Geometric Theory of Spaces of Integral Polynomials and Symmetric Tensor Products

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We investigate the nature of extreme, (weak*)-exposed, and (weak*-) strongly exposed points of the unit ball of spaces of \( n \)-homogeneous integral polynomials on, and \( n \)-fold symmetric products of, a Banach space \( E \). For the space of integral polynomials we show the set of extreme points is contained in the set \( \{ \pm \phi^n, \phi \in E^*, ||\phi|| = 1 \} \). We give Šmul'yan type theorems for spaces of \( n \)-homogeneous polynomials and \( n \)-fold symmetric tensors that characterise weak*-exposed (resp. weak*-strongly exposed) points in terms of Gâteaux (resp. Fréchet) differentiability of the norm on various spaces of tensor products and polynomials. Our study of the geometry of these spaces has many applications: When \( E \) has the Radon-Nikodym property we show that the spaces of \( n \)-homogeneous integral and nuclear polynomials are isomerically isomorphic for each integer \( n \). When the dimensions of \( E \) and \( n \) are both at least 2 then the space of \( n \)-homogeneous polynomials on \( E \) is neither smooth nor rotund. For a certain class of reflexive Banach space the space of \( n \)-homogeneous approximable polynomials on \( E \) is either reflexive or is not isometric to a dual Banach space. We conclude with a Choquet Theorem for a space of homogeneous polynomials.

1. INTRODUCTION

The study of Banach spaces to date can be divided into two broad fields. The first is the isomorphic theory of Banach spaces. This studies properties which are unchanged when the space is renormed with an equivalent norm

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and includes topics such as reflexivity, separability and type. The second is the isometric theory of Banach spaces. This studies topics which depend on the choice of norm given to the space and includes properties such as M-embeddedness, Fréchet differentiability of the norm and uniform convexity. Since the isometric theory of Banach spaces depends on the norm on the space the shape of the unit ball is fundamental to the theory. This leads us to study concepts such as extreme points, denting points and smooth points and to the isometric theory of Banach spaces being referred to as geometry of Banach spaces.

Given a Banach space $E$ we can form the space $\otimes_n E$ of all $n$-fold tensors in $E$. Tensors of the forms $x_1 \otimes x_2 \otimes \cdots \otimes x_n$ are called basic tensors and every $n$-fold tensor can be written as a finite sum of basic tensors. Given a tensor $u = \sum x_i \otimes x_i \otimes \cdots \otimes x_i$ in $\otimes_n E$ we would like to define the norm of $u$ to be $\sum \|x_i\| \|x_i\| \cdots \|x_i\|$. Unfortunately tensors in $\otimes_n E$ do not have unique representations as sums of basic tensors and so the projective norm of $u$ is defined to be the infimum of $\sum \|x_i\| \|x_i\| \cdots \|x_i\|$ over all possible representations of $u$ as sums of basic tensors. We use $\otimes_{n,\pi} E$ to denote the completion of $\otimes_n E$ with respect to this norm. The study of $n$-homogeneous polynomials on Banach spaces has led to the detailed study of a subspace of $\otimes_{n,\pi} E$. We consider the subspace, $\otimes_{n,s} E$, of $\otimes_n E$ consisting of all tensors of the form $\sum_{i=1}^n \pm x_i \otimes x_i \otimes \cdots x_i$, where $\lambda_i = \pm 1$. To make $\otimes_{n,s} E$ into a normed space with a projective type norm there are two possibilities. Either we can give a norm by restricting the projective norm on $\otimes_{n,\pi} E$ to $\otimes_{n,s} E$ or we can define the norm of a vector $u$ to be the infimum of $\sum_{i=1}^n \|x_i\|^2$ over all possible representations of the form $u = \sum \lambda_i x_i \otimes x_i \otimes \cdots \otimes x_i$. These two norms can easily be shown to be equivalent. Thus from an isomorphic point of view it is not important which norm we consider. From an isometric point of view we will show that these two norms in general give rise to spaces with radically different geometric structures. If $E$ is a real Banach space and $n$ is at least 2 it can be shown that the two norms are equal if and only if $E$ is a Hilbert space. We denote the completion of $\otimes_{n,s} E$ with respect to the latter norm by $\otimes_{n,s,\pi} E$ and call it the space of symmetric $n$-fold tensors on $E$.

The spaces $\otimes_{n,\pi} E$ and $\otimes_{n,s,\pi} E$ arise naturally as preduals of the space of $n$-linear maps, $\mathcal{L}^n(E)$, and $n$-homogeneous polynomials, $\mathcal{P}^n(E)$, respectively. Each of these spaces has in turn, the supremum norms of uniform convergence on the $n$-fold product of the unit ball of $E$ and on the unit ball of $E$. An (infinite dimensional) Banach space $E$ is said to be stable if $E$ is isomorphic to $E \times E$. In a recent paper by Díaz and Dineen [10] (see also [3]), it is shown that if $E$ is stable then $\mathcal{L}^n(E)$ and $\mathcal{P}^n(E)$ are isomorphic from which it follows that $\otimes_{n,\pi} E$ and $\otimes_{n,s,\pi} E$ are isomorphic. (Díaz [9] gives an example of a non-stable Banach $E$ such that $\mathcal{L}^n(E)$ and $\mathcal{P}^n(E)$ are not isomorphic.)
In this paper we examine the geometry of the Banach spaces of \( n \)-homogeneous integral polynomials and symmetric \( n \)-fold tensor products. We investigate their extreme, strongly exposed, weak*-exposed and weak*-strongly exposed points. These points have previously been investigated in [31] for the symmetric tensor product of finite dimensional spaces and in [29, 30] for the usual (non-symmetric) tensor product. We show that geometrically the space of symmetric \( n \)-fold tensor products \( \otimes_{n,E} \), is very different from the space of ordinary tensor products, \( \otimes_{n,E} \). Our investigation of the geometry of \( \otimes_{n,E} \) has brought to light certain interesting consequences. We show that when \( \otimes_{n,E} \) does not contain a copy of \( l_1 \) then the spaces of integral and nuclear \( n \)-homogeneous polynomials on \( E \) are isometrically isomorphic. We will show that \( \otimes_{n,E} \) and \( P ([n,E]) \) can never be rotund or smooth for \( \dim E \geq 2 \) and \( n \) is at least 2. Under certain conditions we will show that when \( E \) is a reflexive Banach space the space of \( n \)-homogeneous polynomials on \( E \) which are weakly continuous on bounded sets is either reflexive or not isometric to a dual space. We also give a Choquet type theorem for spaces of \( n \)-homogeneous polynomials. Results similar to some of the results in Sections 3 and 4 have recently been found by Ferrera [18]. Unless stated otherwise all results are for real Banach spaces. We thank Richard Aron, Sean Dineen and Nacho Zalduendo for their discussions during the preparation of this paper.

2. EXTREME POINTS OF THE UNIT BALL OF SPACES OF INTEGRAL POLYNOMIALS

Given a Banach space \( E \) we say that an \( n \)-homogeneous polynomial \( P \) on \( E \) is nuclear if there is a bounded sequence \( (\lambda_j)_{j=1}^{\infty} \subseteq E' \) and a sequence \( (\phi_j)_{j=1}^{\infty} \) in \( l_1 \) such that

\[
P(x) = \sum_{j=1}^{\infty} \lambda_j \phi_j(x)^n
\]

for every \( x \) in \( E \). The space of all nuclear \( n \)-homogeneous polynomials on \( E \) is denoted by \( \mathcal{P}_n(E) \) and becomes a Banach space when the norm of \( P \) is given as the infimum of \( \sum_{j=1}^{\infty} |\lambda_j| |\phi_j|^{n} \) taken over all representations of \( P \) of the form described above. This norm is called the nuclear norm of \( P \) and is denoted by \( \|P\|_N \). Given \( \phi \) in \( E' \) we denote by \( \phi^n \) the \( n \)-homogeneous polynomial which takes \( x \) to \( \phi(x)^n \). When \( E' \) has the approximation property \( \mathcal{A}(E) \) is isometrically isomorphic to \( \otimes_{n,E} \) under the map induced by \( \phi^n \to \phi \otimes \phi \otimes \cdots \otimes \phi \).
A polynomial $P$ on $E$ is said to be integral if there is a regular Borel measure $\mu$ on $(B_E, \sigma(E, E))$ such that

$$P(x) = \int_{B_E} \phi(x)^n \, d\mu(\phi)$$

(1)

for every $x$ in $E$. We write $\mathcal{P}(E)$ for the space of all $n$-homogeneous integral polynomials on $E$. We define the integral norm of an integral polynomial $P$, $\|P\|_I$, as the infimum of $\|\mu\|$ taken over all regular Borel measures which satisfy (1). With the integral norm $\mathcal{P}(E)$ becomes a Banach space. It can be shown that for every Banach space $E$ we have $P_N(E) \subseteq P_d(E) \subseteq P(E)$ and for $P \in \mathcal{P}_d(E)$ we have $\|P\| \leq \|P\|_I \leq \|P\|_N$.

For $\phi \in E^*$ it is easily seen that $\phi^n$ is a nuclear polynomial and that $\|\phi^n\|_N = \|\phi^n\|_I = \|\phi^n\|$.

