0.40, 0.55] \\ c_2 & [0.60, 0.75] & [0.68, 0.80] & [0.75, 0.95] \\ c_3 & [0.80, 0.80] & [0.45, 0.50] & [0.60, 0.70] \end{pmatrix}.$$
In a similar way, the degrees $\rho_j$ of membership and the degrees $\tau_j$ of non-membership for the three criteria $c_j \in C$ to the fuzzy concept “importance” can be obtained, respectively, where $j = 1, 2, 3$. Namely,

$$\begin{pmatrix} (\rho_j, \tau_j) \end{pmatrix}_{1 \times 3} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} (0.25, 0.25) \\ (0.35, 0.40) \\ (0.30, 0.65) \end{pmatrix}.$$ 

Therefore, criteria weights lie in the closed interval as follows,

$$\begin{pmatrix} [\omega'_j, \omega''_j] \end{pmatrix}_{1 \times 3} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} (0.25, 0.75) \\ [0.35, 0.60] \\ [0.30, 0.35] \end{pmatrix}.$$ 

According to Eq. (3), the linear programming can be obtained

$$\begin{align*}
\text{max} & \quad 1.41 \ast \omega_1 + 1.765 \ast \omega_2 + 0.925 \ast \omega_3, \\
\text{s.t.} & \quad 0.25 \leq \omega_1 \leq 0.75, \\
& \quad 0.35 \leq \omega_2 \leq 0.60, \\
& \quad 0.30 \leq \omega_3 \leq 0.35, \\
& \quad \omega_1 + \omega_1 + \omega_3 = 1.
\end{align*} \tag{6}$$

Using Simplex method to solve the above linear programming, its optimal solution can be obtained as $\omega_1 = 0.25, \omega_2 = 0.45, \omega_3 = 0.30$. Then by applying Eq. (2), we can get

$$\begin{align*}
R(a_1) &= (0.375 + 1.35 - 1) \times 0.25 + (0.30 + 1.125 - 1) \times 0.45 + (0.40 + 1.20 - 1) \times 0.30 = 0.5525, \\
R(a_2) &= (0.40 + 1.275 - 1) \times 0.25 + (0.34 + 1.20 - 1) \times 0.45 + (0.225 + 0.75 - 1) \times 0.30 = 0.40425, \\
R(a_3) &= (0.20 + 0.825 - 1) \times 0.25 + (0.375 + 1.425 - 1) \times 0.45 + (0.30 + 1.05 - 1) \times 0.30 = 0.47125.
\end{align*}$$

Therefore, we can see that the alternative $a_1$ is the best choice. And the optimal ranking order of the alternatives is given by $a_1 \succ a_3 \succ a_2$. From the process of calculation, we can see that the method present in this paper is more easier than that in [7].

5. Conclusion

In this paper, we have presented a new method for handling multicriteria fuzzy decision-making problems, where the characteristics of the alternatives are represented by intuitionistic fuzzy sets. The proposed method allows the degrees of satisfiability and non-satisfiability of each alternative with respect to a set of criteria to be represented by intuitionistic fuzzy sets, respectively. Furthermore, the proposed method allows the decision-maker to assign the degree of membership and the degree of non-membership of the criteria to the fuzzy concept “importance.” An example is presented to illustrate the fuzzy decision-making process. From these, we can see that the proposed method can provide a useful way to efficiently help the decision-maker to make his decisions. The proposed method differs from previous approaches for multicriteria fuzzy decision-making not only due to the fact that the proposed method use intuitionistic fuzzy set theory rather than fuzzy set theory, but also due to the degree of importance of the criteria are not constant and the calculation is smaller.

References