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Semidirect products of internal groupoids

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ABSTRACT

We give a characterization of those finitely complete categories with initial object and pushouts of split monomorphisms that admit categorical semidirect products. As an application we examine the case of groupoids with fixed set of objects. Further, we extend this to the internal case.

1. Introduction

A categorical notion of semidirect product has been introduced in [9], for categories where pulling back a split epimorphism along any morphism gives a monadic functor. Hence a natural environment for studying this construction is that of protomodular categories [7], where the functors mentioned above are only required to be conservative. Whenever the category is also pointed, the existence of semidirect products yields canonically the classical equivalence between split epimorphic pairs, or *points*, and internal actions (see [6]).

In their paper, Bourn and Janelidze present some sufficient conditions (recalled in Theorem 1) for a protomodular category to admit semidirect products. These have proved to be useful in dealing with many algebraic contexts, e.g. semi-abelian categories [13]. In [5] the authors proved that the topological models of any semi-abelian variety admit semidirect products. We show that the general definition given in [9], involving all morphisms of the considered category, can be simplified if the category has initial object (Corollary 3), this coming from a fairly general fact that relates conservative and monadic functors. As a concrete example we study the case of groupoids over a fixed set of objects, and further we give an internal version of it.

A notion of semidirect products for groupoids can be found in [10], where the author attributes it to Ehresmann [11]. An important generalization of that is known as the *Grothendieck construction* [12] for groupoids. In a recent paper [3] the authors use the Grothendieck construction in order to formulate a Schreier theory for groupoids. Here a definition of a groupoid extension is given as a fibration of groupoids which is *bijective on objects*, and they justify their requirement by observing that group homomorphisms are bijective on objects when considered as functors.

Here we show that this construction corresponds to the definition given in [9], at least for the special case of fibrations that are *split* extensions, and we give a categorical reason for this. In fact, the category of groupoids is not protomodular, while the category of groupoids with fixed set of objects is. Furthermore it admits semidirect products in the sense of Bourn and Janelidze [9], and they coincide with Brown's construction [10] in this particular context. The internal case is dealt similarly. The paper is organized as follows. After having recalled the categorical definition of semidirect products in Section 2, in Section 3 we give a sufficient condition for its existence. Sections 4 and 5 deal respectively with set-theoretical and internal case studies.

2. The categorical notion of semidirect product

In this section we recall from [9] the categorical notion of semidirect product.

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Let $\mathcal{C}$ be a category. A diagram

$$\begin{array}{ccc} D & \xrightarrow{q} & A \\ \delta\uparrow\downarrow\gamma & & \beta\uparrow\downarrow\alpha \\ E & \xrightarrow[p]{} & B \end{array} \qquad (1)$$

is called a *split commutative square* if $\alpha\beta = 1_B$, $\gamma\delta = 1_E$ and it commutes both upwards and downwards, i.e. $\alpha q = p\gamma$ and $q\delta = \beta p$.

A *split pullback* is a universal such square. More precisely, the diagram (1) is a split pullback of (α, β) along p if, for any other split commutative square

$$\begin{array}{ccc} D' & \xrightarrow{q'} & A \\ \delta'\uparrow\downarrow\gamma' & & \beta\uparrow\downarrow\alpha \\ E & \xrightarrow[p]{} & B, \end{array}$$

there exists a unique morphism $d: D' \to D$ such that

$$\gamma \circ d = \gamma', \qquad d \circ \delta' = \delta, \qquad q \circ d = q'.$$

Dually, the same diagram defines a split pushout of (γ, δ) along p when, for any other split commutative square

$$\begin{array}{ccc} D & \xrightarrow{q'} & A' \\ \delta\uparrow\downarrow\gamma & & \beta'\uparrow\downarrow\alpha' \\ E & \xrightarrow[p]{} & B, \end{array}$$

there exists a unique morphism $a: A \to A'$ such that

$$\alpha' \circ a = \alpha, \qquad a \circ \beta = \beta', \qquad a \circ q = q'.$$

We say that the category $\mathcal{C}$ has split pullbacks (resp. split pushouts) if it admits split pullbacks (resp. split pushouts) along *any* morphism $p: E \to B$.