Apart from the projective norm, there are many other norms that can be placed on $\otimes_{n \geq 1} E$. Given an $n$-fold symmetric tensor $\sum_{i=1}^n \lambda_i x_i \otimes \cdots \otimes x_i$ on $E$ we define its injective norm as

$$\sup_{\phi \in B_E} \left| \sum_{i=1}^n \lambda_i \phi(x_i) \right|$$

This may also be regarded as the norm inherited from $\mathcal{P}(E')$. We denote the completion of $\otimes_{n \geq 1} E$ with respect to this norm by $\tilde{\otimes}_{n \geq 1} E$. It is shown in [12] that the dual of $\otimes_{n \geq 1} E$ is isometrically isomorphic to $(\mathcal{P}(E), \|\|_I)$.

Given a subset $C$ of a vector space $V$ a point $x$ of $C$ said to be an extreme point of $C$ if $x$ cannot be written as a convex combination of points in $C$ which are distinct from $x$ itself. The set of all extreme points of $C$ is denoted by $\text{ext} C$.

**Proposition 1.** For a Banach space $E$ and a positive integer $n$ the set of extreme points of the unit ball of $\mathcal{P}(E)$ is contained in $\{ \pm \phi^n : \phi \in E^*, \|\phi\|_I = 1 \}$.

**Proof.** We can consider $\tilde{\otimes}_{n \geq 1} E$ as a subspace of $C(B_E, \sigma(E, E))$ by defining $\sum_{i=1}^n \lambda_i x_i \otimes \cdots \otimes x_i$ to be $\sum_{i=1}^n \lambda_i \phi(x_i)^n$. Since $C(B_E, \sigma(E, E))$ is a $C(K)$ space very extreme point of the unit ball of $(\tilde{\otimes}_{n \geq 1} E) = \mathcal{P}(E)$ has as extension to an extreme point of the unit ball of $C(B_E, \sigma(E, E))$ (see Lemma V.8.6 of [15]). However, the set of extreme points of the unit ball of $C(B_E, \sigma(E, E))$ is $\{ \pm \delta_E : \phi \in E^*, \|\phi\|_I = 1 \}$. Restricting these to $\mathcal{P}(E)$ we see that our possible choice of extreme points of the unit ball of $\mathcal{P}(E)$ is limited to those given in the statement of the Proposition. ☐
If $E$ is a complex Banach space the above proof is easily modified to show that the (real) extreme points of the unit ball of $\mathcal{B}(E)$ are contained in the set $\{\phi^*; \phi \in E^*, \|\phi\| = 1\}$.

If $E$ is finite dimensional then it follows from [31] that $\text{Ext } B_{\mathcal{P}I(E)}$ is in fact equal to $\{\pm \phi^*; \phi \in E^*, \|\phi\| = 1\}$.

Proposition 1 also puts an upper bound on the sets of exposed points, strongly exposed points, weak*-exposed and weak*-strongly exposed points of the unit ball of $\mathcal{B}(E)$.

In [2] Alencar shows that if $E$ is a Banach space such that $E^*$ has the Radon–Nikodym property (RNP) then $\mathcal{B}(E)$ is isomorphic to $\mathcal{B}_N(E)$. The above Proposition allows us to show much more. Since we shall make constant use the next result in what follows, we interrupt our discussion of extreme points of the unit ball of $\mathcal{B}(E)$ to give the following Theorem.

**Theorem 2.** If $E$ is a Banach space, $n$ is a positive integer and $\bigotimes_{n,s} E$ does not contain a copy of $\ell_1$ then $\mathcal{B}(E)$ and $\mathcal{B}_N(E)$ are isometrically isomorphic.

**Proof.** Suppose that $\phi^*$ is an extreme point of the unit ball of $\mathcal{B}(E)$. It follows that $\phi^*$ is also a unit vector in $\mathcal{B}_N(E)$. Let $P$ be a polynomial in the unit ball of $\mathcal{B}_N(E)$ so that

$$\|\phi^* + P\|_N \leq 1.$$ 

Since $\mathcal{B}_N(E) \subseteq \mathcal{B}(E)$ and $\|Q\|_I \leq \|Q\|_N$ for every $Q$ in $\mathcal{B}_N(E)$ we have

$$\|\phi^* + P\|_I \leq 1$$ 

and hence $P \equiv 0$. This means that

$$\mathcal{E} \text{xt } B_{\mathcal{P}I(E)} \subseteq \mathcal{E} \text{xt } B_{\mathcal{P}I(E)}.$$ 

Since $\ell_1$ is not a subspace of $\bigotimes_{n,s} E$ it follows from [20] that the unit ball of $\mathcal{B}(E)$ is the norm-closure of the convex hull of its extreme points. We always have $B_{\mathcal{P}I(E)} \subseteq B_{\mathcal{P}I(E)}$, thus

$$B_{\mathcal{P}I(E)} = \overline{B_{\mathcal{P}I(E)}},$$ 

$$B_{\mathcal{P}I(E)} \subseteq B_{\mathcal{P}I(E)} \subseteq B_{\mathcal{P}I(E)}.$$ 

It follows from Lemma 5.5 of [6] and the Open Mapping Theorem that $\mathcal{B}(E)$ and $\mathcal{B}_N(E)$ are isomorphic and we may conclude from (2) that $B_{\mathcal{P}I(E)} = B_{\mathcal{P}I(E)}$ i.e. $\mathcal{B}(E)$ is isometrically isomorphic to $\mathcal{B}_N(E)$. 

Theorem 2 is also true for complex Banach spaces, although we have to be a little careful here as Haydon proves his result for real Banach spaces.
If $E$ is a complex Banach space and $\mathfrak{S}_{n,s,t}E$ does not contain a (complex) copy of $\ell_1$ then considering $\mathfrak{S}_{n,s,t}E$ as a real Banach space it will not contain a (real) copy of $\ell_1$ (see [14]). As the (real) dual of $\mathfrak{S}_{n,s,t}E$ considered as a real Banach space is the real Banach space underlying $\mathfrak{S}(^*E)$ we can now apply Haydon’s result as we did in Theorem 2 to get that $\mathfrak{S}(^*E)$ is isometrically isomorphic to $\mathfrak{S}(^*E)$.

If we assume that $E'$ has RNP then it follows from Theorem 1.9 of [29] that $\mathfrak{S}(^*E)$ has RNP for every integer $n$ and hence $\mathfrak{S}_{n,s,t}E$ cannot contain a copy of $\ell_1$. This gives the following result. The corresponding result for multilinear mappings is due to Alencar [1].

**Proposition 3.** Let $E$ be a real or complex Banach space such that $E'$ has RNP. Then $\mathfrak{S}(^*E)$ and $\mathfrak{S}(^*E)$ are isometrically isomorphic for every positive integer $n$.

The James-Hagler space, $JH$, [19] is an example of a Banach space whose dual does not have RNP yet whose injective tensor product does not contain a copy of $\ell_1$. See [25] for the case $n=2$. Hence $JH$ is an example of a Banach space whose dual does not have RNP yet on which the spaces of integral and nuclear polynomials are isometrically isomorphic.

We note that $\mathfrak{S}(^*E)$ is a dual space it follows from the Krein–Milman Theorem that its unit ball will always contain extreme points. The following lemma enables us to show that it contains a considerable number of extreme points.

**Lemma 4.** Let $E$ be a normed space, and let $\phi$ be a unit vector in $E$. Suppose that for every finite dimensional subspace $F$ of $E$ there exists a subspace $\tilde{F}$ of $E$ containing $F$ with the property that $||\phi||_{\tilde{F}} = 1$ and such that $\phi_{\tilde{F}}$ is an extreme point of the unit ball of $\tilde{F}$. Then $\phi$ is an extreme point of the unit ball of $E$.

**Proof.** Suppose that $\phi$ is not an extreme point of $B_E$. Then we can write $\phi$ as $\phi = \lambda \phi_1 + (1-\lambda)\phi_2$ for some $\phi_1$ and $\phi_2$ in the unit ball of $E$, neither of which is equal to $\phi$, and for some $\lambda$, $0 < \lambda < 1$. Choose $x_i$, $i=1, 2$, in the unit ball of $E$ so that $\phi_i(x_i) \neq \phi(x_i)$. Let $F$ be the subspace of $E$ spanned by $x_1$ and $x_2$. On restricting $\phi$ to $F$ we see that

$$\phi|_F = \lambda \phi_1|_F + (1-\lambda)\phi_2|_F$$

but $\phi_i|_F \neq \phi|_F$ for $i=1, 2$ contradicting our assumption that we can find a subspace $\tilde{F}$ of $E$ containing $F$ with the property that $\phi|_F$ is an extreme point of the unit ball of $\tilde{F}$. $\blacksquare$
Proposition 5. Let $E$ be a Banach space and let $n$ be an integer which is greater than or equal to 2. Then the set of extreme points of the unit ball $\mathcal{B}(E)$ contains $\{ \pm \phi^n : \phi \in E^*, \|\phi\| = 1 \text{ and } \phi \text{ attains its norm} \}$.

