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Self-similar solutions for the LSW model with encounters

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ABSTRACT

The LSW model with encounters has been suggested by Lifshitz and Slyozov as a regularization of their classical mean-field model for domain coarsening to obtain universal self-similar long-time behavior. We rigorously establish that an exponentially decaying self-similar solution to this model exist, and show that this solutions is isolated in a certain function space. Our proof relies on setting up a suitable fixed-point problem in an appropriate function space and careful asymptotic estimates of the solution to a corresponding homogeneous problem.

1. Introduction

The classical mean-field theory by Lifshitz and Slyozov [6] and Wagner [16] describes domain coarsening of a dilute system of particles which interact by diffusional mass transfer to reduce their total interfacial area. It is based on the assumption that particles interact only via a common mean-field $\theta = \theta(t)$ which yields a nonlocal transport equation for the number density $f = f(v,t)$ of particles with volume v. It is given by

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial v}\big(\big(-1 + \theta(t) v^{1/3}\big) f\big) = 0, \quad v > 0,\ t > 0, \tag{1}$$

where $\theta(t)$ is determined by the constraint that the total volume of the particles is preserved in time, i.e.

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$$\int_0^\infty v f(v,t)\,dv = \rho. \tag{2}$$

This implies that

$$\theta(t) = \frac{1}{\langle v^{1/3}\rangle} = \frac{\int_0^\infty f(v,t)\,dv}{\int_0^\infty v^{1/3} f(v,t)\,dv}, \tag{3}$$

where $\langle v^k\rangle := m_k/m_0$ and $m_k := \int_0^\infty v^k f(v,t)\,dv$ for $k \geqslant 0$. It is observed in experiments that coarsening systems display statistical self-similarity over long times, that is the number density converges towards a unique self-similar form. Indeed, also the mean-field model (1)–(2) has a scale invariance, which suggests that typical particle volumes grow proportional to t. Going over to self-similar variables one easily establishes that there exists a whole one-parameter family of self-similar solutions. All of the members of this family have compact support and can be characterized by their behavior near the end of their support: One is infinitely smooth, the others behave like a power law. It has been established in [12] (cf. also [3] for asymptotics and numerical simulations, [1] for results on a simplified problem and [13] for some refinements), that the long-time behavior of solutions to (1)–(2) depends sensitively on the initial data, more precisely on their behavior near the end of their support. Roughly speaking, the solution converges to the self-similar solution which behaves as a power law of power $p < \infty$ if and only if the data are regularly varying with power p at the end of their support. The domain of attraction of the infinitely smooth solution is characterized by a more involved condition [13], which we do not give here since it is not relevant for the forthcoming analysis.

This weak selection of self-similar asymptotic states reflects a degeneracy in the mean-field model which is generally believed to be due to the fact that the model is valid only in the regime of vanishing volume fraction of particles [10]. Some effort has been made to derive corrections to the classical mean-field model in order to reflect the effect of positive volume fraction, such as screening induced fluctuations [7,11,14], or to take nucleation into account [2,8,15]. A different approach was already suggested by Lifshitz and Slyozov [6] which is to take the occasional merging of particles ("encounters") into account. This leads to the equation

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial v}\big(\big(-1+\theta(t)v^{1/3}\big)f\big) = J[f], \tag{4}$$

where $J[f]$ is a typical coagulation term, given by

$$J[f] = \frac{1}{2}\int_0^v w(v-v',v')f(v-v',t)f(v',t)\,dv' - \int_0^\infty w(v,v')f(v',t)f(v,t)\,dv',$$

with a suitable rate kernel w specified below. Volume conservation (2) should still be valid, and since

$$\int_0^\infty v\,J[f](v,t)\,dv = 0$$

this requires that θ is again given by (3).

It remains to specify the rate kernel $w(v,v')$ which Lifshitz and Slyozov assume to be dimensionless with respect to rescalings of v, v' and to be additive for large values of v and v'. For simplicity we assume here that

$$w(v,v') = (v+v')/t, \tag{5}$$

that is we obtain a coagulation term with the so-called "additive kernel". Well-posedness of (4) with additive kernel has been established in [5].

As explained before, the model (4), (2) is relevant in the regime that the volume fraction covered by the particles is small and hence we assume that

$$\int v f(v,t)\,dv = \varepsilon \ll 1. \tag{6}$$

The system (4)–(6) can now be written in self-similar variables

$$f(v,t) = \frac{\varepsilon}{t^2}\Phi\left(\frac{v}{t}, \log(t)\right), \qquad z = \frac{v}{t}, \qquad \tau = \log(t), \qquad \theta(t) = \frac{\lambda(\tau)}{t^{1/3}}$$

as

$$\Phi_\tau - z\Phi_z - 2\Phi + \frac{\partial}{\partial z}\big(\big(-1 + \lambda(\tau)z^{1/3}\big)\Phi\big) = \varepsilon J[\Phi](z,\tau), \tag{7}$$

$$\int_0^\infty z\Phi(z,\tau)\,dz = 1, \tag{8}$$

where

$$J[\Phi](z,\tau) = \frac{1}{2}\int_0^z z\Phi(z-z',\tau)\Phi(z',\tau)\,dz' - \Phi(z,\tau)\int_0^\infty (z+z')\Phi(z',\tau)\,dz'$$

and

$$\lambda(\tau) = \frac{\int_0^\infty \Phi(z,\tau)\,dz}{\int_0^\infty z^{1/3}\Phi(z,\tau)\,dz}.$$

Our goal in this paper is to study stationary solutions of (7)–(8) in the regime of small ε. We notice first that the convolution term on the right-hand side of (7) enforces that any solution must have infinite support. We also expect that for small $\varepsilon > 0$ the solution should be close – in an appropriate sense – to one of the self-similar solutions of the LSW model, that is (7) with $\varepsilon = 0$. It can be verified by a stability argument that the only solution of the LSW model for which this is possible is the smooth one which has the largest support.

Indeed, we obtain as our main result, that for any given sufficiently small $\varepsilon > 0$ there exists an exponentially decaying stationary solution to (7)–(8). Moreover, we show this solution to be isolated, i.e., there is no further solution with exponential tail in a sufficiently small neighborhood of the LSW solution. We do not believe, that there exist other exponentially decaying solutions, but our proof does not exclude that. However, we conjecture that there are algebraically decaying stationary solutions as well. We are not yet able to establish a corresponding result for the model discussed in the present paper, but can prove this for a simplified model (see [4]).

For the pure coagulation equation, that is (7) without the drift term, an exponentially decaying stationary solution exists only for $\varepsilon = 1/2$. For every smaller ε there exists a stationary solution with algebraic decay. The domain of attraction of these self-similar solutions has been completely characterized in [9], and can also be related to the regular variation of certain moments of the initial data.

However, the situation here is somewhat different. While the behavior for large volumes v is determined by the coagulation term, the tail introduced by the coagulation term is very small, and the equation behaves – at least in the regime in which we are working – as the LSW model with a small perturbation. Our analysis reflects this fact, since we also treat the coagulation term as a perturbation.

2. Statement of the fixed point problem

In this section we set up a suitable fixed point problem for the construction of stationary solutions to (7)–(8). These solve

$$-z\frac{\partial \Phi}{\partial z} - 2\Phi + \frac{\partial}{\partial z}\big(\big(-1+\lambda z^{1/3}\big)\Phi\big) = \varepsilon J[\Phi](z), \qquad \int_0^\infty z\Phi(z)\,dz = 1, \quad \Phi(z) \geqslant 0, \tag{9}$$

with $z > 0$ and

$$\lambda = \frac{\int_0^\infty \Phi(z)\,dz}{\int_0^\infty z^{1/3}\Phi(z)\,dz}.$$

In the LSW limit $\varepsilon = 0$ there exists a family of solutions with compact support, which can be parametrized by the mean field $\lambda \in [3(\frac{1}{2})^{2/3}, \infty)$. The self-similar solution with the largest support, which is $[0, 1/2]$, is exponentially smooth and is given by

$$\Phi_{\mathrm{LSW}}(z) = \begin{cases} C\exp\left(-\int_0^z \frac{2-\frac{1}{3}\lambda_{\mathrm{LSW}}\xi^{-2/3}}{\xi+1-\lambda_{\mathrm{LSW}}\xi^{1/3}}\,d\xi\right) & \text{for } z \in [0, \frac{1}{2}], \\ 0 & \text{for } z > \frac{1}{2}, \end{cases}$$

with

$$\lambda_{\mathrm{LSW}} := 3\left(\frac{1}{2}\right)^{2/3}$$

and C is a normalization constant chosen such that $\int_0^\infty z\Phi_{\mathrm{LSW}}\,dz = 1$. We denote from now on this solution by Φ_{LSW}. As discussed above there are several physical and mathematical arguments supporting the fact that such a solution is the only stable one under perturbations of the model.

The main goal of this paper is to show the following result.

Theorem 2.1. *For any sufficiently small $\lambda_{\mathrm{LSW}} - \lambda$ there exists a choice for ε such that there exists an exponentially decaying solution to* (9).

The key idea for proving this theorem is to reduce the problem to a standard fixed point problem assuming that (9) is a small perturbation of Φ_{LSW}.

2.1. Formal asymptotics as $\varepsilon \to 0$

The formal asymptotics of the solution of (9) whose existence we prove in this paper was obtained in [6]. We recall it here for convenience. Such a solution Φ is expected to be close to the solution Φ_{LSW} as $\varepsilon \to 0$. Notice, however that Φ_{LSW} vanishes for $z \geqslant \frac{1}{2}$. Therefore, in order to approximate Φ for $z \geqslant \frac{1}{2}$ Lifshitz and Slyozov approximate (9) by means of

$$-z\frac{\partial \Phi}{\partial z} - 2\Phi + \frac{\partial}{\partial z}\big((-1+\lambda z^{1/3})\Phi\big) = \varepsilon J[\Phi_{\mathrm{LSW}}](z). \tag{10}$$

There exists a unique solution of (10) which vanishes for $z \geqslant 1$. Such a function is of order ε in the interval $(\frac{1}{2}, 1)$. However, there is a boundary layer in the region $z \approx \frac{1}{2}$ for λ close to λ_{LSW} where the function Φ experiences an abrupt change. Adjusting the value of λ in a suitable manner it is possible to obtain Φ which is of order one for $z < \frac{1}{2}$. A careful analysis shows that λ must be chosen as

$$\lambda_{\mathrm{LSW}} - \lambda \sim \frac{3\pi^2}{2^{2/3}} \frac{1}{(\log \varepsilon)^2} \tag{11}$$

as $\varepsilon \to 0$. This scaling law was already derived in [6], and is in accordance with our results. Notice that the smallness of Φ for $z \geqslant \frac{1}{2}$ implies that most of the volume of the particles is in the region $z < \frac{1}{2}$.

In order to approximate Φ for $z \geqslant 1$ we would need to use the values of the function Φ for $z \in [\frac{1}{2}, 1]$ obtained by means of (10). Therefore, $\varepsilon J[\Phi]$ becomes of order ε^2 for $z \in [1, \frac{3}{2}]$ and the contribution of this region can be expected to be negligible compared to that of the interval $[\frac{1}{2}, 1]$. A similar argument indicates that the contributions to Φ due to the operator $\varepsilon J(\Phi)$ for $z > \frac{3}{2}$ can be ignored. This procedure can be iterated to obtain in the limit a solution to (7) which decays exponentially fast at infinity. What remains to be established is that such a procedure indeed leads to a converging sequence of solutions. A rigorous proof could be based on such a procedure; we proceed, however, in a slightly different manner.

Before we continue we briefly comment on (11), which give the deviation of the mean-field from the value of the LSW model. This quantity is of particular interest, since its inverse is a measure for the coarsening rate, which is one of the key quantities in the study of coarsening systems. Eq. (11) predicts a much larger deviation than the ones obtained from other corrections to the LSW models. For example, one model which takes the effect of fluctuations into account [14] predicts a deviation of order $O(\varepsilon^{1/4})$. The large deviation predicted by (10) can be attributed to the fact that all particles contribute to the coagulation term and suggests, that encounters are more relevant in the self-similar regimes than fluctuations. We refer to [11] for a more extensive discussion of these issues.