The existence of split pullbacks defines a contravariant pseudofunctor

$$\mathcal{P}t: \mathcal{C}^{op} \to \mathbf{Cat}$$

(the pseudofunctor of *points*) that assigns to a morphism $p: E \to B$ the pullback functor

$$p^*: \mathcal{P}t(B) \to \mathcal{P}t(E),$$

where the category $\mathcal{P}t(B)$ is the category of the points of the comma category $\mathbb{C}$ over B, i.e. the cocomma category 1_B over $\mathbb{C}/B$. This amounts to the category whose objects are the split epimorphisms with codomain B. In fact a morphism from the terminal $1_B: B \to B$ to an object $\alpha: A \to B$ is precisely an arrow $\beta: B \to A$ such that $\alpha\beta = 1_B$.

Hence the following is purely categorical:

Definition 1. A category $\mathcal{C}$ with split pullbacks is said to be a category with semidirect products if, for any arrow $p: E \to B$ in $\mathcal{C}$, the pullback functor p^* (has a left adjoint and) is monadic.

In this case, denoting by T^p the monad defined by this adjunction, given a T^p-algebra (D, ξ) the semidirect product $(D, \xi) \rtimes (B, p)$ is an object in $\mathcal{P}t(B)$ corresponding to (D, ξ) via the canonical equivalence K:

$$\begin{array}{ccc} & & [\mathcal{P}(E)]^{T^p} \\ & \overset{K}{\nearrow} & \downarrow\ \vdash\ \uparrow \\ \mathcal{P}t(B) & \underset{p^*}{\overset{p_!}{\underset{\bot}{\rightleftarrows}}} & \mathcal{P}t(E) \end{array} \qquad (2)$$

Let us observe that, if $\mathcal{C}$ is finitely complete, the pullback functors p^* have left adjoints $p_!$ (for any p in $\mathcal{C}$) if and only if $\mathcal{C}$ has split pushouts. Recall that a category $\mathcal{C}$ is called protomodular [7] when all functors p^* are conservative. Then the following characterization is also given in [9].

Theorem 1. *A finitely complete Barr-exact [1] category is a category with semidirect products if and only if it is protomodular and has split pushouts.*

3. The case of the initial arrows

The classical semidirect product construction is readily recovered when $\mathcal{C}$ is the category of groups. This is in fact a pointed category, i.e. the terminal and the initial object exist and coincide. Then $\mathcal{P}t(0) = \mathcal{C}$ and for $i_B\colon 0 \to B$, the initial arrow of B, the monad $T^{i_B} = T^B$ is indeed the monad of (internal) actions [6]

$$B\flat(-) : \mathcal{C} \to \mathcal{C}$$

given for an object X of $\mathcal{C}$ by the kernel of the morphism $[1, 0]\colon B + X \to B$. Then the equivalence K of diagram (2) reduces to the usual equivalence between points and actions described in [6], and the semidirect product construction gives a tool that realizes this equivalence.

We are going to show that initial arrows play an essential role in defining semidirect products. In fact they are sufficient for getting semidirect products relative to any $p\colon E \to B$. Nevertheless the situation described does not concern only the pointed case. Indeed it can be explained by a more general fact concerning weakly commutative triangles of functors and their monadicity. In fact a straightforward application of Beck's monadicity theorem gives the following:

Proposition 2. *Consider the diagram of categories and functors, commutative up to natural isomorphisms:*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ & \searrow^{H} & \downarrow{\scriptstyle G} \\ & & \mathcal{E}. \end{array}$$

Suppose further that F has a left adjoint.

If H is monadic and G is conservative (i.e. it reflects isomorphisms), then also F is monadic.

(See [2] for the general setting concerning monad morphisms; the result can be obtained also as a consequence of more recent surveys, e.g. [14], V, section 2.9.)