Proof. For the purposes of the Proposition it is convenient to consider $\mathcal{B}(E)$ as the dual of $\bigotimes_{n,n} E$, the uncompleted $n$-fold injective tensor product of $E$, rather than $\bigotimes_{n,s} E$. Given a finite dimensional subspace $X$ of $\bigotimes_{n,s} E$ we can find a finite dimensional subspace $F$ of $E$ so that $X$ is a subspace of $\bigotimes_{n,s} F$. Given a norm attaining unit vector $\phi$ in $E$ we choose $\phi \in E$ so that $\phi(x) = 1$. Let $\bar{F}$ be the subspace spanned by $F$ and $x$. Applying Theorem 4 of [31] we see that $\phi^n |_{\bigotimes_{n,s} \bar{F}}$ is an extreme point of the unit ball of $\bigotimes_{n,s} \bar{F}$). Lemma 4 now implies that $\phi^n$ is an extreme point of the unit ball of $\mathcal{B}(E)$.

Corollary 6. Let $E$ be a reflexive Banach space and let $n$ be integer which is greater than or equal to 2. Then the set of extreme points of the unit ball of $\mathcal{B}(E)$ is precisely the set $\{ \pm \phi^n : \phi \in E^*, \|\phi\| = 1 \}$.

As we mentioned in the introduction the space of $n$-homogeneous integral polynomials and $n$-linear integral mappings on infinite dimensional stable spaces are isomorphic as Banach spaces. From a geometric point of view they are very different. For example, Ruess and Stegall [29, Theorem 1.1] show that the set of extreme points of the unit ball of $\mathcal{L}(E)$ is equal to $\{ \phi_1 \phi_2 \cdots \phi_n : \phi_j \in \mathcal{E}_{\text{ext}} B_E \}$ which in general are very different to the type of points described in Proposition 1.

3. WEAK*-EXPOSED POINTS OF THE UNIT BALL OF SPACES OF INTEGRAL POLYNOMIALS AND $n$-FOLD SYMMETRIC TENSORS

We recall that a unit vector $x$ in a Banach space $E$ is exposed if there is a unit vector $f \in E^*$ so that $f(x) = 1$ and $f(y) < 1$ for $y \in B_E \setminus \{x\}$. We will say that $f$ exposes $B_E$ at $x$. If $E = E'$ is a dual space and the vector $f$ which exposes $x$ is in $F$ we shall say that $x$ is weak* exposed and that $f$ weak*-exposes the unit ball of $E$ at $x$. Note that $f$ in $F$ weak*-exposes the unit ball of $E$ at $x$ if and only if whenever $(x_k)$ is a sequence in $E$ so that $f(x_k)$ converges to 1 then $(x_k)$ converges to $x$ in norm. We will say that $f$ strongly exposes $B_E$ at $x$. When $E = F'$ is a dual space and the vector $f$ which strongly exposes $B_E$ is in $F$ we shall say that $x$ is weak*-strongly exposed and that $f$ weak*-strongly exposes the unit ball of $E$ at $x$. 
The following diagram gives the relationship between each of the sets of points we have defined to date:

\[
\begin{array}{ccc}
\text{strongly-exposed} & \overset{=}\rightarrow & \text{exposed} \\
\uparrow & & \uparrow \\
\text{weak*-strongly-exposed} & \Rightarrow & \text{weak*-exposed} \\
& & \downarrow
\end{array}
\]

In this section we investigate the weak*-exposed points of the unit ball of \( \mathcal{P}(E) \) and (under certain conditions) of the unit ball of \( \bigotimes_{n,s} E \). As the dual of \( \bigotimes_{n,s} E \) is \( \mathcal{P}(E) \) we are tempted to make polynomial versions of each of these four definitions. For example, we could say a point \( x \) in \( E \) is \( n \)- polynomially exposed if there is a \( P \in \mathcal{P}(E) \) such that \( P(x) = 1 \) and \( P(y) < 1 \) for \( y \in B_E \setminus \{x\} \). We shall see however that these would be superfluous definitions.

We will use \( \mathcal{P}_w(E) \) to denote the space of all \( n \)-homogeneous polynomials on \( E \) that are weakly continuous on bounded sets. This space is a Banach space when considered with the norm inherited from \( \mathcal{P}(E) \). If \( E \) has the approximation property then every polynomial in \( \mathcal{P}_w(E) \) is uniformly approximable on the unit ball of \( E \) by finite type polynomials and hence \( \mathcal{P}_w(E) \) is isometrically isomorphic to \( \bigotimes_{n,s} E' \). Therefore if \( E \) is a Banach space such that \( E' \) has the approximation property and \( \mathcal{P}_w(E) \) does not contain a copy of \( l_1 \) then \( \mathcal{P}_w(E)' = \bigotimes_{n,s} E'' \) and \( \mathcal{P}_w(E)' = \mathcal{P}(E') \) (isometrically).

Smul'yan [33, 34] shows that a point \( x \) in \( B_E \) weak*-exposes (resp. weak*-strongly exposes) the unit ball of \( E \) at \( f \) if and only if the norm of \( E \) is Gâteaux differentiable (resp. Fréchet differentiable) at \( x \) with derivative \( f \). We will now derive a Smul'yan type theorem for spaces of integral polynomials and \( n \)-fold symmetric tensor products. The proof of this Theorem is very similar to Proposition 2.1 and Theorem 1.1 of [30]. We find it convenient to distinguish between the odd and even cases.

**Theorem 7.** Given a Banach space \( E \):

(E) Let \( n \) be an even integer. Then for \( F_o \in \bigotimes_{n,s} E \) and \( \phi \in B_E \) with \( \|F_o\| = \|\phi\| = 1 \) the following conditions are equivalent:

(a) The norm of \( \bigotimes_{n,s} E \) is Gâteaux differentiable at \( F_o \) with differential \( \phi' \) (resp. \( -\phi'' \)).
(b)  
(i) \( F_\phi(\phi^n) = 1 \) (resp. \( F_\phi(\phi^n) = -1 \)). 
(ii) There is a real number \( \alpha, -1 < \alpha < 1 \), so that \( F_\phi(\psi^n) > \alpha \) (resp. \( F_\phi(\psi^n) < \alpha \)) for all \( \psi \) in \( B_E \). 
(iii) If \( (\phi_k)_k \) is a sequence in the unit ball of \( E' \) so that \( F_\phi(\phi^n_k) \rightarrow 1 \) (resp. \( F_\phi(\phi^n_k) \rightarrow -1 \)) then \( (\phi_k)_k \) has a subsequence \( (\phi_{n_k})_k \), which converges weak* to \( \pm \phi \).

(c)  
(i) The functional \( \phi \) is unique in \( B_E \) modulo multiplication by \( -1 \) with the property that \( F_\phi(\phi^n) = 1 \) (resp. \( F_\phi(\phi^n) = -1 \)). 
(ii) There is a real number \( \alpha, -1 < \alpha < 1 \), so that \( F_\phi(\psi^n) > \alpha \) (resp. \( F_\phi(\psi^n) < \alpha \)) for all \( \psi \) in \( B_E \).

\textbf{(O)} Let \( n \) be an odd integer. Then for \( F_\phi \) in \( \hat{\otimes}_{n,s,E} \) and \( \phi \in B_E \) with \( \|F_\phi\| = \|\phi\| = 1 \) the following conditions are equivalent:

(a) The norm of \( \hat{\otimes}_{n,s,E} \) is Gâteaux differentiable at \( F_\phi \) with differential \( \phi^n \).

(b)  
(i) \( F_\phi(\phi^n) = 1 \).
(ii) If \( (\phi_k)_k \) is a sequence in the unit ball of \( E' \) so that \( F_\phi(\phi^n_k) \rightarrow 1 \) then \( (\phi_k)_k \) converges weak* to \( \phi \).

(c) The functional \( \phi \) is unique in \( B_E \) with the property that \( F_\phi(\phi^n) = 1 \).

\textbf{Proof.} We consider the case when \( n \) is even, the odd case is analogous but simpler as here we can use the fact that \( -\phi^n = (-\phi)^n \). We shall also assume that \( F_\phi(\phi^n) = 1 \) since the case when it equals \( -1 \) can be dealt with in the same way. By the Theorem of Šmul'yan, (a) implies (c) part (i). Furthermore, if (c) part (ii) (or equivalently (b) part (ii)) fails we can find a sequence \( (\phi_k)_k \) in \( B_E \) so that \( F_\phi(\phi^n_k) \) converges to \( -1 \). By weak* compactness of \( B_E \) we get a subnet \( (\phi_{n_k})_\beta \) of \( (\phi_k)_k \) and \( \psi \) in \( B_E \) so that \( \phi_{n_k} \) converges weak* to \( \psi \). It is easily shown that \( \phi^n \) converges \( \sigma(\hat{\otimes}_{n,s,E}, \hat{\otimes}_{n,s,E}) \) to \( \psi^n \). Therefore, we have

\[
F_\phi(\psi^n) = \lim_{\beta} F_{\phi_{n_k}}(\phi^n_{n_k}) = -1
\]

and hence \( F_\phi(-\psi^n) = 1 \) contradicting the fact that \( F_\phi \) exposes the unit ball of \( P_{\psi}(E) \) at \( \phi^n \).