2.2. Derivation of a fixed point problem

We now transform (9) with the choice of kernel (5) into a fixed point problem. To this end we write our equation as follows

$$-z\frac{\partial \Phi}{\partial z} - 2\Phi + \frac{\partial}{\partial z}\big((-1+\lambda z^{1/3})\Phi\big) = \varepsilon\left[\frac{z}{2}\int_0^v \Phi(z-z')\Phi(z')\,dz' - \Phi(z,t) - m_0 z\Phi(z)\right] \tag{12}$$

with

$$1 = \int_0^\infty z\Phi(z)\,dz, \qquad m_0 = \int_0^\infty \Phi(z)\,dz. \tag{13}$$

It would be natural to proceed as follows: For each given value of ε we select m_0 and λ in order to satisfy (13). However, it turns out to be more convenient to fix λ and then select ε and m_0 such that (13) is satisfied. The reason is that our argument requires to differentiate the function ψ defined below with respect to either λ or ε, but it is easier to control the derivatives with respect to ε.

In the following we always consider $\lambda < \lambda_{\mathrm{LSW}}$ and write

$$\delta := \lambda_{\mathrm{LSW}} - \lambda > 0, \qquad \tilde{\varepsilon} := \varepsilon m_0 > 0.$$

An important role in the fixed point argument is played by the functions $z \mapsto \psi(z; \varepsilon, \tilde{\varepsilon}, \delta)$, which are defined as solution to the following homogeneous problem

$$-\big(1 + z - (\lambda_{\mathrm{LSW}} - \delta) z^{1/3}\big)\psi' = \left(2 - \frac{1}{3}(\lambda_{\mathrm{LSW}} - \delta) z^{-2/3} - \tilde{\varepsilon} z - \varepsilon\right)\psi. \tag{14}$$

Each of these functions ψ is uniquely determined up to a constant to be fixed later. Notice that for $\delta > 0$ the function $\psi(z; \varepsilon, \tilde{\varepsilon}, \delta)$ is defined for all $z \geqslant 0$. If $\delta = 0$, however, the function $\psi(z; \varepsilon, \tilde{\varepsilon}, 0)$ is defined only in the set $z > \frac{1}{2}$, and it becomes singular as $z \to (\frac{1}{2})^+$. Therefore the function ψ changes abruptly in a neighborhood of $z = \frac{1}{2}$ for λ close to λ_{LSW}. More precisely, if ψ takes values of order one for $z < \frac{1}{2}$, then it is of order $\exp(-c/\sqrt{\delta})$ for $z > \frac{1}{2}$, and this transition layer causes most of the technical difficulties.

We can now transform (12) into a fixed point problem for an integral operator. Indeed, using Variation of Constants, and assuming that $\Phi(z)$ decreases sufficiently fast to provide the integrability required in the different formulas, we obtain that each solution to (12) satisfies

$$\Phi(z) = \varepsilon \int_z^\infty \frac{\xi}{(\xi + 1 - (\lambda_{\mathrm{LSW}} - \delta)\xi^{1/3})} \frac{\psi(z; \varepsilon, \tilde{\varepsilon}, \delta)}{\psi(\xi; \varepsilon, \tilde{\varepsilon}, \delta)} (\Phi * \Phi)(\xi)\, d\xi =: I[\Phi; \varepsilon, \tilde{\varepsilon}, \delta](z), \tag{15}$$

where the symmetric convolution operator $*$ is defined by

$$(\Phi_1 * \Phi_2)(z) = \frac{1}{2} \int_0^z \Phi_1(z - y)\Phi_2(y)\, dy = \frac{1}{2} \int_0^z \Phi_2(z - y)\Phi_1(y)\, dy. \tag{16}$$

However, the values of the parameters $(\varepsilon, \tilde{\varepsilon})$ cannot be chosen arbitrarily but must be determined by the *compatibility conditions*

$$\varepsilon \int_0^\infty z I[\Phi; \varepsilon, \tilde{\varepsilon}, \delta]\, dz = \varepsilon, \qquad \varepsilon \int_0^\infty I[\Phi; \varepsilon, \tilde{\varepsilon}, \delta]\, dz = \tilde{\varepsilon}. \tag{17}$$

Notice that the operator $I[\Phi; \varepsilon, \tilde{\varepsilon}, \delta]$ maps the cone of nonnegative functions Φ into itself, and this implies that each solution to (15) is nonnegative.

2.3. *Main results and outline of the proofs*

We introduce the following function space Z of exponentially decaying functions. For arbitrary but fixed constants $\beta_1 > 0$ and $\beta_2 > 1$ let $Z := \{\Phi\colon\ \|\Phi\|_Z < \infty\}$ with

$$\|\Phi\|_Z := \lceil \Phi \rceil_Z + \lfloor \Phi \rfloor_Z, \qquad \lceil \Phi \rceil_Z := \sup_{0 \leqslant z \leqslant 1} \big|\Phi(z)\big|, \qquad \lfloor \Phi \rfloor_Z := \sup_{z \geqslant 1} \big|\Phi(z) \exp(\beta_1 z) z^{\beta_2}\big|. \tag{18}$$

Below in Section 3.1 we prove that $\Phi \in Z$ implies $\Phi * \Phi \in Z$. The particular choice of the parameters β_1 and β_2 affects our smallness assumptions for the parameter δ: The larger β_1 and β_2 are the smaller

δ must be chosen, and the faster the solution will decay. We come back to this issue at the end of the paper, cf. Remark 3.25.

Our (local) existence and uniqueness results relies on the following smallness assumptions concerning δ, ε, $\tilde{\varepsilon}$ and Φ.

Assumption 2.2. Suppose that

1. δ is sufficiently small,
2. both ε and $\tilde{\varepsilon}$ are of order $o(\sqrt{\delta})$,
3. Φ is sufficiently close to Φ_{LSW}, in the sense that $\|\Phi - \Phi_{\mathrm{LSW}}\|_Z$ is small.

Our first main result guarantees that we can choose the parameters ε and $\tilde{\varepsilon}$ appropriately.

Theorem 2.3. *Under Assumption* 2.2 *we can solve* (17)*, i.e., for each Φ there exists a unique choice of* $(\varepsilon, \tilde{\varepsilon})$ *such that the compatibility conditions are satisfied. This solution belongs to*

$$U_\delta = \left\{(\varepsilon, \tilde{\varepsilon}):\ \frac{1}{2}\epsilon_\delta \leqslant \varepsilon \leqslant 2\epsilon_\delta,\ \frac{1}{2}\tilde{\epsilon}_\delta \leqslant \tilde{\varepsilon} \leqslant 2\tilde{\epsilon}_\delta\right\},$$

where $\epsilon_\delta \sim \exp(-c/\sqrt{\delta})$ *and* $\tilde{\epsilon}_\delta \sim (-c/\sqrt{\delta})$ *will be identified in Eq.* (43) *below.*

The solution from Theorem 2.3 depends naturally on the function $h = h[\Phi] = \Phi * \Phi$, and is denoted by $(\varepsilon[h;\delta], \tilde{\varepsilon}[h;\delta])$. In a second step we define an operator $\bar{I}_\delta[\Phi]$ via

$$\bar{I}_\delta[\Phi] := I\big[\Phi;\varepsilon\big[h[\Phi];\delta\big], \tilde{\varepsilon}\big[h[\Phi];\delta\big], \delta\big] \tag{19}$$

with I as in (15), and show that for sufficiently small δ there exists a corresponding fixed point.

Theorem 2.4. *Under Assumption* 2.2 *there exists a nonnegative solution to* $\Phi = \bar{I}_\delta[\Phi]$ *that is isolated in the space* Z.

In order to prove Theorem 2.3 we rewrite the compatibility conditions as a fixed point equation for $(\varepsilon, \tilde{\varepsilon})$ with parameters δ and h. This reads

$$g_1[h;\varepsilon,\tilde{\varepsilon},\delta] = \varepsilon, \qquad g_2[h;\varepsilon,\tilde{\varepsilon},\delta] = \tilde{\varepsilon}, \tag{20}$$

where

$$g_i[h;\varepsilon,\tilde{\varepsilon},\delta] := \varepsilon^2 \int_0^\infty \frac{\xi\, h(\xi)}{(\xi + 1 - (\lambda_{\mathrm{LSW}} - \delta)\xi^{1/3})} \int_0^\xi \gamma_i(z) \frac{\psi(z;\varepsilon,\tilde{\varepsilon},\delta)}{\psi(\xi;\varepsilon,\tilde{\varepsilon},\delta)}\, dz\, d\xi, \tag{21}$$

with $\gamma_1(z) = z$ and $\gamma_2(z) = 1$.

For fixed h the integrals g_1 and g_2 depend extremely sensitive on ε, $\tilde{\varepsilon}$, and δ. Therefore, the crucial part in our analysis are the following asymptotic expressions for g_1, g_2 and their derivatives that we derive within Section 3.2.

Proposition 2.5. *Assumption* 2.2 *implies*

$$\big|g_1[h;\varepsilon,\tilde{\varepsilon},\delta] - \epsilon_\delta\big| = o(\epsilon_\delta), \qquad \big|g_2[h;\varepsilon,\tilde{\varepsilon},\delta] - \tilde{\epsilon}_\delta\big| = o(\tilde{\epsilon}_\delta)$$

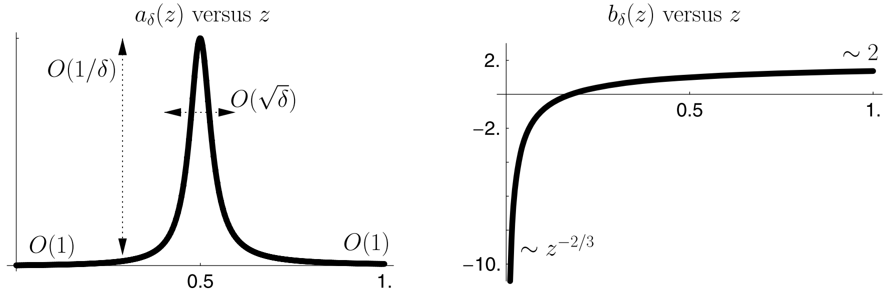

Fig. 1. Sketch of the functions a_δ and b_δ for small δ and $z \in [0, 1]$.

as well as

$$|\varepsilon\partial_\varepsilon g_i - 2g_i| \leqslant o(g_i), \qquad |\tilde{\varepsilon}\partial_{\tilde{\varepsilon}} g_i| \leqslant o(g_i),$$

for $i = 1, 2$.

Exploiting these estimates we prove Theorem 2.3 by means of elementary analysis, see Section 3.2, and as a consequence we derive in Section 3.2 the following result, which in turn implies Theorem 3.3.

Proposition 2.6. *Under Assumption* 2.2 *there exists a small ball around* Φ_{LSW} *in the space* Z *such that the operator* $\bar{I}_\delta[\Phi]$ *is a contraction on this ball.*

We proceed with some comments concerning the uniqueness of solutions. Proposition 2.6 provides a local uniqueness result in the function space Z. Moreover, since we can choose the decay parameters $\beta_1 > 0$ and $\beta_2 > 1$ arbitrarily, compare the discussion in Remark 3.25, we finally obtain local uniqueness in the space of all exponential decaying solutions. However, this does exclude neither the existence of further exponentially decaying solutions being sufficiently far away from Φ_{LSW}, nor the existence of algebraically decaying solutions.

3. Proofs

In what follows c and C are small and large, respectively, positive constants which are independent of δ. Moreover, $o(1)$ denotes a number that converges to 0 as $\delta \to 0$, where this convergence is always uniform with respect to all other quantities under consideration.