Corollary 3. *Let $\mathcal{C}$ be a category with finite limits, pushouts of split monomorphisms and initial object. Then the following statements are equivalent:*

(i) *all pullback functors i_B^* defined by the initial arrows are monadic;*
(ii) *for any morphism p in $\mathcal{C}$, the pullback functor p^* is monadic, i.e. $\mathcal{C}$ admits semidirect products.*

Proof. (ii) implies (i): trivial. The other direction: let $p\colon E \to B$ be an arrow in $\mathcal{C}$. The change of base p^* has a left adjoint, as $\mathcal{C}$ has pushouts of split monomorphisms, so we can apply Proposition 2 to the commutative triangle

$$\begin{array}{ccc} \mathcal{P}t(B) & \xrightarrow{p^*} & \mathcal{P}t(E) \\ & \searrow^{i_B^*} & \downarrow{\scriptstyle i_E^*} \\ & & \mathcal{P}t(0). \end{array}$$

and get the result. □

4. The semidirect product in $\mathcal{G}pd_X$

Given a set X, we denote by $\mathcal{G}pd_X$ the category of groupoids with fixed set X of objects, morphisms being functors that are identities on objects. These categories are the fibres of the fibration

$$(-)_0\colon \mathcal{G}pd \longrightarrow \mathcal{S}et$$

associating with any groupoid its set of objects. The categories $\mathcal{G}pd_X$ are not pointed. Nevertheless the classical literature on groupoids shows that in these fiber categories one can develop some relevant constructions that mimic the classical group-theoretical ones. A categorical reason for this fact is that these categories are quasi-pointed, exact and protomodular, where quasi-pointed [8] means that the unique arrow $0 \to 1$ is a monomorphism. It is easy to see that these categories have split pushouts; hence, by Theorem 1, they have semidirect products. Now we show the explicit construction of semidirect products in these settings, in order to clarify that the categorical definition of semidirect products corresponds to the classical construction in the context of groupoids.

In the sequel, we denote by G both an object of $\mathcal{G}pd_X$ and its set of arrows. More precisely we will describe the reflexive graph underlying a groupoid by a diagram

$$G \underset{c_G}{\overset{d_G}{\underset{\xleftarrow{e_G}}{\rightrightarrows}}} X,$$

d_G, c_G and e_G being respectively the domain, codomain and unit maps, and we will denote by m_G the composition of arrows in G.

Definition 2. Let two groupoids $G, H \in \mathcal{G}pd_X$ be given, with H totally disconnected, i.e. with $d_H = c_H$. Furthermore, let us denote by $G \times_X H$ the pullback of d_G along d_H. Then an (external) action of G on H is a set-theoretical map

$$\bullet\colon G \times_X H \to H,$$

such that for any $h, h_1, h_2 \in H(x, x), g, g_1 \in G(x, y), g_2 \in G(y, z)$, with x, y, z in X one has:

$$\begin{array}{ll} \text{(i) } c_H(g \bullet h) = c_G(g); & \text{(ii) } (g_2 \circ g_1) \bullet h = g_2 \bullet (g_1 \bullet h); \\ \text{(iii) } 1_x \bullet h = h; & \text{(iv) } g \bullet (h_2 \circ h_1) = (g \bullet h_2) \circ (g \bullet h_1). \end{array}$$

Remark 4. Let us point out that an external action of G is the same as a functor from G to the category of groups, which assigns to any object x of G the group of automorphisms $H(x, x)$, or, equivalently, as an internal group in the category of functors from G to the category of sets. This is a restriction of the well known Grothendieck construction. See [3] for a detailed account of this classical subject in a context very close to ours.

The fiber categories $\mathcal{G}pd_X$ are quasi-pointed, with initial object given by the discrete groupoid Δ_X (and terminal object the co-discrete (total) groupoid ∇_X). Let us call the kernel of a morphism $p\colon E \longrightarrow B$ of groupoids the pullback of p along the initial arrow into its codomain.

Definition 3. Given two groupoids G and H in $\mathcal{G}pd_X$, with H totally disconnected, an extension of G by H is a diagram in $\mathcal{G}pd_X$:

$$H \xrightarrow{j} E \xrightarrow{p} G$$

with p full (i.e. surjective on arrows, in this context) and $\operatorname{Ker} p = (H, j)$.