Suppose (c) holds and that (b) part (iii) is not true. Then we can find a weak*-neighbourhood \( V \) of \( 0 \) in \( E' \) and a sequence \( (\phi_k)_{k=1}^\infty \) in \( B_E \) such that
we may suppose that \((\phi_n)_n\) converges weak* to some \(\hat{\phi}\) in \(B_E\), with \(\hat{\phi} \neq \pm \phi\). Since \(F_o\) is weakly continuous on \(\mathcal{P}(E)\), we have that
\[
F_o(\hat{\phi}^*) = \lim_{n \to \infty} F_o(\phi_n^*) = 1.
\]

Thus (c) is not true and we see that (c) implies (b).

Now suppose that (b) holds and that (a) is not true. Then we can find \(F\) in \(\bigotimes_{n \in \mathbb{N}} E\), \(\varepsilon > 0\) and a sequence \((\lambda_k)_k\) of positive real numbers converging to 0 so that
\[
|\|F_o + \lambda_k F\| - \|F_o\| - \lambda_k F(\phi^*)| \geq \varepsilon |\lambda_k|
\]
for every positive integer \(k\). We choose, for every \(k \in \mathbb{N}\), \(\phi_k\), a unit vector in \(E\), and \(\beta_k = \pm 1\) so that
\[
\beta_k(F_o + \lambda_k F)(\phi_k^*) = \|F_o + \lambda_k F\|.
\]

Then we have
\[
1 = \|F_o\| \geq \beta_k F_o(\phi_k^*) = \beta_k(F_o + \lambda_k F)(\phi_k^*) - \beta_k \lambda_k F(\phi_k^*) \geq \|F_o + \lambda_k F\| - |\lambda_k| \|F\|.
\]

Since \((\lambda_k)\) is a null sequence, \(\|F_o + \lambda_k F\| - |\lambda_k| \|F\|\) converges to \(\|F_o\|\) as \(k\) tends to \(\infty\). Thus we have that \(\beta_k F_o(\phi_k^*) \to 1\). It follows from part (ii) of (b) that \(\beta_k = 1\) except for possibly finitely many \(k\). Thus by part (iii) of (b) we have that \((\phi_k)_k\) has a subsequence \((\phi_{k_i})_i\), which converges weak* to \(\phi\). Therefore we have that
\[
\varepsilon |\lambda_{k_i}| \leq \|F_o + \lambda_{k_i} F\| - \|F_o\| - \lambda_{k_i} F(\phi^*)\|
\]
\[
= |(F_o + \lambda_{k_i} F)(\phi_{k_i}^*) - \|F_o\| - \lambda_{k_i} F(\phi^*)|
\]
\[
\leq |\lambda_{k_i}| \|F(\phi_{k_i}^*) - F(\phi^*)\|
\]
for all \(k_i\), which is impossible and shows that (c) implies (a).

Note that if \(n\) is even and \(F_o\) in \(\bigotimes_{n \in \mathbb{N}} E\) weak*-exposes the unit ball of \(\mathcal{P}(E)\) at \(\phi^*\) then \(-F_o\) weak*-exposes the unit ball of \(\mathcal{P}(E)\) at \(-\phi^*\).

The following Theorem allows us to calculate the weak*-exposed points of the unit ball of \(\mathcal{P}(E)\) for a considerable collection of Banach spaces.
**Theorem 8.** Let $E$ be a Banach space containing a sequence $(x_k)_k$ such that if $\phi \in E^*$, then $\phi \equiv 0$ if and only if $\phi(x_k) = 0$ for all $k$. Then for $n \geq 2$ the set of weak*-exposed points of $B_{\sigma(E^*, E)}$ contains $\{ \pm \phi^n; \phi \in E^* \text{ and } \|\phi\| = 1 \text{ and } \phi \text{ attains its norm} \}$. 

**Proof.** Let us suppose, without loss of generality, that each $x_k$ is a unit vector. Let $\phi$ be a norm attaining unit vector in $E$. If $F_\phi$ in $\bigotimes_{n, x, n} E$ weak*-exposes $\phi^n$ then $-F_\phi$ will weak*-expose $-\phi^n$. Thus it is sufficient to show that $\phi^n$ is exposed in $B(\mathcal{H})$. Let $x$ in $E$ be a unit vector such that $\phi(x) = 1$. Then $E$ is the topological direct sum of the span of $x$, $\text{sp}[x]$, and ker $\phi$. Therefore $E'$ is the topological direct sum of $\text{sp}[\phi]$ $(\text{ker } \phi)^\perp$ and $(\text{ker } \phi)' = \text{sp} \{x\}^\perp$, defined by $\psi \mapsto (\psi(x) \phi, \psi - \psi(x) \phi)$. Consider the sequence $\{y_k\}_{k=1}^\infty$ in ker $\phi$ given by $y_k = x_k - \phi(x_k) x$. If $\psi \in (\text{ker } \phi)' = \{x\}^\perp$ then $\psi(y_k) = 0$ for each $k$ implies that $\psi \equiv 0$.

If $n$ is even consider the vector

$$x \otimes x \otimes \cdots \otimes x - C \sum_{i=1}^\infty \frac{1}{i^2} y_i \otimes y_i \otimes \cdots \otimes y_i,$$

in $\bigotimes_{n, x, n} E$ where $C$ is chosen so that $C \sum_{i=1}^\infty \frac{1}{i^2} y_i \otimes y_i \otimes \cdots \otimes y_i$ has norm less than $1/2$. Then

$$\left(x \otimes x \otimes \cdots \otimes x - C \sum_{i=1}^\infty \frac{1}{i^2} y_i \otimes y_i \otimes \cdots \otimes y_i \right)(\phi^n) = (\phi^n)(x) = 1$$

and so $x \otimes x \otimes \cdots \otimes x - C \sum_{i=1}^\infty \frac{1}{i^2} y_i \otimes y_i \otimes \cdots \otimes y_i$ has norm 1. Our choice of $C$ ensures that $(x \otimes x \otimes \cdots \otimes x - C \sum_{i=1}^\infty \frac{1}{i^2} y_i \otimes y_i \otimes \cdots \otimes y_i)(\psi^n)$ can never be less than $-1/2$ for $\psi$ in $B_{\mathcal{H}}$ and so (b), part (ii) of Theorem 7 holds. Furthermore for $\psi \in B_{\mathcal{H}} \setminus \{0\}$ $\psi \neq \pm \phi$ we have $\psi(y_k) \neq 0$ for at least one $k$. This means that

$$\left(x \otimes x \otimes \cdots \otimes x - C \sum_{i=1}^\infty \frac{1}{i^2} y_i \otimes y_i \otimes \cdots \otimes y_i \right)(\psi^n) = (\psi(x))^n - C \sum_{i=1}^\infty \frac{1}{i^2} (\psi(y_i))^n < 1.$$

And thus by Theorem 7 $\phi^n$ is weak*-exposed by $x \otimes x \otimes \cdots \otimes x - C \sum_{i=1}^\infty \frac{1}{i^2} y_i \otimes y_i \otimes \cdots \otimes y_i$.

Denote by $n$ the natural linear projection of $\bigotimes_n E$ onto $\bigotimes_{n, x, n} E$ sending $v_1 \otimes v_2 \otimes \cdots \otimes v_n$ onto $\frac{1}{n} \sum_{\sigma \in S_n} v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)}$. If $n$ is odd we

consider the vector $x \otimes x \cdots \otimes x - D \sum_{j=1}^{n-1} (1/j^2) \pi(x \otimes y_j \otimes \cdots \otimes y_j) = 1$ in $\otimes_{n \in \mathbb{N}, E}$ where $D$ is again chosen so that $D \sum_{j=1}^{n-1} (1/j^2) \pi(x \otimes y_j \otimes \cdots \otimes y_j)$ has norm less than 1/2. As in the even case it can be shown that $x \otimes x \cdots \otimes x - D \sum_{j=1}^{n-1} (1/j^2) \pi(x \otimes y_j \otimes \cdots \otimes y_j)$ weak-*exposes the unit ball of $\mathcal{B}(E)$ at $\phi^*$.]

Note that every separable Banach space has a sequence with the property stated in Theorem 8. In particular, we have the following result.

**Corollary 9.** Let $E$ be a separable reflexive Banach space and $n$ be a positive integer. Then for $n \geq 2$ the set of weak-*exposed points of the unit ball of $\mathcal{B}(E)$ is equal to $\{ \pm \phi^*: \phi \in E^*, \| \phi \| = 1 \}$.

The Banach space $\ell_\infty$ also has the property mentioned in Theorem 8. Theorem 8 fails dramatically for the space $\ell_1(\Gamma)$ when $\Gamma$ is an uncountable index. It follows from Proposition 1 that the weak-*exposed points of $\mathcal{B}(\ell_1(\Gamma))$ are contained in $\{ \pm \phi^*: \phi \in \ell_\infty(\Gamma), \| \phi \| = 1 \}$. The space $\otimes_{n \in \mathbb{N}, \ell_\infty(\Gamma)}$ is isometric to $\mathcal{B}(\ell_\infty(\Gamma))$. It follows from [24] however that every $P$ in $\mathcal{B}(\ell_\infty(\Gamma))$ factors through $c_0(\Gamma_1)$ for some countable subset $\Gamma_1$. If $\phi \in c_0(\Gamma)$, and $P(\phi) = 1$ we will also have $P(\phi) = 1$ where

$$
\hat{\phi}_\gamma = \begin{cases} 
\phi_\gamma & \text{if } \gamma \in \Gamma_1; \\
\psi_\gamma \neq \phi_\gamma & \text{for at least one } \gamma \notin \Gamma_1
\end{cases}
$$

Thus the set of weak-*exposed points of $\mathcal{B}(\ell_\infty(\Gamma))$ is empty.