For the subsequent considerations the following notations are useful. For $\delta > 0$ let

$$a_\delta(z) := \frac{1}{1 + z - \lambda_{\mathrm{LSW}} z^{1/3} + \delta z^{1/3}}, \qquad b_\delta(z) := 2 - \frac{1}{3}\lambda_{\mathrm{LSW}} z^{-2/3} + \frac{1}{3}\delta z^{-2/3}, \tag{22}$$

compare Fig. 1, so that by definition the function $z \mapsto \psi(z; \varepsilon, \tilde{\varepsilon}, \delta)$ solves the homogenous equation

$$\psi' = -a_\delta(z)\big(b_\delta(z) - \tilde{\varepsilon} z - \varepsilon\big)\psi, \quad 0 \leqslant z < \infty, \tag{23}$$

compare (14). For convenience we normalize ψ by

$$\int_0^1 z\psi(z; \varepsilon, \tilde{\varepsilon}, \delta)\, dz = \int_0^1 z\Phi_{\mathrm{LSW}}(z)\, dz = 1, \tag{24}$$

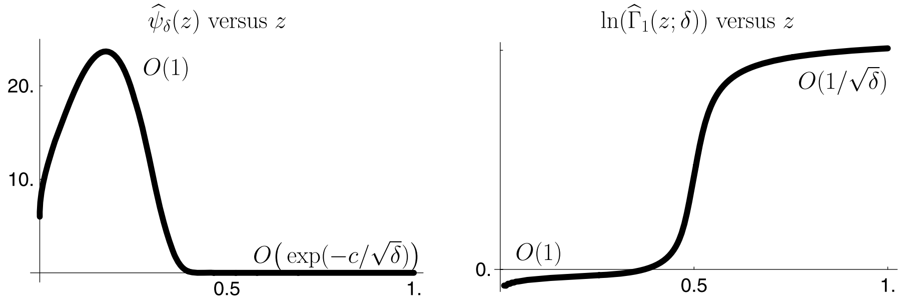

Fig. 2. Sketch of the functions and $\widehat{\psi}_\delta$ and $\ln(\widehat{\Gamma}_1(\cdot;\delta))$ for small δ and $z \in [0,1]$.

and this implies

$$\left|\widehat{\psi}_\delta(z) - \Phi_{\mathrm{LSW}}(z)\right| = o(1), \quad 0 \leqslant z \leqslant 1,$$

with $\widehat{\psi}_\delta(z) = \psi(z; 0, 0, \delta)$, and

$$\frac{1}{|\widehat{\psi}_\delta(z)|} \begin{cases} \leqslant C_\sigma & \text{for } 0 \leqslant z \leqslant \frac{1}{2} - \sigma, \\ \geqslant c_\sigma \exp(c/\sqrt{\delta}) & \text{for } \frac{1}{2} + \sigma \leqslant z \leqslant 1, \end{cases}$$

with $0 < \sigma < \frac{1}{2}$ arbitrary, compare Fig. 2.

Moreover, we define

$$G_i[h; \varepsilon, \tilde{\varepsilon}, \delta] := \int_0^\infty \xi a_\delta(\xi) h(\xi) \Gamma_i(\xi; \varepsilon, \tilde{\varepsilon}, \delta)\, d\xi,$$

$$\Gamma_i(\xi; \varepsilon, \tilde{\varepsilon}, \delta) := \int_0^\xi \gamma_i(z) \frac{\psi(z; \varepsilon, \tilde{\varepsilon}, \delta)}{\psi(\xi; \varepsilon, \tilde{\varepsilon}, \delta)}\, dz = \int_0^\xi \gamma_i(z) \exp\left(\int_z^\xi a_\delta(y)\big(b_\delta(y) - \tilde{\varepsilon} y - \varepsilon\big)\, dy \right) dz, \quad (25)$$

with $\gamma_1(z) = z$, $\gamma_2(z) = 1$, so that the compatibility conditions (21) read

$$\varepsilon = \varepsilon^2 G_1[h; \varepsilon, \tilde{\varepsilon}, \delta], \qquad \tilde{\varepsilon} = \varepsilon^2 G_2[h; \varepsilon, \tilde{\varepsilon}, \delta].$$

In order to prove the existence of solutions to these equations we need careful estimates on the functionals G_i and their derivatives with respect to ε and $\tilde{\varepsilon}$. These are derived within Section 3.2 by exploiting Assumption 2.2, and rely on the following two observations. For for small ε and $\tilde{\varepsilon}$ we can ignore that the functions Γ_1 and Γ_2 depend on these parameters, and for h close to $\Phi_{\mathrm{LSW}} * \Phi_{\mathrm{LSW}}$ we can neglect the contributions of the exponential tails to G_1 and G_2. More precisely, our main approximation arguments are

$$\Gamma_i(\xi; \varepsilon, \tilde{\varepsilon}, \delta) \approx \widehat{\Gamma}_i(\xi; \delta), \qquad G_i[h; \varepsilon, \tilde{\varepsilon}, \delta] \approx \widehat{G}_i[h; \delta] \approx \widehat{G}_i[\Phi_{\mathrm{LSW}} * \Phi_{\mathrm{LSW}}; \delta],$$

where

$$\widehat{\Gamma}_i(\xi; \delta) := \Gamma_i(\xi; 0, 0, \delta), \qquad \widehat{G}_i[h; \delta] := \int_0^1 \xi a_\delta(\xi)\, h(\xi)\, \widehat{\Gamma}_i(\xi; \delta)\, d\xi. \quad (26)$$

Moreover, for small δ we can further approximate $\widehat{G}_i$ by

$$\widehat{G}_i[h;\delta] \approx K_{i,\delta} R_0[h],$$

where

$$K_{i,\delta} := \widehat{\Gamma}_i(1;\delta)$$

and R_0 is the well defined limit for $\delta \to 0$ of the functionals

$$R_\delta[h] := \int_0^1 \varrho_\delta(\xi) h(\xi)\, d\xi, \qquad \varrho_\delta(\xi) := \xi a_\delta(\xi) \exp\left(-\int_\xi^1 a_\delta(y) b_\delta(y)\, dy\right), \tag{27}$$

see Corollary 3.5 below.

3.1. Auxiliary results

3.1.1. Properties of the function spaces

Here we prove that the convolution operator $*$ from (16) is continuous and maps the space Z into itself, and we derive further useful estimates.

Lemma 3.1. *For arbitrary $\Phi_1, \Phi_2 \in Z$ we have $\Phi_1 * \Phi_2 \in Z$, and there exists a constant C such that*

$$\|\Phi_2 * \Phi_2 - \Phi_1 * \Phi_1\|_Z \leqslant C\big(\|\Phi_2\|_Z + \|\Phi_1\|_Z\big)\|\Phi_2 - \Phi_1\|_Z.$$

Moreover, for each $n \in \mathbb{N}$ there exists a constant C_n such that

$$\frac{1}{z^n}\int_z^\infty \xi^n \big|\Phi(\xi)\big|\, d\xi + \frac{1}{z^n}\int_z^\infty \xi^n(\xi - z)\big|\Phi(\xi)\big|\, d\xi \leqslant C_n \frac{\exp(-\beta_1 z)}{z^{\beta_2}} \lfloor \Phi \rfloor_Z \tag{28}$$

for $z \geqslant 1$, and hence

$$\int_1^\infty \xi^n \big|\Phi(\xi)\big|\, d\xi \leqslant C_n \lfloor \Phi \rfloor_Z \tag{29}$$

for all $\Phi \in Z$.

Proof. Let $\Phi_i \in Z$ be arbitrary. Definition (18) provides

$$\big|(\Phi_2 * \Phi_1)(z)\big| \leqslant C\|\Phi_1\|_Z \|\Phi_2\|_Z, \quad 0 \leqslant z \leqslant 2,$$

where we used that $\sup_{0\leqslant z\leqslant 2}|\Phi(z)| \leqslant C\|\Phi\|_Z$. For $z \geqslant 2$ we estimate

$$\int_{z-1}^z \frac{\exp(-\beta_1 y)}{y^{\beta_2}}\, dy = \int_0^1 \frac{\exp(-\beta_1(z-y))}{(z-y)^{\beta_2}}\, dy \leqslant \frac{\exp(-\beta_1 z)}{(z-1)^{\beta_2}} \frac{\exp(\beta_1) - 1}{\beta_1} \leqslant C\frac{\exp(-\beta_1 z)}{z^{\beta_2}}$$

and

$$\int_1^{z-1} \frac{\exp(-\beta_1(z-y))}{(z-y)^{\beta_2}} \frac{\exp(-\beta_1 y)}{y^{\beta_2}}\, dy \leqslant \exp(-\beta_1 z) \int_1^{z-1} \frac{dy}{(z-y)^{\beta_2}\, y^{\beta_2}} \leqslant C \frac{\exp(-\beta_1 z)}{z^{\beta_2}}.$$

These results imply $\Phi_2 * \Phi_1 \in Z$ with $\|\Phi_2 * \Phi_1\|_Z \leqslant C\|\Phi_1\|_Z\|\Phi_2\|_Z$, and we conclude

$$\begin{aligned}\|\Phi_2 * \Phi_2 - \Phi_1 * \Phi_1]\|_Z &\leqslant \big\|\Phi_2 * (\Phi_2 - \Phi_1)\big\|_Z + \big\|\Phi_1 * (\Phi_2 - \Phi_1)\big\|_Z \\ &\leqslant C\big(\|\Phi_2\|_Z + \|\Phi_1\|_Z\big)\big(\|\Phi_2 - \Phi_1\|_Z\big).\end{aligned}$$

Finally, the estimates (28) and (29) follow from $\Phi(z) \leqslant \lfloor\Phi\rfloor_Z z^{-\beta_2}\exp(-\beta_1 z)$ and elementary estimates for integrals. □

Remark 3.2. All constants C in Lemma 3.1 depend on the parameters β_1 and β_2 that appear in the definition of the function space Z, compare (18). More precisely, we have $C \to \infty$ as $\beta_1 \to \infty$ or $\beta_2 \to \infty$.

3.1.2. Properties of a_δ and b_δ

All subsequent estimates rely on the following properties of the functions a_δ and b_δ which are illustrated in Fig. 1.

Lemma 3.3. *Let $\delta \leqslant 1$ and σ be arbitrary with $0 < \sigma < \frac{1}{2}$. Then,*

1. *a_δ is uniformly positive on $[0,1]$, where*

$$\max a_\delta(y) = \frac{2^{1/3}}{\delta}\big(1 + O(\delta)\big) \quad \textit{and} \quad \operatorname{argmax} a_\delta(y) = \frac{1}{2} + O(\delta)$$

denote the maximum and the maximizer, respectively,

2. *b_δ is uniformly integrable on $[0,1]$ and nonnegative for $z \geqslant (\frac{1}{2})^{5/2}$,*
3. *$z a_\delta(z) \to 1$ and $b_\delta(z) \to 2$ as $z \to \infty$,*
4. *a_δ can be expanded with respect to δ on $[0, \frac{1}{2} - \sigma] \cup [\frac{1}{2} + \sigma, 1]$, i.e.,*

$$\big|a_\delta(z) - a_0(z)\big| \leqslant C_\sigma \delta$$

for $z \leqslant 1$ with $|z - \frac{1}{2}| > \sigma$ and some constant C_σ depending on σ,

5. *$\int_0^1 a_\delta(y)\, dy = \frac{\kappa}{\sqrt{\delta}}(1 + o(1))$ for some constant κ given in the proof.*

Proof. Assertions 1–4 follow immediately from the definitions of a_δ and b_δ, compare (22). With $y = \frac{1}{2} + \sqrt{\delta}\eta$ we find

$$\sqrt{\delta} a_\delta(y) = \frac{2}{2^{2/3} + \frac{4}{3}\eta^2}\big(1 + o(1)\big).$$

This expansion implies2

$$\sqrt{\delta}\int_0^1 a_\delta(y)\, dy = \big(1 + o(1)\big)\int_{-\infty}^{\infty} \frac{2}{2^{2/3} + \frac{4}{3}\eta^2}\, d\eta = \big(1 + o(1)\big)\frac{\sqrt{3}\pi}{2^{1/3}},$$

and the proof is complete. □

3.1.3. Properties of ϱ_δ and R_δ

Lemma 3.4. *For $\delta \leqslant 1$ we have $\int_0^1 \varrho_\delta(\xi)\,d\xi \leqslant C$, where ϱ_δ is defined in* (27)*. Moreover, for each $0 < \sigma < \frac{1}{2}$ there exist constants c_σ and C_σ such that*

1. $\varrho_\delta(\xi) \geqslant c_\sigma$ *for* $\frac{1}{2} + \sigma \leqslant \xi \leqslant 1$,
2. $\varrho_\delta(\xi) \leqslant o(1) C_\sigma$ *for* $0 \leqslant \xi \leqslant \frac{1}{2} - \sigma$.