An extension of G by H is said to be split if p has a section in $\mathcal{G}pd_X$, i.e. a functor $s\colon G \to E$ such that $ps = 1_G$.

Given G and H in $\mathcal{G}pd_X$, with H totally disconnected, and an action $\bullet\colon G \times_X H \to H$ of G on H, one can define the semidirect product of H and G with respect to $\bullet$ by means of the following explicit construction: $H \rtimes G$ is the groupoid whose set of objects is X and whose arrows from an object x to an object y are pairs (h, g) with $g\colon x \to y$ in G and $h\colon y \to y$ in H. Composition is given by the formula

$$(h_2, g_2) \circ (h_1, g_1) = (h_2 \circ (g_2 \bullet h_1), g_2 \circ g_1).$$

This yields a split extension:

$$H \xrightarrow{j} H \rtimes G \underset{p}{\overset{s}{\leftrightarrows}} G,$$

where j, p and s are defined on arrows in the following way:

$$j(h) = (h, 1_{d_H(h)}), \qquad p(h, g) = g, \qquad s(g) = (1_{c_G(g)}, g).$$

Conversely, consider a split extension:

$$\text{(E)} \qquad H \xrightarrow{j} E \underset{p}{\overset{s}{\leftrightarrows}} G.$$

As in the case of groups, this defines an action of G on H, given by conjugation:

$$g \bullet h = s(g) \circ j(h) \circ s(g)^{-1},$$

and the semidirect product extension given by this action is isomorphic to the split extension (E).

The construction described above yields the equivalence between (external) actions and split extensions, and this will prove directly that the category $\mathcal{G}pd_X$ has semidirect products in the sense of [9].

Let $i_B\colon \Delta_X \to B$ be the unique groupoid morphism from Δ_X to B. The pullback functor $i_B^*\colon \mathcal{P}t(B) \to \mathcal{P}t(\Delta_X)$ is the kernel functor, given explicitly by a diagram

$$\begin{array}{ccc} \operatorname{Ker}\alpha & \xrightarrow{pr_2} & A \\ {\scriptstyle i_{\operatorname{Ker}\alpha}}\uparrow\downarrow{\scriptstyle pr_1} & & {\scriptstyle \beta}\uparrow\downarrow{\scriptstyle \alpha} \\ \Delta_X & \xrightarrow[i_B]{} & B, \end{array}$$

where pr_1 and pr_2 denote the pullback projections. Notice that pr_1 is the functor which sends every arrow of $\operatorname{Ker}\alpha$ to the identity of its domain.

It is worth observing that, since kernels in $\mathcal{G}pd_X$ are points over Δ_X, this implies that they are totally disconnected groupoids; moreover all such groupoids are kernels.

The left adjoint of i_B^* is the pushout functor $i_{B*}\colon \mathcal{P}t(\Delta_X) \to \mathcal{P}t(B)$. For an object (D, π_D, i_D) in $\mathcal{P}t(\Delta_X)$, its image under i_B^* is given by the split pushout diagram

$$\begin{array}{ccc} D & \xrightarrow{j_2} & B+D \\ i_D \Big\uparrow\Big\downarrow \pi_D & & j_1 \Big\uparrow\Big\downarrow \rho \\ \Delta_X & \xrightarrow[i_B]{} & B, \end{array}$$

where j_1 and j_2 are the pushout injections and $\rho = [1_B, i_B \circ \pi_D]$ is given by the universal property of pushouts. This obviously extends to morphisms.