In [4], Aron and Berner show that any $n$-homogeneous polynomial $P$ on a Banach space $E$ can be extended to an $n$-homogeneous polynomial on the bidual of $E$. We denote this canonical extension of $P$ by $\tilde{P}$.

Using Theorem 7 we have the following characterisation of the weak-*exposed points of $\otimes_{n \in \mathbb{N}, E}$ when $\otimes_{n \in \mathbb{N}, E}$ does not contain a copy of $\ell_1$ and $E^*$ has the approximation property.

**Corollary 10.** Let $E$ be a Banach space so that $E^*$ has the approximation property and $\mathcal{B}(\ell_\infty(\Gamma))$ does not contain a copy of $\ell_1$. Let $n$ be an even integer. Then for $P_n$ in $\mathcal{B}(E)$ and $x \in B_{\ell_\infty}$ with $\|P_n\| = \|x\| = 1$ the following conditions are equivalent:

(a) The norm of $\mathcal{B}(\ell_\infty(\Gamma))$ is Gâteaux differentiable at $P_n$ with differential $\delta_n$ (resp. $-\delta_n$).

(b) (i) $\tilde{P}_\gamma(x) = 1$ (resp. $\tilde{P}_\gamma(x) = -1$).

(ii) There is a real number $\alpha$, $-1 < \alpha < 1$, so that $\tilde{P}_\gamma(y) > \alpha$ (resp. $\tilde{P}_\gamma(y) < \alpha$) for all $y$ in $B_{\ell_\infty}$. 

(iii) If \((x_k)_k\) is a sequence in the unit ball of \(E^*\) so that \(\hat{P}_f(x_k) \to 1\) (resp. \(\hat{P}_f(x_k) \to -1\)) then \((x_k)_k\) has a subsequence \((x_{k_l})_l\) which converges weak* to \(\pm x\).

(c) The vector \(x\) is unique in \(B_{E^*}\) modulo multiplication by \(-1\) with the property that \(\hat{P}_f(x) = 1\) (resp. \(\hat{P}_f(x) = -1\)).

(ii) There is a real number \(\alpha, -1 < \alpha < 1\), so that \(\hat{P}_f(y) = \alpha\) (resp. \(\hat{P}_f(y) = -\alpha\)) for all \(y\) in \(B_{E^*}\).

(O) Let \(n\) be an odd integer. Then for \(P_o\) in \(P_\text{id}(E)\) and \(x \in B_{E^*}\) with \(\|P_o\| = \|x\| = 1\) the following conditions are equivalent:

(a) The norm of \(P_\text{id}(E)\) is Gâteaux differentiable at \(P_o\) with differential \(\delta_x\).

(b) The vector \(x\) is unique in \(B_{E^*}\) with the property that \(\hat{P}_f(x) = 1\).

As with extreme points, the weak*-exposed points of \(\mathcal{B}_I(E)\) and \(\bigotimes_{n,s} E\) behave very differently to those of \(\mathcal{L}(E)\) and \(\bigotimes_{n,s} E\). Theorem 2.3 of [30] shows that the set of weak*-exposed points of the unit ball of \(\mathcal{L}(E)\) is

\[ \{ \phi_1 \phi_2 \cdots \phi_n; \phi_i \text{ is weak*-exposed in the unit ball of } E \} \]

which is different to the type of points we found in Theorem 7.

4. WEAK*-STRONGLY EXPOSED AND STRONGLY EXPOSED POINTS OF THE UNIT BALL OF SPACES OF INTEGRAL POLYNOMIALS AND \(n\)-FOLD SYMMETRIC TENSORS

Let us now turn to strongly exposed points and weak*-strongly exposed of \(\mathcal{B}_I(E)\) and \(\bigotimes_{n,s} E\). The proofs of our main theorems are very closely modeled on Theorem 1.1 of [30].

**Theorem 11.** Given a Banach space \(E\):

(E) Let \(n\) be an even integer. Then for \(F_o\) in \(\bigotimes_{n,s} E\) and \(\phi \in B_{E^*}\) with \(\|F_o\| = \|\phi\| = 1\) the following conditions are equivalent:
(a) The norm of $\hat{\otimes}_{n,s,E}$ is Fréchet differentiable at $F_o$ with differential $\phi^*$ (resp. $-\phi^*$).

(b) 

(i) $F_o(\phi^*) = 1$ (resp. $F_o(\phi^*) = -1$).

(ii) There is a real number $\alpha, -1 < \alpha < 1$, so that $F_o(\psi^*) > \alpha$ (resp. $F_o(\psi^*) < \alpha$) for all $\psi$ in $B_E$.

(iii) If $(\phi_k)_k$ is a sequence in the unit ball of $E'$ so that $F_o(\phi_k^*) \to 1$ (resp. $F_o(\phi_k^*) \to -1$) then $(\phi_k)_k$ has a subsequence $(\phi_{k_i})_i$ which converges in norm to $\pm \phi^*$.

(c) $F_o$ weak*-strongly exposes the unit ball of $\mathcal{P}(E)$ at $\phi^*$ (resp. $-\phi^*$).

(0) Let $n$ be an odd integer. Then for $F_o$ in $\hat{\otimes}_{n,s,E}$ and $\phi \in B_E$ with \[\|F_o\| = \|\phi\| = 1\] the following conditions are equivalent:

(a) The norm of $\hat{\otimes}_{n,s,E}$ is Fréchet differentiable at $F_o$ with differential $\phi^*$.

(b) 

(i) $F_o(\phi^*) = 1$.

(ii) If $(\phi_k)_k$ is a sequence in the unit ball of $E'$ so that $F_o(\phi_k^*) \to 1$ then $(\phi_k)_k$ has a subsequence $(\phi_{k_i})_i$, which converges in norm to $\phi$.

(c) $F_o$ weak*-strongly exposes the unit ball of $\mathcal{P}(E)$ at $\phi^*$.

Proof. Again we will consider the even case and assume that $F_o(\phi^*) = 1$. The alternative cases can all be dealt with in an analogous fashion. By the Theorem of Šmul'yan (a) and (c) are equivalent. Suppose (c) holds. By Theorem 7 parts (i) and (ii) of (b) hold. Suppose $(\phi_k)_k$ is a sequence in $B_E$ such that $F_o(\phi_k^*) \to 1$. From the definition of weak*-strongly exposed point $(\phi_k^*)_k$ converges in norm to $\phi^*$. In particular, $(\phi_k^*)_k$ is norm Cauchy so, applying Lemma 1.2 of [30], we can find a subsequence $(\phi_{k_i})_i$ of $(\phi_k)_k$, which is norm Cauchy and thus converges in norm to $\pm \phi$.

Now suppose that (b) holds and (a) is false. Then we can find $\epsilon > 0$ and a sequence $(F_k)_k$ in $\hat{\otimes}_{n,s,E}$ converging to 0 so that

$$|\|F_o + F_k\| - \|F_o\| - F_o(\phi^*)| \geq \epsilon \|F_k\|$$

for every positive integer $k$. We choose, for every $k \in \mathbb{N}$, $\phi_k$, a unit vector in $E'$, and $\beta_k = \pm 1$ so that

$$\beta_k(F_o + F_k)(\phi_k^*) > \|F_o + F_k\| - \frac{1}{k} \|F_k\|.$$
Thus we have
\[
1 = \|F_o\| \\
\geq \beta_k F_o(\phi_k^n) \\
= \beta_k (F_o + F_k)(\phi_k^n) - \beta_k F_k(\phi_k^n) \\
> \|F_o + F_k\| - \frac{1}{k} \|F_k\| - \|F_k\|.
\]

Since \(\|F_o + F_k\| - \frac{1}{k} \|F_k\| - \|F_k\|\) converges to \(\|F_o\|\) as \(k\) tends to \(\infty\) we have that \(\beta_k F_o(\phi_k^n) \to 1\). Part (ii) of (b) now implies that \(\beta_k = 1\) for all but finitely many \(k\) and so we may assume that \(F_o(\phi_k^n)\) converges to 1. Part (iii) of (b) now implies \((\phi_k)_k\) has a subsequence \((\phi_{k_i})_i\) which converges to \(\pm \phi\). Thus we have
\[
e \|F_{k_i}\| < \|F_o + F_{k_i}\| - \|F_o\| - F_{k_i}(\phi^n) \\
< (F_o + F_{k_i})(\phi_{k_i}^n) + \frac{1}{k_i} \|F_{k_i}\| - \|F_o\| - F_{k_i}(\phi^n) \\
\leq \left| F_o(\phi_{k_i}^n) - F_o(\phi^n) + \|F_{k_i}\| \left( \|\phi_{k_i}^n - \phi^n\| + \frac{1}{k_i} \right) \right| \\
\leq \|F_{k_i}\| \left( \|\phi_{k_i}^n - \phi^n\| + \frac{1}{k_i} \right)
\]
and we arrive at a contradiction.

When \(E\) is a Banach space such that \(\mathcal{B}_n(^n E)\) does not contain a copy of \(\ell_1\) and \(E^*\) has the approximation property we can translate Theorem 11 into the language of symmetric \(n\)-fold tensor products as follows.

