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Analogs of Cauchy–Poincaré and Fan–Pall interlacing theorems for J-Hermitian and J-normal matrices

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ABSTRACT

The interlacing theorem of Cauchy–Poincaré states that the eigenvalues of a principal submatrix A_0 of a Hermitian matrix A interlace the eigenvalues of A. Fan and Pall obtained an analog of this theorem for normal matrices. In this note we investigate analogs of Cauchy–Poincaré and Fan–Pall interlacing theorems for J-Hermitian and J-normal matrices. The corresponding inverse spectral problems are also considered.

1. Introduction

We consider $\mathbb{C}^n$ with an indefinite inner product $[\cdot,\cdot]$ defined as $[x, y] := y^*Jx$, $x, y \in \mathbb{C}^n$, where $J = I_r \oplus -I_{n-r}$. Let M_n be the associative algebra of $n \times n$ complex matrices. A matrix $A \in M_n$ is said to be *J-normal* if $A^\# A = AA^\#$, where $A^\#$ is the *J-adjoint* of A defined by $[Ax, y] = [x, A^\# y]$, for any $x, y \in \mathbb{C}^n$, i.e., $A^\# = JA^*J$. If A is invertible and $A^{-1} = A^\#$, then A is *J-unitary*. The J-unitary matrices form a locally

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compact group $U_{r,n-r}$, called the *J-unitary group*. A matrix $A \in M_n$ is said to be *J-Hermitian* if $A = A^{\#}$, that is, $A = JA^*J$.

The J-Hermitian matrices appear in many problems of physics, such as in relativistic quantum mechanics or in the theory of algebraic models in quantum physics [4,9,10]. Due to its applications and also on its own interest, the study of J-Hermitian matrices has deserved the attention of some researchers [1,7]. In contrast with Hermitian matrices whose spectrum is real, the spectrum of J-Hermitian matrices is symmetric relatively to the real axis. Henceforth, this property prevents the derivation of spectral inequalities for these matrices, except for some particular classes.

We denote by $\sigma(A)$ the spectrum of $A \in M_n$ (counting multiplicities). Given $\lambda \in \sigma(A)$, we say that $\lambda \in \sigma_J^+(A)$ (resp. $\lambda \in \sigma_J^-(A)$) and has multiplicity k if there exist k J-orthonormal eigenvectors $x_j, Ax_j = \lambda x_j, j = 1, \ldots, k$, such that $[x_j, x_j] > 0$ (resp. $[x_j, x_j] < 0$). We notice that a J-normal matrix A such that the equality $\sigma(A) = \sigma_J^-(A) \cup \sigma_J^+(A)$ holds is J-unitarily diagonalizable, that is, diagonalizable under a J-unitary matrix [6]. In this note we focus on $\mathcal{H}_J$, the class of J-Hermitian matrices with real and separated spectrum. We recall that the spectrum of A is *separated* if there exist two disjoint intervals I^+ and I^- in $\mathbb{R}$ with $\sigma_J^+(A) = (\alpha_1, \ldots, \alpha_r) \subset I^+$ and $\sigma_J^-(A) = (\alpha_{r+1}, \ldots, \alpha_n) \subset I^-$. The matrices of $\mathcal{H}_J$ are J-unitarily diagonalizable. We will also be concerned with J-normal matrices which are J-unitarily diagonalizable. We study analogs of the famous Cauchy–Poincaré interlacing theorem (recalled below) for matrices in $\mathcal{H}_J$. In [7], this problem was investigated in a more general but, consequently, rather involved approach. Let $A = A^* \in M_n$ be a Hermitian matrix and let $A(n|n)$ be its principal $(n-1) \times (n-1)$ submatrix obtained by deleting the last row and column. Let $\sigma(A) = (\alpha_1 \geqslant \cdots \geqslant \alpha_n)$ and $\sigma(A(n|n)) = (\mu_1 \geqslant \cdots \geqslant \mu_{n-1})$ be the ordered lists of eigenvalues of A and $A(n|n)$, respectively. The Cauchy–Poincaré interlacing theorem [3] states that these sequences interlace each other, that is,

$$\alpha_1 \geqslant \mu_1 \geqslant \alpha_2 \geqslant \mu_2 \geqslant \cdots \geqslant \alpha_{n-1} \geqslant \mu_{n-1} \geqslant \alpha_n. \tag{1}$$

It is known that the converse is also true, that is, for any two sequences $(\alpha_j)_1^n$ and $(\mu_j)_1^{n-1}$ of real numbers satisfying (1), there exists a (non-unique) Hermitian matrix A of order n such that $\sigma(A) = (\alpha_j)_1^n$ and $\sigma(A(n|n)) = (\mu_j)_1^{n-1}$. In [5] an analog of Cauchy–Poincaré interlacing theorem for the case of normal matrices was obtained by Fan and Pall.