The unit and the counit of the adjunction have components

$$\eta_{(D,\pi_D,i_D)}\colon D \to \operatorname{Ker}[1_B, i_B \circ \pi_D], \qquad \epsilon_{(A,\alpha,\beta)} = [\beta, pr_2]\colon B + \operatorname{Ker}\alpha \to A.$$

Following [6], we denote by $B\flat(-)$ the monad canonically associated with this adjunction. For a point $D = (D, \pi_D, i_D)$ in $\mathcal{P}t(\Delta_X)$, one has $B\flat D = \operatorname{Ker}([1_B, i_B \circ \pi_D])$. It is useful to give an explicit description of the multiplication μ of the monad. Arrows in $B\flat D$ are generated by formal composites

$$y \xrightarrow{b^{-1}} x \xrightarrow{d} x \xrightarrow{b} y\ ,$$

with b in B and d in D. Furthermore

$$B\flat(B\flat D) = \operatorname{Ker}([1_B, i_B \circ \pi_{B\flat D}]\colon B + (B\flat D) \to B),$$

so the arrows in $B\flat(B\flat D)$ are generated by formal composites $b \circ k \circ b^{-1}$ with b in B and k in $B\flat D$. Rewriting k, one sees that $B\flat(B\flat D)$ is indeed generated by strings of the form $b \circ (\overline{b} \circ d \circ \overline{b}^{-1}) \circ b^{-1}$. Hence μ is given by the assignment

$$b \circ (\overline{b} \circ d \circ \overline{b}^{-1}) \circ b^{-1} \mapsto (b \circ \overline{b}) \circ d \circ (b \circ \overline{b})^{-1}.$$

We denote by $\mathcal{P}t(\Delta_X)^B$ the category of algebras of the monad $B\flat(-)$. Then we get a comparison functor

$$K\colon \mathcal{P}t(B) \to \mathcal{P}t(\Delta_X)^B$$

$$(A, \alpha, \beta) \mapsto ((\operatorname{Ker}\alpha, pr_1, i_{\operatorname{Ker}\alpha}), i_B^*(\epsilon_A)).$$

Let us notice that on morphisms, K is just the restriction to kernels. Using the relationship between $B\flat(-)$-algebras and external B-actions, one gets the following:

Proposition 5. *The comparison K is an equivalence of categories.*

Proof. Let us observe first that every arrow $f\colon (A, \alpha, \beta) \to (A', \alpha', \beta')$ in $\mathcal{P}t(B)$ is uniquely determined by its restriction to kernels $i_B^*(f)$. In fact this follows from the fact that the kernel injection and the section of a split extension are jointly epimorphic. Then the comparison functor K is full and faithful. As a matter of fact, $i_B^*(f_1) = i_B^*(f_2)$ determines a unique $f_1 = f_2$ in $\mathcal{P}t(B)$, so K is faithful. Moreover, if $g\colon K(A, \alpha, \beta) \to K(A', \alpha', \beta')$ is a morphism of algebras, it determines a functor

$$\overline{g}\colon \operatorname{Ker}\alpha \rtimes B \to \operatorname{Ker}\alpha' \rtimes B,$$

defined by

$$\overline{g}(n, b) = (g(n), b),$$

which is indeed a morphism of points.

Then, denoting by $\varphi\colon \operatorname{Ker}\alpha \rtimes B \to A$ (and respectively φ' for (A', α', β')) the canonical isomorphism with the semidirect product, we get

$$K(\varphi' \circ \overline{g} \circ \varphi^{-1}) = g$$

and so K is full.

Finally, let us consider an object $(D, h) \in \mathcal{P}t(\Delta_X)^B$, with $D = (D, \pi_D, i_D)$. Then $h\colon B\flat D \to D$ is an algebra structure for the monad $B\flat(-)$ and so the following diagram commutes:

$$\begin{array}{ccccc} B\flat(B\flat D) & \xrightarrow{B\flat h} & B\flat D & \xleftarrow{\eta_D} & D \\ \mu_D \Big\downarrow & & h \Big\downarrow & \diagup\!\!\diagup & \\ B\flat D & \xrightarrow[h]{} & D. & & \end{array}$$

In terms of elements, this means that

$$h(b_2 \circ h(b_1 \circ d \circ b_1^{-1}) \circ b_2^{-1}) = h((b_2 \circ b_1) \circ d \circ (b_2 \circ b_1)^{-1}), \quad h(d) = d.$$

This gives an external action of B on D:

$$b \bullet d = h(b \circ d \circ b^{-1}),$$

and hence an object $(D \rtimes B, p, s) \in \mathcal{P}t(B)$. It is a straightforward calculation to show that $K(D \rtimes B, p, s)$ is isomorphic to (D, h) and so K is essentially surjective on objects, and this concludes the proof. $\square$

5. The internal case

Let us observe that, since the constructions of the external actions and of the semidirect products in the previous section involve only finite limits, they are invariant under the Yoneda embedding. Hence the equivalence between (external) actions and split extensions is Yoneda invariant, as we will show in detail in this section.