**Corollary 12.** Let \(E\) be a Banach space so that \(E^*\) has the approximation property and \(\ell_1\) is not a subspace of \(\mathcal{B}_n(^n E)\).

\(E\) Let \(n\) be an even integer. Then for \(P_o\) in \(\mathcal{B}_n(^n E)\) with \(\|P_o\| = 1\) and \(x \in B_{E'}\) the following conditions are equivalent:

(a) The norm of \(\mathcal{B}_n(^n E)\) is Fréchet differentiable at \(P_o\) with differential \(\delta_o\) (resp. \(-\delta_o\)).

(b) 
(i) \(\bar{P}_o(x) = 1\) (resp. \(\bar{P}_o(x) = -1\)).

(ii) There is a real number \(\alpha\), \(-1 < \alpha < 1\), so that \(\bar{P}_o(y) > \alpha\) (resp. \(\bar{P}_o(y) < \alpha\)) for all \(y\) in \(B_{E'}\).
(iii) If \((x_k)_k\) is a sequence in the unit ball of \(E^*\) so that \(\tilde{P}_o(x_k) \to 1\) (resp. \(\tilde{P}_o(x_k) \to -1\)) then \((x_k)_k\) has a subsequence \((x_{k_i})_i\), which converges in norm to \(\pm x\).

(c) \(\tilde{P}_o\) weak*-strongly exposes the unit ball of \(\hat{\otimes}_{n,s,E} E^*\) at \(\delta_x\) (resp. \(-\delta_x\)).

(\(O\)) Let \(n\) be an odd integer. Then for \(P_o\) in \(\mathcal{P}_o(E)\) with \(\|P_o\| = 1\) and \(x \in B_E\), the following conditions are equivalent:

(a) The norm of \(\mathcal{P}_o(E)\) is Fréchet differentiable at \(P_o\) with differential \(\delta_x\).

(b) (i) \(P_o(x) = 1\).

(ii) If \((x_k)_k\) is a sequence in the unit ball of \(E^*\) so that \(\tilde{P}_o(x_k) \to 1\) then \((x_k)_k\) has a subsequence \((x_{k_i})_i\) which converges in norm to \(x\).

(c) \(\tilde{P}_o\) weak*-strongly exposes the unit ball of \(\hat{\otimes}_{n,s,E} E^*\) at \(\delta_x\).

**Corollary 13.** Let \(E\) be a Banach space such that \(E^*\) has the approximation property and \(\mathcal{P}_o(E)\) does not contain a copy of \(l_1\). If \(\varphi\) in \(E^*\) weak*-strongly exposes the unit ball of \(E^*\) at \(x\) then for \(n \geq 2\), \(\varphi^n\) (resp. \(-\varphi^n\)) weak*-strongly exposes the unit ball of \(\hat{\otimes}_{n,s,E} E^*\) at \(\delta_x\) (resp. \(-\delta_x\)).

Analogous to Corollary 12 we have the following characterisation of strongly exposed points of \(\hat{\otimes}_{n,s,E}\).

**Theorem 14.** Let \(E\) be a Banach space.

(\(E\)) Let \(n\) be an even integer. Then for \(P_o\) in \(\mathcal{P}(E)\) and \(x \in B_E\) with \(\|P_o\| = \|x\| = 1\) the following conditions are equivalent:

(a) The norm of \(\mathcal{P}(E)\) is Fréchet differentiable at \(P_o\) with differential \(\delta_x\) (resp. \(-\delta_x\)).

(b) (i) \(P_o(x) = 1\) (resp. \(P_o(x) = -1\)).

(ii) There is a real number \(\alpha, -1 < \alpha < 1\), so that \(P_o(y) > \alpha\) (resp. \(P_o(y) < \alpha\)) for all \(y\) in \(B_E\).

(iii) If \((x_k)_k\) is a sequence in the unit ball of \(E\) so that \(P_o(x_k) \to 1\) (resp. \(P_o(x_k) \to -1\)) then \((x_k)_k\) has a subsequence \((x_{k_i})_i\) which converges in norm to \(\pm x\).

(c) \(\tilde{P}_o\) strongly exposes the unit ball of \(\hat{\otimes}_{n,s,E} E\) at \(\delta_x\) (resp. \(-\delta_x\)).
Let \( n \) be an odd integer. Then for \( P_\alpha \in \mathcal{P}(^*E) \) and \( x \in B_E \) with \( \|P_\alpha\| = \|x\| = 1 \) the following conditions are equivalent:

(a) The norm of \( \mathcal{P}(^*E) \) is Fréchet differentiable at \( P_\alpha \) with differential \( x \).

(b)

(i) \( P_\alpha(x) = 1 \).

(ii) If \( (x_k) \) is a sequence in the unit ball of \( E \) so that \( P_\alpha(x_k) \to 1 \) then \( (x_k) \) has a subsequence \( (x_{k_i}) \) which converges in norm to \( x \).

(c) \( P_\alpha \) strongly exposes the unit ball of \( \dot{\mathcal{D}}_{n,s,E} \) at \( x \).

Proof. Once more we will only show the even part and the case where the differential of \( P_\alpha \) in \( x \). Since the set of strongly exposed points of \( \dot{\mathcal{D}}_{n,s,E} \) is equal to the set of weak*-strongly exposed points of \( \mathcal{P}(^*E) \), using the Theorem of Šmul'yan, we see that (c) is equivalent to (a). Now suppose (c) holds. Then \( P_\alpha(x) \) is clearly 1. If part (ii) of (b) fails then we can choose a sequence \( (x_k) \) in \( B_E \) so that \( P_\alpha(x_k) \to 1 \). From the definition of a strongly exposed point we conclude that \( \delta_\omega \) converges in norm to \(-\delta_\omega\). By Lemma 1.2 of [30] we can find a subsequence \( (x_{k_i}) \), which converges to some \( \tilde{x} \) in \( E \). But \( \dot{P}(\tilde{x} - \delta_\omega) = 1 \) contradicting (c). Suppose \( (x_k) \) is a sequence in \( B_{E^*} \) so that \( P_\alpha(x_k) \to 1 \). It follows from the definition of strongly exposed point that \( (\delta_\omega(x_k)) \) converges in norm to \( \delta_\omega \). In particular, \( (\delta_\omega(x_k)) \) is norm Cauchy, so applying Lemma 1.2 of [30] we can find a subsequence \( (x_{k_i}) \) of \( (x_k) \), which is norm Cauchy. It is clear that this subsequence must converges to \( \pm x \) in norm and so (c) implies (b).

Now suppose (b) holds and (a) is false. Then we can find \( \varepsilon > 0 \) and a sequence \( (P_k) \) in \( \mathcal{P}(^*E) \) converging to 0 so that

\[
|\|P_\alpha + P_k\| - \|P_\alpha\| - P_\alpha(x)| \geq \varepsilon \|P_k\|
\]

for every positive integer \( k \). Choose, for every \( k \in \mathbb{N} \) a unit vector in \( E \), \( x_k \), and \( \beta_k = \pm 1 \) so that

\[
\beta_k(P_\alpha + P_k)(x_k) \geq \|P_\alpha + P_k\| - \frac{1}{k} \|P_k\|.
\]

Thus we have

\[
1 = \|P_\alpha\| \\
\geq \beta_k P_\alpha(x_k) \\
= \beta_k(P_\alpha + P_k)(x_k) - \beta_k P_k(x_k) \\
> \|P_\alpha + P_k\| - \frac{1}{k} \|P_k\| - \|P_k\|.
\]
Since \( \|P_\sigma + P_k\| - \frac{1}{k} \|P_\sigma\| - \|P_k\| \) converges to \( \|P_\sigma\| \) as \( k \) tends to \( \infty \) we have that \( \beta_k P_\sigma(x_k) \to 1 \). Applying part (ii) of (b) we see that \( \beta_k \) is 1 except for possibly finitely many \( k \). Thus by part (iii) of (b) we get a subsequence \( (x_{k_i})_i \) of \( (x_k)_k \) so that \( x_{k_i} \) converges in norm to \( x \). Hence we have

\[
\epsilon \|P_{k_i}\| \leq \|P_\sigma + P_{k_i}\| - \|P_\sigma\| - P_{k_i}(x)
\]

\[
\leq \left( \|P_\sigma + P_{k_i}(x_{k_i}) + \frac{1}{k_i} \|P_{k_i}\| - \|P_\sigma\| - P_{k_i}(x) \right)
\]

\[
\leq \|P_{k_i}\| \left( \|\delta_{x_k} - \delta_x\| + \frac{1}{k_i} \right)
\]

for all \( i \), a contradiction. Thus (b) implies (a).

**Corollary 15.** If \( \phi \) in \( E' \) strongly exposes the unit ball of \( E \) at \( x \) then \( \phi^* \) (resp., \( -\phi^* \)) strongly exposes the unit ball of \( \otimes_{n,n,n} E \) at \( \delta_x \) (resp., \( -\delta_x \)).