Proof. The existence of c_σ and C_σ is provided by Lemma 3.3, so there remains to show

$$\int_{1/2-\sigma_0}^{1/2+\sigma_0} \varrho_\delta(\xi)\,d\xi \leqslant C$$

for fixed but small σ_0. According to Lemma 3.3 there exist constants c and C such that

$$\varrho_\delta(\xi) \leqslant C a_\delta(\xi) \exp\left(-c \int_\xi^1 a_\delta(y)\,dy\right) = \frac{C}{c} \frac{d}{d\xi} \exp\left(-c \int_\xi^1 a_\delta(y)\,dy\right) d\xi$$

for all ξ with $|\xi - \frac{1}{2}| \leqslant \sigma_0$, and we conclude

$$\int_{1/2-\sigma_0}^{1/2+\sigma_0} \varrho_\delta(\xi)\,d\xi \leqslant \frac{C}{c} \exp\left(-c \int_{1/2+\sigma_0}^{1} a_\delta(y)\,dy\right) - \exp\left(-c \int_{1/2-\sigma_0}^{1} a_\delta(y)\,dy\right),$$

which gives the desired result. □

Corollary 3.5. *The functionals R are uniformly Lipschitz continuous with respect to $h \in Z$ for $\delta \leqslant 1$ with*

$$R_\delta[h] \xrightarrow{\delta\to 0} R_0[h] := \int_{1/2}^{1} \xi a_0(\xi) h(\xi) \exp\left(-\int_\xi^1 a_0(y) b_0(y)\,dy\right) d\xi$$

for all $h \in Z$.

3.1.4. Properties of ψ

Lemma 3.6. *The estimates*

$$\frac{\psi(z; \varepsilon_2, \tilde{\varepsilon}_2, \delta)}{\psi(z; \varepsilon_1, \tilde{\varepsilon}_1, \delta)} \leqslant \exp\left(C \frac{|\varepsilon_2 - \varepsilon_1| + |\tilde{\varepsilon}_2 - \tilde{\varepsilon}_1|}{\sqrt{\delta}}\right) \tag{30}$$

are satisfied for $0 \leqslant z \leqslant 1$, $\delta \leqslant 1$, and arbitrary $(\varepsilon_1, \tilde{\varepsilon}_1)$, $(\varepsilon_2, \tilde{\varepsilon}_2)$.

Proof. The Variation of Constants formula provides

$$\psi(z; \varepsilon_2, \tilde{\varepsilon}_2, \delta) = d\,\psi(z; \varepsilon_1, \tilde{\varepsilon}_1, \delta) \exp\left(\int_0^z a_\delta(y)\big(\varepsilon_2 - \varepsilon_1 + (\tilde{\varepsilon}_2 - \tilde{\varepsilon}_1) y\big)\,dy\right)$$

for some factor d which depends on ε_1, ε_2, $\tilde{\varepsilon}_1$, $\tilde{\varepsilon}_2$, and δ. Thanks to

$$\int_0^z a_\delta(y)\,dy + \int_0^z a_\delta(y)y\,dy \leqslant \frac{C}{2\sqrt{\delta}}$$

for $0 \leqslant z \leqslant 1$ we conclude

$$d\widetilde{C}^{-1/2}\psi(z;\varepsilon_1,\tilde{\varepsilon}_1,\delta) \leqslant \psi(z;\varepsilon_2,\tilde{\varepsilon}_2,\delta) \leqslant d\widetilde{C}^{1/2}\psi(z;\varepsilon_1,\tilde{\varepsilon}_1,\delta),$$

where $\widetilde{C}$ denotes the constant on the r.h.s. in (30). Finally, the normalization condition (24) yields

$$\widetilde{C}^{-1/2} \leqslant d \leqslant \widetilde{C}^{1/2},$$

and the proof is complete. □

3.1.5. Properties of $\widehat{\Gamma}_i$ and Γ_i

Recall definition (26), which implies that the functions $\xi \mapsto \widehat{\Gamma}_i(\xi;\delta)$, $i = 1, 2$, are strictly increasing and satisfy the ODE

$$\partial_\xi \widehat{\Gamma}_i(\xi;\delta) = \gamma_i(\xi) + a_\delta(\xi)b_\delta(\xi)\widehat{\Gamma}_i(\xi;\delta) > 0, \quad \widehat{\Gamma}_i(0;\delta) = 0. \tag{31}$$

Lemma 3.7. *For all $\delta \leqslant 1$ we have*

$$c\exp\left(\frac{c}{\sqrt{\delta}}\right) \leqslant K_{i,\delta} \leqslant C\exp\left(\frac{C}{\sqrt{\delta}}\right), \tag{32}$$

and

$$cK_{2,\delta} \leqslant K_{1,\delta} \leqslant K_{2,\delta}. \tag{33}$$

Proof. Exploiting the properties of a_δ and b_δ from Lemma 3.3 we find

$$\widehat{\Gamma}_i(1;\delta) \leqslant \int_0^1 \gamma_i(z)\exp\left(\int_0^{\frac{1}{4}} a_\delta(y)b_\delta(y)\,dy\right)\exp\left(\int_{\frac{1}{4}}^1 a_\delta(y)b_\delta(y)\,dy\right)dz \leqslant C\exp\left(\frac{C}{\sqrt{\delta}}\right),$$

as well as

$$\widehat{\Gamma}_i(1;\delta) \geqslant c\int_0^{1/4} \gamma_i(z)\exp\left(c\int_{1/4}^1 a_\delta(y)\,dy\right)dz \geqslant c\exp\left(\frac{c}{\sqrt{\delta}}\right),$$

and this gives (32) since $K_{i,\delta} = \widehat{\Gamma}_i(1;\delta)$ by definition. The inequality $K_{1,\delta} \leqslant K_{2,\delta}$ is obvious as $\gamma_1(z) \leqslant \gamma_2(z)$ for all $0 \leqslant z \leqslant 1$. Moreover, there exists a constant $\tilde{c}$ such that

$$\widehat{\Gamma}_1\left(\frac{1}{4};\delta\right) \geqslant \tilde{c}\,\widehat{\Gamma}_2\left(\frac{1}{4};\delta\right),$$

where we used that a_δ, b_δ can be expanded in powers of δ on $[0, \frac{1}{4}]$, and the ODE (31) implies that $\widehat{\Gamma}_1$ is a supersolution to the equation for $c\widehat{\Gamma}_2(\xi;\delta)$ on $[\frac{1}{4}, 1]$, where $c = \min\{\tilde{c}, \frac{1}{4}\}$. Hence we proved (33). □

Lemma 3.8. *Let $\delta \leqslant 1$. Then*

$$\widehat{\Gamma}_i(\xi;\delta) \leqslant C K_{i,\delta}\xi^4, \quad 1 \leqslant \xi < \infty, \tag{34}$$

and

$$\widehat{\Gamma}_i(\xi;\delta) = K_{i,\delta}\left(\exp\left(-\int_\xi^1 a_\delta(y)b_\delta(y)\,dy\right) - o(1)\sqrt{\delta}\right), \quad 0 \leqslant \xi \leqslant 1. \tag{35}$$

In particular,

1. $\widehat{\Gamma}_i(\xi;\delta) \leqslant C_\sigma$ *for* $\xi \leqslant \frac{1}{2} - \sigma$,
2. $\widehat{\Gamma}_i(\xi;\delta) \geqslant c_\sigma K_{i,\delta}$ *for* $\frac{1}{2} + \sigma \leqslant \xi \leqslant 1$,

with $0 < \sigma < \frac{1}{2}$ *arbitrary.*

Proof. We start with $\xi \geqslant 1$. Lemma 3.3 provides

$$\exp\left(\int_1^\xi a_\delta(y)b_\delta(y)\,dy\right) \leqslant \exp\bigl(C + 3\ln(\xi)\bigr) \leqslant C\xi^3$$

and we conclude

$$\begin{aligned}\widehat{\Gamma}_i(\xi;\delta) &= \int_0^1 \gamma_i(z)\exp\left(\int_z^\xi a_\delta(y)b_\delta(y)\,dy\right)dz + \int_1^\xi \gamma_i(z)\exp\left(\int_z^\xi a_\delta(y)b_\delta(y)\,dy\right)dz \\ &\leqslant C\xi^3\int_0^1 \gamma_i(z)\exp\left(\int_z^1 a_\delta(y)b_\delta(y)\,dy\right)dz + C\xi^3\int_1^\xi \gamma_i(z)\,dz \leqslant C\xi^4(K_{i,\delta}+1),\end{aligned}$$

which implies (34) thanks to (32). Now consider $0 \leqslant \xi \leqslant 1$, and let $\xi_0 = (\frac{1}{2})^{5/2} < \frac{1}{2}$ so that $b_\delta(y) \geqslant 0$ for all $y \geqslant \xi_0$, compare Lemma 3.3. For $0 \leqslant \xi \leqslant \xi_0$ we have

$$\begin{aligned}0 &\leqslant \int_\xi^1 \gamma_i(z)\exp\left(\int_z^\xi a_\delta(y)b_\delta(y)\,dy\right) \\ &= \int_\xi^{\xi_0} \gamma_i(z)\exp\left(-\int_\xi^z a_\delta(y)b_\delta(y)\,dy\right) + \int_{\xi_0}^1 \gamma_i(z)\exp\left(-\int_\xi^z a_\delta(y)b_\delta(y)\,dy\right)dz\end{aligned}$$

$$\leqslant \int_{\xi}^{\xi_0} \gamma_i(z) \exp\left(\int_0^{\xi_0} a_\delta(y)\left|b_\delta(y)\right| dy\right) dz + \int_{\xi_0}^{1} \gamma_i(z)\, dz \leqslant C,$$

whereas for $\xi_0 \leqslant \xi \leqslant 1$ we find

$$0 \leqslant \int_{\xi}^{1} \gamma_i(z) \exp\left(\int_z^{\xi} a_\delta(y) b_\delta(y)\, dy\right) dz \leqslant \int_{\xi}^{1} \gamma_i(z)\, dz \leqslant C.$$

Therefore, for all $0 \leqslant \xi \leqslant 1$ we have

$$\begin{aligned}
\widehat{\Gamma}_i(\xi;\delta) &= \int_0^1 \gamma_i(z) \exp\left(\int_z^{\xi} a_\delta(y) b_\delta(y)\, dy\right) dz - \int_{\xi}^{1} \gamma_i(z) \exp\left(\int_z^{\xi} a_\delta(y) b_\delta(y)\, dy\right) dz \\
&\geqslant \int_0^1 \gamma_i(z) \exp\left(\int_z^{1} a_\delta(y) b_\delta(y)\, dy\right) \exp\left(-\int_{\xi}^{1} a_\delta(y) b_\delta(y)\, dy\right) dz - C \\
&= K_{i,\delta} \exp\left(-\int_{\xi}^{1} a_\delta(y) b_\delta(y)\, dy\right) - C,
\end{aligned}$$

and (35) follows due to (32). The remaining assertions are direct consequences of (35) and Lemma 3.3. □

From now on we assume that both ε and $\tilde{\varepsilon}$ are small with respect to δ.

Assumption 3.9. Suppose $\varepsilon = o(\sqrt{\delta})$ and $\tilde{\varepsilon} = o(\sqrt{\delta})$.

Lemma 3.10. *Under Assumption* 3.9 *the estimates*

$$\Gamma_i(\xi;\varepsilon,\tilde{\varepsilon},\delta) \leqslant \widehat{\Gamma}_i(\xi;\delta), \quad 0 \leqslant \xi < \infty,$$

and

$$\left|\Gamma_i(\xi;\varepsilon,\tilde{\varepsilon},\delta) - \widehat{\Gamma}_i(\xi;\delta)\right| = o(1)\widehat{\Gamma}_i(\xi;\delta), \quad 0 \leqslant \xi \leqslant 1,$$

hold for $i = 1, 2$ and all $\delta \leqslant 1$.