Given a category $\mathcal{E}$ with finite limits and an object X of $\mathcal{E}$, we denote by $\mathcal{G}pd_X(\mathcal{E})$ the category of internal groupoids in $\mathcal{E}$ with fixed object X of objects. These categories are the fibres of the fibration

$$(-)_0 \colon \mathcal{G}pd(\mathcal{E}) \to \mathcal{E}$$

associating with any groupoid its object of objects.

Every category $\mathcal{G}pd_X(\mathcal{E})$ has an initial object Δ_X, as before.

Definition 4. Given two groupoids $G, H \in \mathcal{G}pd_X(\mathcal{E})$, with H totally disconnected (i.e. $d_H = c_H$), an (external) action of G on H is an arrow in $\mathcal{E}$:

$$\bullet \colon G \times_X H \to H$$

(again we denote by G and H both the objects of $\mathcal{G}pd_X(\mathcal{E})$ and their objects of arrows) where $G \times_X H$ denotes the pullback:

$$\begin{array}{ccc} G \times_X H & \longrightarrow & H \\ \downarrow & & \downarrow{\scriptstyle d_H} \\ G & \xrightarrow[d_G]{} & X, \end{array}$$

such that the following diagrams commute:

$$\begin{array}{ccc} G \times_X H & \xrightarrow{\bullet} & H \\ {\scriptstyle pr_1}\downarrow & (1) & \downarrow{\scriptstyle d_H} \\ G & \xrightarrow[c_G]{} & X \end{array} \qquad \begin{array}{ccc} G \times_X G \times_X H & \xrightarrow{m_G \times 1_H} & G \times_X H \\ {\scriptstyle 1_G \times \bullet}\downarrow & (2) & \downarrow{\scriptstyle \bullet} \\ G \times_X H & \xrightarrow[\bullet]{} & H \end{array}$$

$$\begin{array}{ccc} X \times H & \xrightarrow{e_G \times 1_H} & G \times_X H \\ {\scriptstyle (d_H, 1_H)}\uparrow & (3) & \downarrow{\scriptstyle \bullet} \\ H & = & H \end{array} \qquad \begin{array}{ccccc} G \times_X (H \times_X H) & & \xrightarrow{1_G \times m_H} & & G \times_X H \\ {\scriptstyle (1_G \times pr_1, 1_G \times pr_2)}\downarrow & & (4) & & \downarrow{\scriptstyle \bullet} \\ (G \times_X H) \times_X (G \times_X H) & \xrightarrow[\bullet \times \bullet]{} & H \times_X H & \xrightarrow[m_H]{} & H, \end{array}$$

where $G \times_X G \times_X H$ denotes the pullback of d_G and $c_G pr_1 \colon G \times_X H \to X$, and the other pullbacks are defined accordingly.

Remark 6. The viewpoint outlined in Remark 4 can be extended to the internal case. Let us notice that here an internal $\mathcal{E}$-valued functor $G \to \mathcal{E}$ from an internal category G in $\mathcal{E}$ to the base category $\mathcal{E}$ can be described (see, for example, [4], Definition 8.2.1) as a pair

$$G \times_X H \xrightarrow{\bullet} H \xrightarrow{d_H = c_H} X$$

satisfying suitable axioms. These axioms are expressed precisely by diagrams (1), (2) and (3) above.