Once again the weak*-strongly exposed and strongly exposed points of \( \mathcal{B}(E) \) and \( \otimes_{n,n,n} E \) are very different from those of \( \mathcal{B}(E) \) and \( \otimes_{n,n,n} E \). In [30] Ruess and Stegall show that the set of weak*-strongly exposed points of the unit ball of \( \mathcal{B}(E) \) is the set \( \{ \phi_1 \phi_2 \cdots \phi_n; \phi_i \text{ is weak*-strongly exposed in the unit ball of } E \} \) while Heinrich [21] and Ruess and Stegall [30] show that the set of strongly exposed points of the unit ball of \( \otimes_{n,n,n} E \) is the set \( \{ x_1 \otimes x_2 \otimes \cdots \otimes x_n; x_i \text{ strongly exposes the unit ball of } E \} \). These sets are different to those described above.

For \( \Gamma \) be an uncountable index set it follows from Theorem 1 of [24] that any \( P \in \mathcal{B}^{*}(c_0(\Gamma)) \) depends only on countably many indices and hence factors through \( c_0(\Gamma_1) \) for some countable set \( \Gamma_1 \). Thus if \( x \in c_0(\Gamma) \) with \( P(x) = 1 \) then \( P(\tilde{x}) = 1 \) where

\[
\tilde{x}_\gamma = \begin{cases} x_\gamma & \text{if } \gamma \in \Gamma_1; \\ y_\gamma \neq x_\gamma & \text{for at least one } \gamma \not\in \Gamma_1.
\end{cases}
\]

Thus \( \otimes_{n,n,n} c_0(\Gamma) \) has no strongly exposed points.

An \( n \)-homogeneous polynomial \( P \) on a Banach space \( E \) is said to be a separating polynomial if \( \inf \{ P(x) : \|x\| = 1 \} > 0 \). If \( P \) is a separating polynomial on \( E \) and \( (x_k)_k \subseteq E \) then \( P(x_k) \to 0 \) implies that \( x_k \) converges to 0. It is shown in [8] that if \( E \) admits a separating \( n \)-homogeneous polynomial then \( E \) is super-reflexive, has type 2 and has cotype \( n \). (Note \( n \) must be even.) In particular, \( E \) admits a separating 2-homogeneous polynomial if and only if \( E \) is isomorphic to a Hilbert space. Other examples of Banach spaces with separating \( n \)-homogeneous polynomial are to be found in [16]. The property of admitting a separating \( n \)-homogeneous polynomial is inherited by subspaces.
Proposition 16. Let $E$ be a Banach space which admits a separating $n$-homogeneous polynomial $P$. Then for all $m \geq n$ the set of strongly exposed points of the unit ball of $\mathcal{P}_{m,n,E}$ is the set $\{ \pm \delta_x : \|x\| = 1 \}$.

Proof. Given $x \in E$ choose $\phi$ in $B_{E^*}$ so that $\phi(x) = 1$. Let $q$ be the canonical projection of $E$ onto the kernel of $\phi$, $(q(y) = y - \phi(y)x).$ Define $Q$ in $\mathcal{P}(mE)$ by

$$Q(y) = \phi^m(y) - \phi^{m-n}(y) \frac{1}{2 \|P\|} P(q(y))$$

$$= \phi^{m-n}(y) \left( \frac{1}{2 \|P\|} P(q(y)) \right).$$

Then $Q(x) = 1$. If $m$ is even then $Q(y) \geq -\frac{1}{2} \|y\|$ for $y \in B_E$. Furthermore, if $(x_k) \in B_E$ and $Q(x_k) \to 1$ then $\phi(x_k)^* \to 1$ and $P(q(x_k)) \to 0$. By choosing a subsequence (when $n$ is even) we may suppose that $\phi(x_k)$ converges to $\pm 1$. Since $P$ is a separating polynomial $q(x_k) = x_k - \phi(x_k)x \to 0$. Thus

$$\lim_{k \to \infty} x_k = \lim_{k \to \infty} \frac{1}{2 \|P\|} P(q(x_k)) = \pm x$$

and so by Theorem 14 $Q$ strongly exposes $\delta_x$. In addition we see that $-Q$ will strongly expose $-\delta_x$. 

5. APPLICATIONS TO SPACES OF HOMOGENEOUS POLYNOMIALS

A Banach space $E$ is said to be smooth if its norm is Gâteaux differentiable at every non-zero point while it is said to be rotund or strictly convex if every point of the unit sphere is an extreme point. In [31] it is shown that $\mathcal{P}(nE)$ is not rotund when $n \geq 2$ for $E$ any Banach space with a norm-one projection onto a subspace of dimension at least 2 which is a dual space. We now show that this result holds in general.

Proposition 17. Let $E$ be a Banach space of dimension at least 2. Then for $n \geq 2$ the spaces $\mathcal{P}_{n,n,E}$ and $\mathcal{P}(nE)$ are neither smooth nor rotund.

Proof. Since $\mathcal{P}_{n,n,E}$ is a closed subspace of $\mathcal{P}(nE)$ and smoothness and rotundness are hereditary properties it suffices to prove the result for $\mathcal{P}_{n,n,E}$. Let us suppose that $\mathcal{P}_{n,n,E}$ is smooth. Then it would follow from the result of Šmul'yan that every norm-attaining $P$ in the unit sphere
of $\mathcal{A}(E)$ is a weak* exposed point of the unit ball of $\mathcal{A}(E)$. Applying the Bishop–Phelps Theorem it now follows that $\mathcal{A}(E)$ is the norm closure of multiples of its weak*-exposed points. Proposition 1 and Lemma 1.2 of [29] show that this set is contained in $\{ \pm \phi^*: \phi \in E^*, \|\phi\| = 1\}$. This implies that $\mathcal{A}(E) = \{ \pm \phi^*: \phi \in E^*\}$ which is impossible and so $\mathcal{A}_{n,x,x} E$ cannot be smooth.

To see that $\mathcal{A}_{n,x,x} E$ cannot be rotund choose $x, y \in E$ and $\phi \in E^*$ so that

$$\|x\| = \|y\| = \|\phi\| = \phi(x) = 1, \quad \phi(y) = 0.$$ 

If $n$ is even let $F = x \otimes x \otimes \cdots \otimes x - \frac{1}{2} y \otimes y \otimes \cdots \otimes y$ and $G = x \otimes x \otimes \cdots \otimes x - \frac{1}{2} \pi(x \otimes y \otimes \cdots \otimes y)$ while if $n$ odd let $F = x \otimes x \otimes \cdots \otimes x - \frac{1}{2} \pi(x \otimes y \otimes \cdots \otimes y)$ and $G = x \otimes x \otimes \cdots \otimes x - \frac{1}{2} \pi(x \otimes y \otimes \cdots \otimes y)$ where $\pi$ is as defined as in Theorem 8. Then $\|F\| = \|G\| = \|F + G\| = 1$ and so $\mathcal{A}_{n,x,x} E$ cannot be rotund.

A bounded net $(P_n)_n$ of $n$-homogeneous polynomials on a Banach space $E$ is said to converge pointwise to $P$ in $\mathcal{A}(E)$ if $P_n(x)$ converges to $P(x)$ for every $x$ in $E$. We shall say that the bounded net $(P_n)_n$ converges Aron–Berner pointwise to $P$ if $\tilde{P}_n(z)$ converges to $\tilde{P}(z)$ for every $z$ in $E^*$. In [5] it shown that if $E$ has the metric approximation property then the pointwise and Aron–Berner pointwise topologies coincide at $P$ on the unit sphere of $\mathcal{A}(E)$ if and only if $P$ has a unique norm preserving extension to $E^*$. Given a Banach space $E$ we denote by $\mathcal{A}(E)$ the space of all approximable $n$-homogeneous polynomials on $E$ defined as the closure of the space of finite type polynomials in the norm topology. It is easy to see that $\mathcal{A}(E)$ may be identified with $\mathcal{A}_{n,x,x} E$. We note that part (a) of the following Proposition was proved in [13].

**Proposition 18.** Let $E$ be a Banach space and $n$ be a positive integer.

(a) A bounded sequence $(P_k)_k$ in $\mathcal{A}(E)$ converges weakly to $P$ in $\mathcal{A}(E)$ if and only if $(P_k)_k$ converges Aron–Berner pointwise to $P$.

(b) A bounded subset of $\mathcal{A}(E)$ is weakly relatively compact if and only if it is relatively countably compact for the Aron–Berner pointwise topology.

**Proof.** We showed in Theorem 1 that the extreme points of the unit ball of $\mathcal{A}(E)' = \mathcal{A}(E^*)$ are contained in the set $\{ \pm \delta_z: z \in E^*, \|z\| = 1\}$. Part (a) now follows from a result of Rainwater, [28], while part (b) is a consequence of a Theorem of Bourgain and Talagrand, [7].
When $E'$ has the approximation property $\mathcal{P}_d(^nE)$ coincides with $\mathcal{P}_d(^nE)$ and thus we have

**Corollary 19.** Let $E$ be a Banach space so that $E'$ has the approximation property and let $n$ be a positive integer.

(a) A bounded sequence $(P_k)_k$ in $\mathcal{P}_d(^nE)$ converges weakly to $P$ in $\mathcal{P}_d(^nE)$ if and only if $(P_k)_k$ converges Aron–Berner pointwise to $P$.

(b) A bounded subset of $\mathcal{P}_d(^nE)$ is weakly relatively compact if and only if it is relatively countably compact for the Aron–Berner pointwise topology.