Proof. The first assertion follows from definition (25) and the positivity of a_δ. With $0 \leqslant z \leqslant \xi \leqslant 1$ we find

$$0 \leqslant \int_z^{\xi} a_\delta(y)(\tilde{\varepsilon} y + \varepsilon)\, dy \leqslant (\tilde{\varepsilon} + \varepsilon) \int_0^1 a_\delta(y) \leqslant C \frac{\varepsilon + \tilde{\varepsilon}}{\sqrt{\delta}},$$

and

$$\left|\exp\left(-\int_z^\xi a_\delta(y)(\tilde{\varepsilon}y+\varepsilon)\,dy\right)-1\right| \leqslant \exp\left(\int_z^\xi a_\delta(y)(\tilde{\varepsilon}y+\varepsilon)\,dy\right)-1 \leqslant \exp\left(C\frac{\varepsilon+\tilde{\varepsilon}}{\sqrt{\delta}}\right)-1=o(1)$$

gives the second assertion. □

3.2. *Solving the fixed point equation for* $(\varepsilon, \tilde{\varepsilon})$

Lemma 3.11. *For* $\delta \leqslant 1$ *and all* h *sufficiently close to* $\Phi_{\mathrm{LSW}} * \Phi_{\mathrm{LSW}}$ *the following estimates are satisfied*

$$\|h\|_Z \leqslant 2\|\Phi_{\mathrm{LSW}} * \Phi_{\mathrm{LSW}}\|_Z, \qquad \widehat{G}_i[h;\delta] = \big(1 \pm o(1)\big)\, K_{i,\delta} R_\delta[h], \quad c \leqslant R_\delta[h] \leqslant C. \tag{36}$$

Proof. Eq. (35) provides

$$\begin{aligned}
\widehat{G}_i[h;\delta] &= \int_0^1 \xi a_\delta(\xi) h(\xi) \widehat{\Gamma}_i(\xi;\delta)\,d\xi \\
&= K_{i,\delta} \int_0^1 \xi a_\delta(\xi) h(\xi) \exp\left(-\int_\xi^1 a_\delta(y) b_\delta(y)\,dy\right) d\xi - o(1)\sqrt{\delta} K_{i,\delta} \int_0^1 \xi a_\delta(\xi) h(\xi)\,d\xi \\
&= K_{i,\delta}\big(R_\delta[h] \pm o(1)\lceil h \rceil_Z\big).
\end{aligned} \tag{37}$$

Recall that $\Phi_{\mathrm{LSW}} * \Phi_{\mathrm{LSW}}$ is strictly positive on $[0, \frac{7}{8}]$, and suppose that $\|h - \Phi_{\mathrm{LSW}} * \Phi_{\mathrm{LSW}}\|_Z$ is sufficiently small such that $h(\xi) \leqslant 2\|\Phi_{\mathrm{LSW}} * \Phi_{\mathrm{LSW}}\|_Z$ for all $0 \leqslant \xi \leqslant 1$ and

$$h(\xi) \geqslant \frac{1}{2}(\Phi_{\mathrm{LSW}} * \Phi_{\mathrm{LSW}})(\xi) \geqslant c, \quad \frac{5}{8} \leqslant \xi \leqslant \frac{7}{8}.$$

Thanks to Lemma 3.4 this estimate implies

$$0 < c \int_{5/8}^{7/8} \varrho_\delta(\xi)\,d\xi \leqslant R_\delta[h] \leqslant 2\|\Phi_{\mathrm{LSW}} * \Phi_{\mathrm{LSW}}\|_Z R_\delta[1] < \infty.$$

In particular, $R_\delta[h] \pm o(1)\lceil h \rceil_Z = (1 \pm o(1))R_\delta[h]$, which is the third claimed estimate, and using (37) we find the second inequality from (36). □

From now on we make the following assumption on the function h.

Assumption 3.12. Let δ be sufficiently small, and $\varepsilon + \tilde{\varepsilon} = o(\sqrt{\delta})$. Moreover, suppose that h is sufficiently close to $\Phi_{\mathrm{LSW}} * \Phi_{\mathrm{LSW}}$ in the sense that all estimates from (36) as well as

$$\lfloor h \rfloor_Z = o(1) \tag{38}$$

are satisfied.

Remark 3.13.

1. The condition $\lfloor h \rfloor_Z = o(1)$ arises naturally. In fact, below we consider functions Φ with $\|\Phi - \Phi_{\mathrm{LSW}}\|_Z = o(1)$, and this implies $\lfloor \Phi * \Phi \rfloor_Z \leqslant \|\Phi * \Phi - \Phi_{\mathrm{LSW}} * \Phi_{\mathrm{LSW}}\|_Z = o(1)$.
2. The condition $\|\Phi - \Phi_{\mathrm{LSW}}\|_Z = o(1)$ depends crucially on the parameters parameter β_1 and β_2 from (18), because for fixed Φ the quantity $\lfloor \Phi \rfloor_Z$ grows as $\beta_1 \to \infty$ or $\beta_2 \to \infty$. This will effect the final choice for δ, see condition (46) below.

3.2.1. Properties of G_i, $\widehat{G}_i$, and their derivatives

In order to estimate G_i we split the ξ-integration in (25) as follows

$$G_{i,1}[h;\varepsilon,\tilde{\varepsilon},\delta] := \int_0^1 \xi a_\delta(\xi) h(\xi) \Gamma_i(\xi;\varepsilon,\tilde{\varepsilon},\delta)\, d\xi,$$

$$G_{i,2}[h;\varepsilon,\tilde{\varepsilon},\delta] := \int_1^\infty \xi a_\delta(\xi) h(\xi) \Gamma_i(\xi;\varepsilon,\tilde{\varepsilon},\delta)\, d\xi,$$

so that $G_i = G_{i,1} + G_{i,2}$.

Lemma 3.14. *Assumption* 3.12 *implies*

$$\big|G_i[h;\varepsilon,\tilde{\varepsilon},\delta] - \widehat{G}_i[h;\delta]\big| \leqslant o(1)\, \widehat{G}_i[h;\delta],$$

and

$$\varepsilon\big|\partial_\varepsilon G_i[h;\varepsilon,\tilde{\varepsilon},\delta]\big| \leqslant o(1)\widehat{G}_i[h;\delta], \qquad \tilde{\varepsilon}\big|\partial_{\tilde{\varepsilon}} G_i[h;\varepsilon,\tilde{\varepsilon},\delta]\big| \leqslant o(1)\widehat{G}_i[h;\delta],$$

for both $i = 1$ and $i = 2$.

Proof. Lemma 3.10 implies

$$\big|G_{i,1}[h;\varepsilon,\tilde{\varepsilon},\delta] - \widehat{G}_i[h;\delta]\big| \leqslant o(1)\widehat{G}_i[h;\delta], \tag{39}$$

and using Lemma 3.3 and Lemma 3.8 we find

$$\begin{aligned} G_{i,2}[h;\varepsilon,\tilde{\varepsilon},\delta] &\leqslant \int_1^\infty \xi a_\delta(\xi) h(\xi) \widehat{\Gamma}_i(\xi;\delta)\, d\xi \leqslant C K_{i,\delta} \int_1^\infty h(\xi)\xi^4\, d\xi \\ &\leqslant C K_{i,\delta} \lfloor h \rfloor_Z \leqslant o(1)\widehat{G}_i(h;\delta), \end{aligned} \tag{40}$$

where we additionally used (29) and (38). Combining (39) and (40) yields the desired estimates for G_i. In order to control the derivatives we compute

$$\partial_\varepsilon G_{i,1}[h;\varepsilon,\tilde{\varepsilon},\delta] = \int_0^1 \xi a_\delta(\xi) h(\xi) \partial_\varepsilon \Gamma_i(\xi;\varepsilon,\tilde{\varepsilon},\delta)\, d\xi,$$

$$\partial_{\tilde{\varepsilon}} G_{i,1}[h;\varepsilon,\tilde{\varepsilon},\delta] = \int_0^1 \xi a_\delta(\xi) h(\xi) \partial_{\tilde{\varepsilon}} \Gamma_i(\xi;\varepsilon,\tilde{\varepsilon},\delta)\, d\xi,$$

as well as similar formulas for the derivatives of $G_{i,2}$, where

$$\partial_\varepsilon \Gamma_i(\xi;\varepsilon,\tilde{\varepsilon},\delta) = \int_0^\xi \gamma_i(z) \exp\left(\int_z^\xi a_\delta(y)\big(b_\delta(y) - \tilde{\varepsilon} y - \varepsilon\big)\, dy\right)\left(-\int_z^\xi a_\delta(y)\, dy\right) dz,$$

$$\partial_{\tilde{\varepsilon}} \Gamma_i(\xi;\varepsilon,\tilde{\varepsilon},\delta) = \int_0^\xi \gamma_i(z) \exp\left(\int_z^\xi a_\delta(y)\big(b_\delta(y) - \tilde{\varepsilon} y - \varepsilon\big)\, dy\right)\left(-\int_z^\xi a_\delta(y) y\, dy\right) dz.$$

For $0 \leqslant z \leqslant \xi \leqslant 1$ we estimate

$$\left|\int_z^\xi a_\delta(y)(y+1)\, dy\right| \leqslant \int_0^1 a_\delta(y)(y+1)\, dy \leqslant \frac{C}{\sqrt{\delta}},$$

so that

$$\big|\partial_\varepsilon \Gamma_i(\xi;\varepsilon,\tilde{\varepsilon},\delta)\big| \leqslant \frac{C}{\sqrt{\delta}} \widehat{\Gamma}_i(\xi;\delta), \qquad \big|\partial_{\tilde{\varepsilon}} \Gamma_i(\xi;\varepsilon,\tilde{\varepsilon},\delta)\big| \leqslant \frac{C}{\sqrt{\delta}} \widehat{\Gamma}_i(\xi;\delta)$$

due to Lemma 3.10. Multiplication with $\xi a_\delta(\xi)\, h(\xi)$ and integration over $0 \leqslant \xi \leqslant 1$ provide

$$\varepsilon \big|\partial_\varepsilon G_{i,1}[h;\varepsilon,\tilde{\varepsilon},\delta]\big| \leqslant o(1) \widehat{G}_i[h;\delta], \qquad \tilde{\varepsilon} \big|\partial_\varepsilon G_{i,1}[h;\varepsilon,\tilde{\varepsilon},\delta]\big| \leqslant o(1) \widehat{G}_i[h;\delta],$$

where we used $(\varepsilon + \tilde{\varepsilon})/\sqrt{\delta} = o(1)$. For $\xi \geqslant 1$ we estimate

$$\left|\int_z^\xi a_\delta(y)(y+1)\, dy\right| \leqslant \int_0^1 a_\delta(y)(y+1)\, dy + \int_1^\xi a_\delta(y)(y+1)\, dy \leqslant \frac{C}{\sqrt{\delta}} + C\xi \leqslant \frac{C}{\sqrt{\delta}} \xi,$$

and exploiting Lemma 3.8 and Lemma 3.10 we find

$$\big|\partial_\varepsilon \Gamma_i(\xi;\varepsilon,\tilde{\varepsilon},\delta)\big| \leqslant \frac{C}{\sqrt{\delta}} \xi^5 K_{i,\delta}, \qquad \big|\partial_{\tilde{\varepsilon}} \Gamma_i(\xi;\varepsilon,\tilde{\varepsilon},\delta)\big| \leqslant \frac{C}{\sqrt{\delta}} \xi^5 K_{i,\delta}.$$

Finally, (29) combined with (36) gives

$$\varepsilon \big|\partial_\varepsilon G_{i,2}[h;\varepsilon,\tilde{\varepsilon},\delta]\big| \leqslant o(1) \widehat{G}_i[h;\delta], \qquad \tilde{\varepsilon} \big|\partial_{\tilde{\varepsilon}} G_{i,2}[h;\varepsilon,\tilde{\varepsilon},\delta]\big| \leqslant o(1) \widehat{G}_i[h;\delta],$$

and the proof is complete. □

3.2.2. Choosing ε and $\tilde{\varepsilon}$

Lemma 3.15. *The following assertions hold true under Assumption* 3.12.