Moreover one can easily show that an internal $\mathcal{E}$-valued functor

$$G \times_X H \longrightarrow H \longrightarrow X$$

induces another functor

$$G \times_X (H \times_X H) \longrightarrow H \times_X H \longrightarrow X.$$

Then in terms of internal $\mathcal{E}$-valued functors, diagram (4) above expresses the fact that the composition in H, m_H, is indeed a natural transformation between internal $\mathcal{E}$-valued functors. Finally, groupoid axioms for H make the whole structure an internal group in the category of internal $\mathcal{E}$-valued functors from G.

Definition 5. Given two groupoids G and H in $\mathcal{G}pd_X(\mathcal{E})$, with H totally disconnected, a split extension of G by H is a 4-tuple (E, p, j, s), where $E \in \mathcal{G}pd_X(\mathcal{E})$, $p \colon E \longrightarrow G$, $j \colon H \longrightarrow E$ and $s \colon G \longrightarrow E$ are arrows in $\mathcal{G}pd_X(\mathcal{E})$, such that j is a kernel of p and $p \circ s = 1_G$.

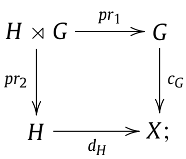
One can easily show that whenever a groupoid H of $\mathcal{G}pd_X(\mathcal{E})$ is the kernel of some arrow, it is totally disconnected.

Let us consider two groupoids G and H in $\mathcal{G}pd_X(\mathcal{E})$, with H totally disconnected, and an action $\bullet \colon G \times_X H \to H$ of G on H.

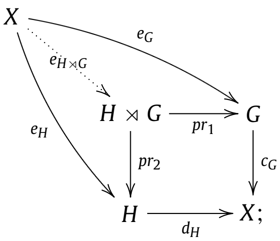
Definition 6. The semidirect product $H \rtimes G$ of H and G with respect to the action $\bullet$ is the groupoid in which:

(a) the object of objects is X;

(b) the object of arrows is the pullback (in $\mathcal{E}$):



(c) $d_{H\rtimes G} = d_G \circ pr_1$;
(d) $c_{H\rtimes G} = c_G \circ pr_1$;
(e) $e_{H\rtimes G} = (e_H, e_G)$ as described by the following diagram:



(f) the composition $m_{H\rtimes G}\colon (H \rtimes G) \times_X (H \rtimes G) \to H \rtimes G$ is

$$(m_H \circ (pr_2 \circ pr_1, (pr_1 \circ pr_1, pr_2 \circ pr_2)), m_G \circ (pr_1 \circ pr_1, pr_1 \circ pr_2));$$

(g) the inverse $inv_{H\rtimes G}\colon H \rtimes G \to H \rtimes G$ is the arrow

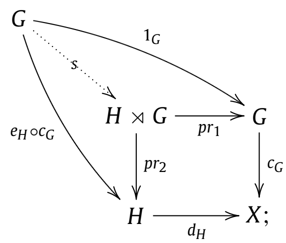
$$(\bullet \circ (inv_G \circ pr_1, inv_H \circ pr_2), inv_G \circ pr_1).$$

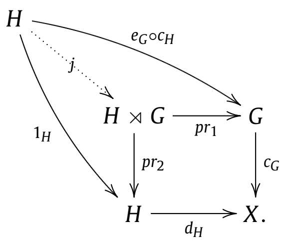
These data do indeed define an internal groupoid when $\mathcal{E}$ is the category of sets; this result extends to a finitely complete category $\mathcal{E}$, as one can show by means of the Yoneda embedding.

Given an action of G on H, we can construct a split extension

$$H \xrightarrow{\;j\;} H \rtimes G \underset{p}{\overset{s}{\leftrightarrows}} G,$$

where $p = pr_1$ is the first projection, $s = (e_H \circ c_G, 1_G)\colon G \to H \rtimes G$ and $j = (1_H, e_G \circ c_H)\colon H \to H \rtimes G$ are defined by the universal property of pullbacks:





It is clear that $p \circ s = 1_G$, while the fact that j is a kernel of p is proved by means of the Yoneda embedding.