Given Banach spaces $E$ and $F$ we denote by $\mathcal{L}(E, F)$ (resp. $\mathcal{K}(E, F)$) the space of all continuous linear (resp. compact linear) maps from $E$ into $F$. When $E$ and $F$ are reflexive Banach space then Holub, [23], shows that $\mathcal{L}(E, F) = \mathcal{K}(E, F)$ is reflexive with the converse being true when either $E$ or $F$ has the approximation property. Furthermore, Feder and Saphar [17] (see also [30]) show that given reflexive Banach spaces $E$ and $F$ then either $\mathcal{K}(E, F)$ is reflexive or is not isometric to a dual space. The space of approximable polynomials, $\mathcal{P}_d(^nE)$, is the polynomial analogue of $\mathcal{K}(E, F)$ and we now prove a polynomial Feder–Saphar type Theorem.

**Lemma 20.** Let $E$ be a weakly sequentially complete Banach space. Then each weak*-exposed point of $B_{E^*}$ belongs to $B_E$ and is an exposed point of $B_{E^*}$.

**Proof.** Suppose that $\phi$ in $E'$ weak*-exposes the unit ball of $E^*$ at $z_\circ$. Since $\|\phi\| = 1$ there is a sequence $(z_k)_k$ in $E$ so that $\phi(z_k) \to 1$. As $z_n$ is weak*-exposed by $\phi$, $(z_k)_k$ converges weak* to $z_\circ$. Since $E$ is weakly sequentially complete we conclude that $z_\circ$ must belongs to $E$.

**Theorem 21.** Let $E$ be a reflexive Banach spaces with one of the following conditions holding:

(a) The unit sphere of $E'$ has a Fréchet differentiable norm.

(b) $E$ is separable and $\mathcal{P}_d(^nE')$ is weakly sequentially complete.

Then $\mathcal{P}_d(^nE)$ is either reflexive or not isometric to a dual space.

**Proof.** (a) Suppose there is a Banach space $Z$ so that $Z'$ is isometric to $\mathcal{P}_d(^nE)$. The set of strongly exposed points of $B_{Z^*}$ is equal to the set of weak*-strongly exposed points of the unit ball of $\mathcal{P}_d(^nE')$. Since the norm is Fréchet differentiable it follows from Corollary 15 that this set is equal...
to \( \{ \pm \phi^*; \phi \in E, \| \phi \| = 1 \} \). Since \( \{ \pm \phi^*; \phi \in E, \| \phi \| = 1 \} \) is the set of extreme points of \( B_{Z^*} \) and \( Z \) has RNP we have that

\[
B_{Z^*} = \tilde{\Gamma} (\text{strongly exposed points of } B_{Z^*})
= \tilde{\Gamma} \{ \pm \phi^*; \phi \in E, \| \phi \| = 1 \} = B_{Z^*},
\]

and thus \( Z \) is reflexive.

(b) Again we suppose there is a Banach space \( Z \) so that \( Z' \) is isometric to \( \mathcal{B}(E)^* \). Being a closed subspace of \( \mathcal{B}(E) \), \( Z \) is weakly sequentially complete and therefore by Lemma 20 the set of exposed points of \( B_{Z^*} \) contains the set of weak*-exposed points of the unit ball of \( \mathcal{B}(E^*_p) \). By Corollary 9 this is equal to the set \( \{ \pm \phi^*; \phi \in E, \| \phi \| = 1 \} \) and so the set of exposed points of \( B_{Z^*} \) is equal to \( \{ \pm \phi^*; \phi \in E, \| \phi \| = 1 \} \). Thus as \( Z \) as RNP we have that

\[
B_{Z^*} = \tilde{\Gamma} (\text{exposed points of } B_{Z^*}) \supseteq \tilde{\Gamma} \{ \pm \phi^*; \phi \in E, \| \phi \| = 1 \} = B_{Z^*},
\]

which proves that \( Z \) is reflexive.

In [32] Schatten shows when \( H \) is a Hilbert space \( \mathcal{K}(H, H) \) is not isometric to a dual space. We have the following Corollary, the first part of which is well known.

**Corollary 22.** If \( n < p \) then \( \mathcal{A}(\ell_p^n) \) is reflexive while if \( n \geq p \), \( \mathcal{A}(\ell_p^n) \) is not isometric to a dual space.

Condition (b) of Theorem 21 holds in particular when \( E \) is a reflexive Banach space with an unconditional finite dimensional decomposition (see [11] and [22]). Pisier [27] shows that the projective tensor product of two weakly sequentially complete Banach spaces need not be weakly sequentially complete. We do not know if the projective tensor product of a reflexive Banach space and a weakly sequentially complete space or even of two reflexive space is weakly sequentially complete.

We conclude the paper with a Choquet Representation Theorem for spaces of \( n \)-homogeneous polynomials. A subset \( A \) of a real Banach space \( X \) is said to be symmetric if \( A = -A \). We define the symmetric \( \sigma \)-algebra on \( (B_{E^*}, \sigma(E^*, E)) \) as the sub \( \sigma \)-algebra of symmetric Borel subsets of \( (B_{E^*}, \sigma(E^*, E)) \).

**Proposition 23.** Let \( E \) be a Banach space so that \( E^* \) is separable and \( E^* \) has the approximation property. If \( n \) is an odd (resp. even) integer then for any \( x \) in \( B_{E^*} \) there is a regular probability measure (resp. regular measure of
variation at most 2) \( \mu_x \) on \((B_{E'}, \sigma(E'', E'))\) (resp. on the symmetric \( \sigma \)-algebra on \((B_{E'}, \sigma(E'', E'))\)) with support contained in the unit sphere of \( E'' \) so that

\[
\tilde{P}(x) = \int_{B_{E'}} \tilde{P}(y) \, d\mu_x(y)
\]

for all \( P \) in \( \mathcal{P}_a(*E) \).

**Proof.** Since \( E' \) is separable \( B_{\mathcal{P}_1(*E')} \) endowed with the weak* topology is a compact convex metrizable subset of \((\mathcal{P}_1(*E'), \sigma(\mathcal{P}_1(*E'), \mathcal{P}_a(*E'))\).

Consider \( x \) in \( B_{E'} \). By the Choquet Representation Theorem (see [26]) there exists a probability measure \( \tilde{\mu}_x \) on \((B_{\mathcal{P}_1(*E')}, \sigma(\mathcal{P}_1(*E'), \mathcal{P}_a(*E'))\) with support contained in \( \text{ext} B_{\mathcal{P}_1(*E')} \subseteq \{ \pm \delta_y : y \in E'', \|y\| = 1 \} \) so that

\[
\tilde{P}(x) = \int_{B_{E'}} \tilde{P}(y) \, d\tilde{\mu}_x(\delta_y)
\]

(resp.

\[
\tilde{P}(x) = \int_{B_{E'}} \tilde{P}(y) \, d\tilde{\mu}_x(\delta_y) + \int_{B_{E'}} -\tilde{P}(y) \, d\tilde{\mu}_x(-\delta_y)
\]

for all \( P \) in \( \mathcal{P}_a(*E) \). It is readily shown that the map \( E'' \to \mathcal{P}_1(*E'), y \to \delta_y \) is \( \sigma(E'', E') - \sigma(\mathcal{P}_1(*E'), \mathcal{P}_a(*E')) \) continuous.

If \( n \) is odd this map is injective and therefore is a homeomorphism onto its image. In particular \( \mu_x(A) = \tilde{\mu}_x(\delta_y : y \in A) \) defines a probability measure on \( B_{E'} \) with support contained in the unit sphere of \( E'' \) so that

\[
\tilde{P}(x) = \int_{B_{E'}} \tilde{P}(y) \, d\mu_x(y)
\]

for all \( P \) in \( \mathcal{P}_a(*E) \).

If \( n \) is even, then

\[
\tilde{P}(x) = \int_{B_{E'}} \tilde{P}(y) \, d\mu_x(\delta_y) + \int_{B_{E'}} -\tilde{P}(y) \, d\mu_x(-\delta_y)
\]

\[
= \int_{B_{E'}} \tilde{P}(y) \, \tilde{\nu}_x(\delta_y),
\]

where \( \tilde{\nu}_x(A) = \tilde{\mu}_x(A) - \tilde{\mu}_x(-A) \) is a measure on \( \{ \delta_y : y \in E'' \} \). Given a symmetric Borel set \( A \) in \((B_{E'}, \sigma(E'', E'))\) the set \( \{ \delta_y : y \in A \} \) is Borel in \((B_{\mathcal{P}_1(*E')}, \sigma(\mathcal{P}_1(*E'), \mathcal{P}_a(*E'))\). If we let \( \mu_x(A) = \tilde{\nu}_x(\delta_y : y \in A) \) we obtain a
measure of finite variation $\mu_x$ on the symmetric $\sigma$-algebra of $(B_{E^*}, \sigma(E^*, E'))$ with support contained in the unit sphere of $E^*$ so that

$$\tilde{P}(x) = \int_{B_{E^*}} P(y) d\mu_x(y)$$

for all $P$ in $\mathcal{P}_w^n(E^*)$.

Note added in proof. We have recently learnt that D. Carando and V. Dimant (J. Math. Anal. Appl. 241 (2000), 107–201) (Duality in spaces of nuclear and integral polynomials) have also obtained Theorem 3.

REFERENCES