1. *For small δ and $i = 1, 2$ we have*

$$\left|g_i[h;\varepsilon,\tilde{\varepsilon},\delta] - \varepsilon^2\widehat{G}_i[h;\delta]\right| \leqslant o(1)\varepsilon^2\widehat{G}_i[h;\delta] \tag{41}$$

as well as

$$\left|\varepsilon\partial_\varepsilon g_i[h;\varepsilon,\tilde{\varepsilon},\delta] - 2g_i[h;\varepsilon,\tilde{\varepsilon},\delta]\right| + \left|\tilde{\varepsilon}\partial_{\tilde{\varepsilon}} g_i[h;\varepsilon,\tilde{\varepsilon},\delta]\right| \leqslant o(1)\varepsilon^2\widehat{G}_i[h;\delta]. \tag{42}$$

2. *For each $\alpha > 1$ there exists $\delta_\alpha > 0$ such that for all $\delta \leqslant \delta_\alpha$ each solution $(\varepsilon, \tilde{\varepsilon})$ to the compatibility conditions* (20) *must belong to*

$$U[h;\alpha,\delta] := \left\{(\varepsilon,\tilde{\varepsilon}) \frac{1}{\alpha}\varepsilon_{\text{app}}[h;\delta] \leqslant \varepsilon \leqslant \alpha\varepsilon_{\text{app}}[h;\delta],\ \frac{1}{\alpha}\tilde{\varepsilon}_{\text{app}}[h;\delta] \leqslant \tilde{\varepsilon} \leqslant \alpha\tilde{\varepsilon}_{\text{app}}[h;\delta]\right\},$$

where

$$\varepsilon_{\text{app}}[h;\delta] := 1/\widehat{G}_1[h;\delta], \qquad \tilde{\varepsilon}_{\text{app}}[h;\delta] := \varepsilon_{\text{app}}[h;\delta]\widehat{G}_2[h;\delta]/\widehat{G}_1[h;\delta]$$

have the same order of magnitude thanks to (33)*.*

3. *For small δ and given h (bounded by Assumption* 3.12*) there exists a solution $(\varepsilon[h;\delta], \tilde{\varepsilon}[h;\delta])$ to* (20)*. Moreover, this solution is unique under the constraints $\varepsilon,\ \tilde{\varepsilon} = o(\sqrt{\delta})$.*

Proof. Let δ be sufficiently small, and h and $\alpha > 1$ be fixed. The estimates (41) and (42) are provided by Lemma 3.14 and imply

$$\frac{g_1(\varepsilon,\tilde{\varepsilon})}{\varepsilon} = \big(1 \pm o(1)\big)\frac{\varepsilon}{\varepsilon_{\text{app}}}, \qquad g_2(\varepsilon,\tilde{\varepsilon}) = \big(1 \pm o(1)\big)\left(\frac{\varepsilon}{\varepsilon_{\text{app}}}\right)^2\tilde{\varepsilon}_{\text{app}},$$

where $g_i(\varepsilon,\tilde{\varepsilon})$, ε_{app}, and $\tilde{\varepsilon}_{\text{app}}$ are shorthands for $g_i[h,\varepsilon,\tilde{\varepsilon},\delta]$, $\varepsilon_{\text{app}}[h;\delta]$, and $\tilde{\varepsilon}_{\text{app}}[h;\delta]$, respectively. Therefore, $(g_1(\varepsilon,\tilde{\varepsilon}), g_2(\varepsilon,\tilde{\varepsilon})) = (\varepsilon,\tilde{\varepsilon})$ implies $\varepsilon = (1 \pm o(1))\varepsilon_{\text{app}}$, and in turn $\tilde{\varepsilon} = (1 \pm o(1))\tilde{\varepsilon}_{\text{app}}$, so each solution to (20) must be an element of $U[h;\alpha,\delta]$.

Now suppose $(\varepsilon,\tilde{\varepsilon}) \in U[h,\alpha,\delta]$. For $\varepsilon = \frac{1}{\alpha}\varepsilon_{\text{app}}$ and $\varepsilon = \alpha\varepsilon_{\text{app}}$ we have

$$g_1(\varepsilon,\tilde{\varepsilon}) = \big(1 \pm o(1)\big)\frac{1}{\alpha}\varepsilon < \varepsilon \quad \text{and} \quad g_1(\varepsilon,\tilde{\varepsilon}) = \big(1 \pm o(1)\big)\alpha\varepsilon > \varepsilon,$$

respectively, and (42) implies $\partial_\varepsilon g_1 > 0$. Therefore, for each $\tilde{\varepsilon}$ there exists a unique solution $\varepsilon = \varepsilon(\tilde{\varepsilon})$ to $g_1 = \varepsilon$, i.e.,

$$g_1\big(\varepsilon(\tilde{\varepsilon}),\tilde{\varepsilon}\big) = \varepsilon(\tilde{\varepsilon}) = \big(1 \pm o(1)\big)\varepsilon_{\text{app}},$$

and differentiation with respect to $\tilde{\varepsilon}$ shows

$$\left|\frac{d\varepsilon}{d\tilde{\varepsilon}}\right| = |\partial_{\tilde{\varepsilon}} g_1|/|\partial_\varepsilon g_1 - 1| = o(1)\frac{\varepsilon}{\tilde{\varepsilon}} = o(1)\frac{\varepsilon_{\text{app}}}{\tilde{\varepsilon}_{\text{app}}} = o(1)$$

since $c\varepsilon_{\text{app}} \leqslant \tilde{\varepsilon}_{\text{app}} \leqslant C\varepsilon_{\text{app}}$ thanks to Lemma 3.11. Now let $\tilde{g}_2(\tilde{\varepsilon}) := g_2(\varepsilon(\tilde{\varepsilon}),\tilde{\varepsilon})$, and note that

$$\tilde{g}_2 = \big(1 \pm o(1)\big)\tilde{\varepsilon}_{\text{app}}, \qquad \left|\frac{\tilde{g}_2}{d\tilde{\varepsilon}}\right| = |\partial_\varepsilon g_2| \left|\frac{d\varepsilon}{d\tilde{\varepsilon}}\right| + |\partial_{\tilde{\varepsilon}} g_2| = o(1)\frac{\tilde{g}_2}{\tilde{\varepsilon}} = o(1).$$

Thus, for small δ the function $\tilde{g_2}$ is contractive with

$$\tilde{g}_2\left(\frac{1}{\alpha}\tilde{\varepsilon}_{\text{app}}\right) > \frac{1}{\alpha}\tilde{\varepsilon}_{\text{app}}, \qquad \tilde{g}_2(\alpha\tilde{\varepsilon}_{\text{app}}) < \alpha\tilde{\varepsilon}_{\text{app}},$$

and hence there exists a unique solution to $\tilde{\varepsilon} = \tilde{g}_2(\tilde{\varepsilon})$. □

In Section 3.3 below we consider functions h close to $\Phi_{\text{LSW}} * \Phi_{\text{LSW}}$, and then the following result, which follows from Lemma 3.11, Lemma 3.15 and Corollary 3.5, becomes useful.

Corollary 3.16. *Suppose that*

$$\|h - \Phi_{\text{LSW}} * \Phi_{\text{LSW}}\|_Z = o(1).$$

Then the solution $(\varepsilon[h;\delta], \tilde{\varepsilon}[h;\delta])$ *from Lemma* 3.15 *satisfies*

$$\varepsilon_{\text{app}}[h;\delta] = \big(1 \pm o(1)\big)\epsilon_\delta, \qquad \tilde{\varepsilon}_{\text{app}}[h;\delta] = \big(1 \pm o(1)\big)\tilde{\epsilon}_\delta,$$

where

$$\epsilon_\delta := 1/K_{1,\delta} R_0[\Phi_{\text{LSW}} * \Phi_{\text{LSW}}], \qquad \tilde{\epsilon}_\delta := \epsilon_\delta K_{2,\delta}/K_{1,\delta}. \tag{43}$$

In particular, the assertions from Theorem 2.3 *and Proposition* 2.5 *are satisfied.*

3.2.3. Continuity of ε and $\tilde{\varepsilon}$

Lemma 3.17. *The solution from Lemma* 3.15 *depends Lipschitz-continuously on h. More precisely, for arbitrary* h_1, h_2 *that fulfil Assumption* 3.12 *we have*

$$\big|\varepsilon[h_2;\delta] - \varepsilon[h_1;\delta]\big| + \big|\tilde{\varepsilon}[h_2;\delta] - \tilde{\varepsilon}[h_1;\delta]\big| \leqslant C\big(\varepsilon[h_2;\delta] + \varepsilon[h_1;\delta]\big)\|h_2 - h_1\|_Z.$$

Proof. We fix δ, abbreviate $\varepsilon_i = \varepsilon[h_i;\delta]$ and $\tilde{\varepsilon}_i = \tilde{\varepsilon}[h_i;\delta]$, and for arbitrary $\tau \in [0,1]$ and $(\varepsilon, \tilde{\varepsilon}) \in U_\delta$ we write $h(\tau) = \tau h_2 + (1-\tau)h_1$, as well as

$$\bar{g}_i(\varepsilon, \tilde{\varepsilon}, \tau) = g_i\big[h(\tau); \varepsilon, \tilde{\varepsilon}, \delta\big], \qquad \varepsilon(\tau) = \varepsilon\big[h(\tau);\delta\big], \qquad \tilde{\varepsilon}(\tau) = \tilde{\varepsilon}\big[h(\tau);\delta\big],$$

so that

$$\varepsilon(\tau) = \bar{g}_1\big(\varepsilon(\tau), \tilde{\varepsilon}(\tau), \tau\big), \qquad \tilde{\varepsilon}(\tau) = \bar{g}_2\big(\varepsilon(\tau), \tilde{\varepsilon}(\tau), \tau\big) \tag{44}$$

hold by construction. For fixed $(\varepsilon, \tilde{\varepsilon})$ we estimate $\partial_\tau \bar{g}_i$ as follows

$$\begin{aligned}
\big|\partial_\tau \bar{g}_i(\varepsilon, \tilde{\varepsilon}, \tau)\big| &\leqslant g_i\big[|h_2 - h_1|; \varepsilon, \tilde{\varepsilon}, \delta\big] = \varepsilon^2 \int_0^\infty \xi a_\delta(\xi)\big|h_2(\xi) - h_1(\xi)\big|\Gamma_i(\xi; \varepsilon, \tilde{\varepsilon}, \delta) \\
&\leqslant \varepsilon^2 \lceil h_2 - h_1 \rceil_Z \widehat{G}_i(1;\delta) + \varepsilon^2 \lfloor h_2 - h_1 \rfloor_Z K_{i,\delta}, \\
&\leqslant C\varepsilon^2 K_{i,\delta}\|h_2 - h_1\|_Z,
\end{aligned}$$

compare (39) and (40), and Lemma 3.11 combined with Lemma 3.15 provides

$$\left|\partial_\tau g_i\big(\varepsilon(\tau), \tilde{\varepsilon}(\tau), \tau\big)\right| \leqslant C\varepsilon(\tau)\|h_2 - h_1\|_Z = C(\varepsilon_2 + \varepsilon_1)\|h_2 - h_1\|_Z.$$

Differentiating (44) with respect to τ yields

$$\varepsilon' = \partial_\varepsilon \bar{g}_1 \varepsilon' + \partial_{\tilde{\varepsilon}} \bar{g}_1 \tilde{\varepsilon}' + \partial_\tau \bar{g}_1, \qquad \tilde{\varepsilon}' = \partial_\varepsilon \bar{g}_2 \varepsilon' + \partial_{\tilde{\varepsilon}} \bar{g}_2 \tilde{\varepsilon}' + \partial_\tau \bar{g}_2,$$

where $'$ denotes $\frac{d}{d\tau}$. Moreover, Lemma 3.15 combined with $\bar{g}_1 = \varepsilon$, $\bar{g}_2 = \tilde{\varepsilon}$, and $\varepsilon/\tilde{\varepsilon} = O(1)$ provides

$$\partial_\varepsilon \bar{g}_1 = \big(2 + o(1)\big), \qquad \partial_\varepsilon \bar{g}_2 = O(1), \qquad \partial_{\tilde{\varepsilon}} \bar{g}_1 = o(1), \qquad \partial_{\tilde{\varepsilon}} \bar{g}_2 = o(1),$$

and we conclude that

$$\begin{pmatrix} -1 + o(1) & o(1) \\ O(1) & 1 + o(1) \end{pmatrix} \begin{pmatrix} \varepsilon' \\ \tilde{\varepsilon}' \end{pmatrix} \sim 1 \begin{pmatrix} \partial_\tau \bar{g}_1 \\ \partial_\tau \bar{g}_2 \end{pmatrix}.$$