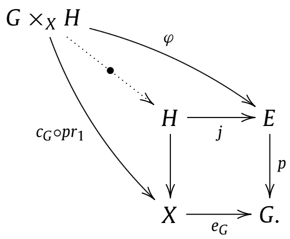
Conversely, given a split extension

$$H \xrightarrow{\;j\;} E \underset{p}{\overset{s}{\leftrightarrows}} G,$$

let φ denote the arrow

$$m_E \circ (1_E \times m_E) \circ (s \circ pr_1, j \circ pr_2, inv_E \circ s \circ pr_1)\colon G \times_X H \to E.$$

We observe that the object of arrows of the pullback of i_G along p in $\mathcal{G}pd_X(\mathcal{E})$ is the pullback of e_G along p in $\mathcal{E}$, so an action of G on H can be defined using the universal property of pullbacks:



The arrow $m_E \circ (j \times s)$ establishes an isomorphism between $H \rtimes G$ and E.

Proposition 7. *Let $\mathcal{E}$ be a category with finite limits such that the category $\mathcal{G}pd_X(\mathcal{E})$ has pushouts of split monomorphisms for any object $X \in \mathcal{E}$. Then $\mathcal{G}pd_X(\mathcal{E})$ has semidirect products.*

Proof. Since $\mathcal{G}pd_X(\mathcal{E})$ has initial object Δ_X, thanks to Corollary 3 it suffices to show that, for any object $B \in \mathcal{G}pd_X(\mathcal{E})$, the pullback functor i_B^* is monadic. All these functors have a left adjoint, since the category $\mathcal{G}pd_X(\mathcal{E})$ has pushouts of split monomorphisms, and they are conservative, since the category $\mathcal{G}pd_X(\mathcal{E})$ is protomodular. So, by Beck's theorem, we only have to show that, for any object B, the functor i_B^* creates coequalizers of parallel arrows f, g in $\mathcal{P}t(B)$ such that the pair $\overline{f} = i_B^*(f)$, $\overline{g} = i_B^*(g)$ has a split coequalizer in $\mathcal{P}t(\Delta_X)$.

Let us display f and g above as follows:

$$\begin{array}{ccc} A & \overset{f}{\underset{g}{\rightrightarrows}} & A' \\ \beta\uparrow\downarrow\alpha & & \beta'\uparrow\downarrow\alpha' \\ B & = & B. \end{array}$$

Letting $K = \mathrm{Ker}\,\alpha$ and $K' = \mathrm{Ker}\,\alpha'$, we know that there are (external) actions of B on K and K' such that $A \simeq K \rtimes B$ and $A' \simeq K' \rtimes B$. In particular, as objects of $\mathcal{E}$, $A \simeq K \times_X B$ and $A' \simeq K' \times_X B$, where the pullbacks are those of d_K and c_B and of $d_{K'}$ and c_B, respectively. As we already observed, these isomorphisms make the condition needed to apply Beck's theorem Yoneda invariant, so the general case reduces to the case of $\mathcal{E}$ being the category of sets. □

Let us observe that, for any finitely complete category $\mathcal{E}$, the category $\mathcal{G}p(\mathcal{E})$ of internal groups in $\mathcal{E}$ is isomorphic to the category $\mathcal{G}pd_1(\mathcal{E})$, where 1 is the terminal object of $\mathcal{E}$. Hence we have the following:

Corollary 8. *Let $\mathcal{E}$ be a category with finite limits such that the category $\mathcal{G}p(\mathcal{E})$ has pushouts of split monomorphisms. Then $\mathcal{G}p(\mathcal{E})$ has semidirect products.*

In particular, the category of topological groups admits semidirect products (as already shown in [5]), as does the category of topological groupoids with fixed space of objects; other examples are the categories of Lie groups and of Lie groupoids with fixed objects. In particular, in the topological case, given two topological groupoids G and H with the same space of objects and with H totally disconnected, and an action of G on H, the topology on the arrow-object of the semidirect product $H \rtimes G$ is the initial one with respect to the canonical maps from $H \rtimes G$ to G and H. The composition of arrows, constructed as in the set-theoretical case, is continuous since it is obtained by composing continuous maps.

We conclude by observing that our proof of the existence of semidirect products can be easily generalized to other internal structures, such as internal rings or internal Lie algebras.

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