Finally, we find

$$|\varepsilon_2 - \varepsilon_1| + |\tilde{\varepsilon}_2 - \tilde{\varepsilon}_1| = \int_0^1 |\varepsilon' + \tilde{\varepsilon}'|\, d\tau \leqslant C(\varepsilon_1 + \varepsilon_2)\, \|h_2 - h_1\|_Z,$$

which is the desired result. □

3.3. *Solving the fixed point equation for Φ*

For each $h \in Z$ and arbitrary parameters $(\varepsilon, \tilde{\varepsilon})$ we define the function

$$J[h; \varepsilon, \tilde{\varepsilon}, \delta](z) := \psi(z; \varepsilon, \tilde{\varepsilon}, \delta) \int_z^\infty \frac{\xi a_\delta(\xi) h(\xi)}{\psi(\xi; \varepsilon, \tilde{\varepsilon}, \delta)}\, d\xi,$$

which is related to the fixed problem for Φ via

$$\bar{I}_\delta[\Phi] = \varepsilon[\Phi * \Phi; \delta] J\big[\Phi * \Phi, \varepsilon[\Phi * \Phi; \delta], \tilde{\varepsilon}[\Phi * \Phi; \delta], \delta\big],$$

compare (19). Notice that the exponental decay of h implies that the function $J[h; \varepsilon, \tilde{\varepsilon}, \delta]$ is well defined and has finite moments, i.e.,

$$\int_0^\infty J[h; \varepsilon, \tilde{\varepsilon}, \delta](z)\, dz < \infty, \qquad \int_0^\infty z J[h; \varepsilon, \tilde{\varepsilon}, \delta](z)\, dz < \infty.$$

Moreover, below we show, cf. Corollary 3.20, that the operator $J[\cdot; \varepsilon, \tilde{\varepsilon}; \delta]$ maps the space Z into itself.

3.3.1. Approximation of the operator J

In this section we show that the operator J can be approximated by

$$J_{\mathrm{app}}[h;\varepsilon,\tilde{\varepsilon},\delta](z) := \chi_{[0,1]}(z)\psi(z;\varepsilon,\tilde{\varepsilon},\delta)\int_0^\infty \xi\, J[h;\varepsilon,\tilde{\varepsilon},\delta](\xi)\,d\xi,$$

that means all contributions coming from

$$J_{\mathrm{res}}[h;\varepsilon,\tilde{\varepsilon},\delta] := J[h;\varepsilon,\tilde{\varepsilon},\delta] - J_{\mathrm{app}}[h;\varepsilon,\tilde{\varepsilon},\delta]$$

can be neglected. To prove this we split the operator J into three parts $J = J_1 + J_2 + J_3$ with

$$\begin{aligned}
J_1[h;\varepsilon,\tilde{\varepsilon},\delta](z) &:= +\chi_{[0,1]}(z)\psi(z;\varepsilon,\tilde{\varepsilon},\delta)\int_0^\infty \frac{\xi a_\delta(\xi)h(\xi)}{\psi(\xi;\varepsilon,\tilde{\varepsilon},\delta)}\,d\xi,\\
J_2[h;\varepsilon,\tilde{\varepsilon},\delta](z) &:= -\chi_{[0,1]}(z)\psi(z;\varepsilon,\tilde{\varepsilon},\delta)\int_0^z \frac{\xi a_\delta(\xi)h(\xi)}{\psi(\xi;\varepsilon,\tilde{\varepsilon},\delta)}\,d\xi,\\
J_3[h;\varepsilon,\tilde{\varepsilon},\delta](z) &:= +\chi_{[1,\infty)}(z)\psi(z;\varepsilon,\tilde{\varepsilon},\delta)\int_z^\infty \frac{\xi a_\delta(\xi)h(\xi)}{\psi(\xi;\varepsilon,\tilde{\varepsilon},\delta)}\,d\xi.
\end{aligned} \tag{45}$$

Notice that $J_{\mathrm{app}}[h;\varepsilon,\tilde{\varepsilon},\delta]$, $J_1[h;\varepsilon,\tilde{\varepsilon},\delta]$, and $J_2[h;\varepsilon,\tilde{\varepsilon},\delta]$ are supported in $[0,1]$, whereas the support of $J_3[h;\varepsilon,\tilde{\varepsilon},\delta]$ equals $[1,\infty)$. Moreover, the next result shows that J_1 does not contribute to the residual operator J_{res}.

Remark 3.18. For all $h \in Z$ and all parameters $(\varepsilon,\tilde{\varepsilon},\delta)$ we have

$$\begin{aligned}
J_{\mathrm{res}}[h;\varepsilon,\tilde{\varepsilon},\delta](z) &= J_2[h;\varepsilon,\tilde{\varepsilon},\delta](z)\\
&\quad - \psi(z;\varepsilon,\tilde{\varepsilon},\delta)\left(\int_0^1 \xi\, J_2[h;\varepsilon,\tilde{\varepsilon},\delta](\xi)\,d\xi + \int_1^\infty \xi\, J_3[h;\varepsilon,\tilde{\varepsilon},\delta](\xi)\,d\xi\right),
\end{aligned}$$

for $0 \leqslant z \leqslant 1$, as well as

$$J_{\mathrm{res}}[h;\varepsilon,\tilde{\varepsilon},\delta](z) = J_3[h;\varepsilon,\tilde{\varepsilon},\delta](z),$$

for $z \geqslant 1$.

Proof. The second assertion is a direct consequence of (45). Now let $0 \leqslant z \leqslant 1$. Due to the normalization condition $\int_0^1 \xi\psi(\xi;\varepsilon,\tilde{\varepsilon},\delta)\,d\xi = 1$ we have

$$\int_0^\infty z J_1[h;\varepsilon,\tilde{\varepsilon},\delta](z)\,dz = \int_0^\infty \frac{\xi a_\delta(\xi)h(\xi)}{\psi(\xi;\varepsilon,\tilde{\varepsilon},\delta)}\,d\xi$$

and this implies

$$\psi(z;\varepsilon,\tilde{\varepsilon},\delta)\int_0^1 \xi J_1[h;\varepsilon,\tilde{\varepsilon},\delta](\xi)\,d\xi = J_1[h;\varepsilon,\tilde{\varepsilon},\delta](z).$$

Moreover, by definition we have

$$J_{\mathrm{res}}[h;\varepsilon,\tilde{\varepsilon},\delta](z) = J_1[h;\varepsilon,\tilde{\varepsilon},\delta](z) + J_2[h;\varepsilon,\tilde{\varepsilon},\delta](z) - \psi(z;\varepsilon,\tilde{\varepsilon},\delta)\sum_{i=1}^{3}\int_0^\infty \xi J_i[h;\varepsilon,\tilde{\varepsilon},\delta](\xi)\,d\xi,$$

and the combination of both results yields the first assertion. □

In the next step we estimate the operators J_2 and J_3 as well as their derivatives with respect to $(\varepsilon,\tilde{\varepsilon})$.

Lemma 3.19. *For all $h \in Z$ and all parameters $(\varepsilon,\tilde{\varepsilon},\delta)$ that satisfy Assumption* 3.9 *we find*

$$\left|J_2[h;\varepsilon,\tilde{\varepsilon},\delta](z)\right| + \left|\partial_\varepsilon J_2[h;\varepsilon,\tilde{\varepsilon},\delta](z)\right| + \left|\partial_{\tilde{\varepsilon}} J_2[h;\varepsilon,\tilde{\varepsilon},\delta](z)\right| \leqslant \frac{C}{\delta}\lceil h\rceil_Z$$

for all $0 \leqslant z \leqslant 1$, *as well as*

$$\left|J_3[h;\varepsilon,\tilde{\varepsilon},\delta](z)\right| + \left|\partial_\varepsilon J_3[h;\varepsilon,\tilde{\varepsilon},\delta](z)\right| + \left|\partial_{\tilde{\varepsilon}} J_3[h;\varepsilon,\tilde{\varepsilon},\delta](z)\right| \leqslant C\lfloor h\rfloor_Z \frac{\exp(-\beta_1 z)}{z^{\beta_2}}$$

for all $z \geqslant 1$.

Proof. Let $0 \leqslant z \leqslant 1$. The definition of J_2 provides

$$J_2[h;\varepsilon,\tilde{\varepsilon},\delta](z) = -\int_0^z \xi a_\delta(\xi)h(\xi)\exp\left(\int_z^\xi a_\delta(y)\big(b_\delta(y)-\varepsilon-\tilde{\varepsilon}y\big)\,dy\right)d\xi,$$

and we estimate

$$\begin{aligned}
\left|J_2[h;\varepsilon,\tilde{\varepsilon},\delta](z)\right| &\leqslant \lceil h\rceil_Z \exp\left(C\frac{\varepsilon+\tilde{\varepsilon}}{\sqrt{\delta}}\right)\int_0^z \xi a_\delta(\xi)\exp\left(-\int_\xi^z a_\delta(y)\min\{0,b_\delta(y)\}\,dy\right)d\xi \\
&\leqslant \lceil h\rceil_Z C\int_0^z \xi a_\delta(\xi)\exp\left(-\int_0^1 a_\delta(y)\min\{0,b_\delta(y)\}\,dy\right)d\xi \\
&\leqslant \lceil h\rceil_Z C\int_0^1 \xi a_\delta(\xi)\,d\xi \leqslant \frac{C}{\sqrt{\delta}}\lceil h\rceil_Z.
\end{aligned}$$

Moreover,

$$\begin{aligned}\left|\partial_\varepsilon J_2[h;\varepsilon,\tilde{\varepsilon},\delta](z)\right| &\leqslant \left|\int_0^z \xi a_\delta(\xi)h(\xi)\exp\left(\int_z^\xi a_\delta(y)\big(b_\delta(y)-\varepsilon-\tilde{\varepsilon}y\big)\,dy\right)\left(\int_z^\xi a_\delta(y)\,dy\right)d\xi\right| \\ &\leqslant \frac{1}{\sqrt{\delta}}J_2\big[|h|;\varepsilon,\tilde{\varepsilon},\delta\big](z) \leqslant \frac{C}{\delta}\lceil h\rceil_Z,\end{aligned}$$

and the estimate for $\partial_{\tilde{\varepsilon}} J_2[h;\varepsilon,\tilde{\varepsilon},\delta](z)$ is entirely similar. Now let $z \geqslant 1$. Then,

$$\begin{aligned}\left|J_3[h;\varepsilon,\tilde{\varepsilon},\delta](z)\right| &\leqslant \int_z^\infty \xi a_\delta(\xi)\left|h(\xi)\right|\exp\left(\int_z^\xi a_\delta(y)\big(b_\delta(y)-\varepsilon-\tilde{\varepsilon}y\big)\,dy\right)d\xi \\ &\leqslant C\int_z^\infty \left|h(\xi)\right|\exp\left(\int_z^\xi a_\delta(y)b_\delta(y)\,dy\right)d\xi \\ &\leqslant C\int_z^\infty \left|h(\xi)\right|\exp(C+3\ln\xi-3\ln z)\,d\xi = \frac{C}{z^3}\int_z^\infty \xi^3\left|h(\xi)\right|d\xi,\end{aligned}$$

as well as

$$\begin{aligned}&\left|\partial_{\tilde{\varepsilon}} J_3[h;\varepsilon,\tilde{\varepsilon},\delta](z)\right| + \left|\partial_{\tilde{\varepsilon}} J_3[h;\varepsilon,\tilde{\varepsilon},\delta](z)\right| \\ &\quad\leqslant \int_z^\infty \xi a_\delta(\xi)\left|h(\xi)\right|\exp\left(\int_z^\xi a_\delta(y)\big(b_\delta(y)-\varepsilon-\tilde{\varepsilon}y\big)\,dy\right)\left(\int_z^\xi a_\delta(y)(y+1)\,dy\right)d\xi \\ &\quad\leqslant C\int_z^\infty \left|h(\xi)\right|(\xi-z)\exp(C+3\ln\xi-3\ln z)\,d\xi = \frac{C}{z^3}\int_z^\infty \xi^3(\xi-z)\left|h(\xi)\right|d\xi.\end{aligned}$$

Finally, using (28) completes the proof. □

As a consequence of Remark 3.18 and Lemma 3.19 we obtain estimates for the residual operator. In particular, $h \in Z$ implies $J_{\mathrm{res}}[h;\varepsilon,\tilde{\varepsilon},\delta] \in Z$, and hence $J_{\mathrm{app}}[h;\varepsilon,\tilde{\varepsilon},\delta] \in Z$.

Corollary 3.20. *The assumptions from Lemma* 3.19 *imply*

$$J_{\mathrm{res}}[h;\varepsilon,\tilde{\varepsilon},\delta] \in Z, \qquad \partial_\varepsilon J_{\mathrm{res}}[h;\varepsilon,\tilde{\varepsilon},\delta] \in Z, \qquad \partial_{\tilde{\varepsilon}} J_{\mathrm{res}}[h;\varepsilon,\tilde{\varepsilon},\delta] \in Z$$

with

$$\big\lceil J_{\mathrm{res}}[h;\varepsilon,\tilde{\varepsilon},\delta]\big\rceil_Z \leqslant \frac{C}{\delta}\lceil h\rceil_Z + C\lfloor h\rfloor_Z, \qquad \big\lfloor J_{\mathrm{res}}[h;\varepsilon,\tilde{\varepsilon},\delta]\big\rfloor_Z \leqslant C\lfloor h\rfloor_Z$$

as well as

$$\big\|\partial_\varepsilon J_{\mathrm{res}}[h;\varepsilon,\tilde{\varepsilon},\delta]\big\|_Z + \big\|\partial_{\tilde{\varepsilon}} J_{\mathrm{res}}[h;\varepsilon,\tilde{\varepsilon},\delta]\big\|_Z \leqslant \frac{C}{\delta}\|h\|_Z.$$

In particular, for fixed h we have

$$\big\| J_{\mathrm{res}}[h;\varepsilon_2,\tilde{\varepsilon}_2,\delta] - J_{\mathrm{res}}[h;\varepsilon_1,\tilde{\varepsilon}_1,\delta]\big\|_Z \leqslant \frac{C}{\delta}\|h\|_Z\big(|\varepsilon_2-\varepsilon_1| + |\tilde{\varepsilon}_2-\tilde{\varepsilon}_1|\big).$$

Proof. All assertions are direct consequences of Remark 3.18 and Lemma 3.19. □

3.3.2. Setting for the fixed point problem

Here we introduce suitable subsets of the function space Z that allow to apply the contraction principle for the operator $\bar{I}_\delta$. For this reason we introduce the functions

$$\widehat{\Phi}_\delta(z) := \chi_{[0,1]}(z)\psi(z;0,0,\delta),$$

which satisfy $\|\widehat{\Phi}_\delta - \Phi_{\mathrm{LSW}}\|_Z \to 0$ as $\delta \to 0$, see (23) and (24).

Definition 3.21. Let μ_δ be a number with

$$\mu_\delta = o(1), \qquad \frac{1}{\delta K_{1,\delta}} = o\big(\mu_\delta^2\big), \qquad \|\widehat{\Phi}_\delta - \Phi_{\mathrm{LSW}}\|_Z = o\big(\mu_\delta^2\big), \tag{46}$$

and define the sets

$$\mathcal{Y}_\delta = \big\{\Phi \in Z\colon \|\Phi - \widehat{\Phi}_\delta\|_Z \leqslant \mu_\delta^2\big\}, \qquad \mathcal{Z}_\delta = \big\{h \in Z\colon \|h - \widehat{\Phi}_\delta * \widehat{\Phi}_\delta\|_Z \leqslant \mu_\delta\big\}.$$

Lemma 3.22. *For all sufficiently small δ the following assertions are satisfied.*

1. $\Phi_{\mathrm{LSW}} \in \mathcal{Y}_\delta$,
2. $\Phi \in \mathcal{Y}_\delta$ *implies* $\Phi * \Phi \in \mathcal{Z}_\delta$,
3. *each* $h \in \mathcal{Z}_\delta$ *satisfies Assumption* 3.12, *hence there exist the solutions* $\varepsilon[h;\delta]$ *and* $\tilde{\varepsilon}[h;\delta]$ *from Lemma* 3.15.

Proof. The first assertion holds by construction. For $\Phi \in \mathcal{Y}_\delta$ we have

$$\begin{aligned}\|\Phi * \Phi - \widehat{\Phi}_\delta * \widehat{\Phi}_\delta\|_Z &= \big\|2\Phi * (\Phi - \widehat{\Phi}_\delta) + (\Phi - \widehat{\Phi}_\delta) * (\Phi - \widehat{\Phi}_\delta)\big\|_Z \\ &\leqslant 2\big\|\Phi * (\Phi - \widehat{\Phi}_\delta)\big\|_Z + \big\|(\Phi - \widehat{\Phi}_\delta) * (\Phi - \widehat{\Phi}_\delta)\big\|_Z \leqslant C\|\Phi - \widehat{\Phi}_\delta\|_Z = o(\mu_\delta).\end{aligned}$$

This implies $\|\widehat{\Phi}_\delta * \widehat{\Phi}_\delta - \Phi_{\mathrm{LSW}} * \Phi_{\mathrm{LSW}}\|_Z \leqslant C\|\widehat{\Phi}_\delta - \Phi_{\mathrm{LSW}}\|_Z \leqslant C{\mu_\delta}^2 = o(\mu_\delta)$, and for all $h \in \mathcal{Z}_\delta$ we find

$$\|h - \Phi_{\mathrm{LSW}} * \Phi_{\mathrm{LSW}}\|_Z \leqslant \|h - \widehat{\Phi}_\delta * \widehat{\Phi}_\delta\|_Z + C\|\widehat{\Phi}_\delta - \Phi_{\mathrm{LSW}}\|_Z = o(\mu_\delta) = o(1).$$

Therefore,

$$\lfloor h \rfloor_Z = \lfloor \Phi_{\mathrm{LSW}} * \Phi_{\mathrm{LSW}} \rfloor_Z + o(\mu_\delta) = o(1),$$

and the proof is complete. □

3.3.3. Contraction principle for Φ

Recall that the solution $(\varepsilon, \tilde{\varepsilon})[h;\delta]$ from Lemma 3.15 satisfies

$$\frac{1}{\varepsilon[h;\delta]} = \int_0^\infty \xi\, J\big[h; \varepsilon[h;\delta], \tilde{\varepsilon}[h;\delta], \delta\big].$$

Therefore we define $I[h;\delta] = I_{\text{app}}[h;\delta] + I_{\text{res}}[h;\delta]$ with

$$I_{\text{app}}[h;\delta](z) := \varepsilon[h;\delta]\, J_{\text{app}}\big[h; \varepsilon[h;\delta], \tilde{\varepsilon}[h;\delta], \delta\big],$$
$$I_{\text{res}}[h;\delta](z) := \varepsilon[h;\delta]\, J_{\text{res}}\big[h; \varepsilon[h;\delta], \tilde{\varepsilon}[h;\delta], \delta\big],$$

and this implies $\bar{I}_\delta[\Phi] = I[\Phi * \Phi; \delta]$, with $\bar{I}_\delta$ as in (19), as well as

$$I_{\text{app}}[h;\delta](z) = \chi_{[0,1]}(z)\, \psi\big(z; \varepsilon[h;\delta], \tilde{\varepsilon}[h;\delta], \delta\big).$$

In particular, $\bar{I}_\delta[\Phi]$ is close to $\widehat{\Phi}_\delta$ with error controlled by ε and $\tilde{\varepsilon}$, provided that δ is small and h is close to $\Phi_{\text{LSW}} * \Phi_{\text{LSW}}$.

Lemma 3.23. *For sufficiently small δ the operator I maps $\mathcal{Z}_\delta$ into $\mathcal{Y}_\delta$ and is Lipschitz continuous with arbitrary small constant. More precisely,*

$$\big\| I[h_2;\delta] - I[h_1;\delta] \big\|_Z \leqslant o(1) \|h_2 - h_1\|_Z$$

for all $h_1, h_2 \in \mathcal{Z}_\delta$.

Proof. For each $h \in \mathcal{Z}_\delta$ Lemma 3.17 provides

$$\big\| I_{\text{res}}[h;\delta] \big\|_Z \leqslant C \frac{\varepsilon[h;\delta]}{\delta} \|h\|_Z \leqslant \frac{C}{\delta\, K_{1,\delta}} \|h\|_Z,$$

where we used Lemma 3.11 and Lemma 3.15. Moreover, (46) implies

$$\big\| I_{\text{res}}[h;\delta] \big\|_Z \leqslant o\big(\mu_\delta^2\big) \|h\|_Z = o\big(\mu_\delta^2\big) \|\widehat{\Phi}_\delta * \widehat{\Phi}_\delta\|_Z = o\big(\mu_\delta^2\big).$$

From Lemma 3.6 we derive

$$\widetilde{C}^{-1}\, \widehat{\Phi}_\delta(z) \leqslant \psi\big(z; \varepsilon[h;\delta], \tilde{\varepsilon}[h;\delta], \delta\big) \leqslant \widetilde{C}\, \widehat{\Phi}_\delta(z),$$

for all $0 \leqslant z \leqslant 1$ with

$$\widetilde{C}[h;\delta] = \exp\left(C \frac{\varepsilon[h;\delta] + \tilde{\varepsilon}[h;\delta]}{\sqrt{\delta}} \right) = \exp\big(o\big(\mu_\delta^2\big)\big).$$

We conclude

$$\big| I_{\text{app}}[h;\delta](z) - \widehat{\Phi}_\delta(z) \big| = \big(\widetilde{C}[h;\delta] - 1\big) \widehat{\Phi}_\delta(z) = o\big(\mu_\delta^2\big),$$

and find $\|I[h;\delta]-\widehat{\Phi}_\delta\|_Z = o(\mu_\delta^2)$, which implies $I[h;\delta]\in\mathcal{Y}_\delta$ for small δ. The Lipschitz continuity of I_{res} follows from Lemma 3.17 and Corollary 3.20, and the Lipschitz continuity of I_{app} is a consequence of Lemma 3.6. Moreover, using (46) and the same estimates as above we find that the Lipschitz constants are of order $o(\mu_\delta^2)$. □

Now we can prove Theorem 2.4 and Proposition 2.6 from Section 2.

Corollary 3.24. *The operator $\bar{I}_\delta$ is a contraction of $\mathcal{Y}_\delta$, and thus there exists a unique fixed point in $\mathcal{Y}_\delta$. Moreover, this fixed point is nonnegative since the cone of nonnegative function is invariant under the action of $\bar{I}_\delta$.*

Proof. By construction, we have $\bar{I}_\delta[\Phi] = I[\Phi * \Phi;\delta]$ and all assertions follow from Lemma 3.22 and Lemma 3.23. □

Finally, we discuss the influence of the parameters (β_1,β_2) that control the decay behavior of the solution, see (18).

Remark 3.25. Consider another pair of parameters $(\tilde{\beta}_1,\tilde{\beta}_2)$ with $\tilde{\beta}_1>\beta_1$ and $\tilde{\beta}_2>\beta_2$, and denote by $\widetilde{\mathcal{Y}}_\delta$ the corresponding set from Lemma 3.22. Our previous results imply, compare Remark 3.2 and Remark 3.13, that $\widetilde{\mathcal{Y}}_\delta \subset \mathcal{Y}_\delta$ for all small δ, and this yields the following two assertions.

(i) For small but fixed δ the solution that is found with the parameters $(\tilde{\beta}_1,\tilde{\beta}_2)$ equals the solution for (β_1,β_2).
(ii) The smaller δ is the larger we can choose the parameters (β_1,β_2), i.e., the faster the solution decays.

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