This note is organized as follows. In Section 2 some preliminary results are presented. In Section 3 we state an indefinite version of Cauchy–Poincaré interlacing theorem for the class $\mathcal{H}_J$ of J-Hermitian matrices. In Section 4 we derive an analog result for J-normal matrices which are J-unitarily diagonalizable. We also investigate the corresponding inverse spectral problems.

2. Preliminaries

Throughout we use the following notation. For fixed integers n and k, $1 \leqslant k \leqslant n$, $Q_{k,n}$ denotes the set of all strictly increasing sequences of k integers from 1 to n. For $w, \tau \in Q_{k,n}$, the $k \times k$ submatrix of $A \in M_n$ with rows and columns indexed by the elements of w and τ, respectively, is denoted by $A[w|\tau]$. If $w = \tau$, we simply write $A[w]$. The $(n-1) \times (n-1)$ submatrix obtained by deleting row i and column j of A is denoted by $A(i|j)$.

Let A, B be two square complex matrices of orders n and m, $m < n$. We say that B is *imbeddable* in A if there exists a matrix V of type $n \times m$ such that $V^{\#}V = I_m$ and $V^{\#}AV = B$.

The following result extends Malamud's Proposition 3.1 in [11].

Theorem 2.1. *Let $(\alpha_k)_1^n$ and $(\mu_j)_1^{n-1}$ be two sequences of complex numbers such that $\{\alpha_1, \ldots, \alpha_r\} \cap \{\alpha_{r+1}, \ldots, \alpha_n\} = \emptyset$. Define*

$$p(\lambda) := \frac{\prod_{k=1}^{n-1}(\lambda - \mu_k)}{\prod_{k=1}^{n}(\lambda - \alpha_k)}.$$

Then the following conditions are equivalent:

(*i*) *The singularities of the rational function p are* $\alpha_1, \ldots, \alpha_n$, *being* α_k *either a removable singularity of p with* $\mathrm{Res}_{\alpha_k} p(\lambda) = 0$ *or* α_k *is a simple pole of p with*

$$\mathrm{Res}_{\alpha_k} p(\lambda) < 0, \quad \text{if } k = 1, \ldots, r; \quad \mathrm{Res}_{\alpha_k} p(\lambda) > 0, \quad \text{if } k = r+1, \ldots, n. \tag{2}$$

(*ii*) *There exists a J-normal* (*and J-unitarily diagonalizable*) *matrix A such that* $\sigma_J^+(A) = (\alpha_1, \ldots, \alpha_r)$, $\sigma_J^-(A) = (\alpha_{r+1}, \ldots, \alpha_n)$, *and* $\sigma(A(n|n)) = (\mu_1, \ldots, \mu_{n-1})$.

Proof. We prove (ii) $\Rightarrow$ (i). Consider the J-orthonormal basis constituted by the vectors $e_k = (\delta_{1k}, \delta_{2k}, \ldots, \delta_{nk})$, where δ_{ij} denotes the Kronecker symbol (i.e., $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise). Then $e_k^* J e_l = \varepsilon_k \delta_{kl}$, with $\varepsilon_1 = \cdots = \varepsilon_r = 1, \varepsilon_{r+1} = \cdots = \varepsilon_n = -1$. Let A be a J-normal matrix such that $\sigma_J^+(A) = (\alpha_1, \ldots, \alpha_r)$, $\sigma_J^-(A) = (\alpha_{r+1}, \ldots, \alpha_n)$. As counting multiplicities the equality $\sigma(A) = \sigma_J^+(A) \cup \sigma_J^-(A)$ holds, A is J-unitarily diagonalizable. Assume moreover that $\sigma(A(n|n)) = (\mu_1, \ldots, \mu_{n-1})$. Consider the function

$$q(\lambda) := -e_n^* J(\lambda I_n - A)^{-1} e_n.$$

Thus, $q(\lambda)$ is the (n, n)th entry of the matrix $(\lambda I_n - A)^{-1}$ and we easily find that

$$q(\lambda) = \frac{\det(\lambda I_{n-1} - A(n|n))}{\det(\lambda I_n - A)} = \frac{\prod_{k=1}^{n-1}(\lambda - \mu_k)}{\prod_{k=1}^{n}(\lambda - \alpha_k)} = p(\lambda).$$

Consider the J-orthonormal basis constituted by the vectors ξ_k, where ξ_k is an eigenvector of A associated with α_k. Writing $e_n = \sum_{k=1}^{n} x_k \xi_k, x_k \in \mathbb{C}$, and having in mind that $\xi_k^* J \xi_l = \varepsilon_k \delta_{kl}$, we obtain

$$p(\lambda) = \frac{\prod_{k=1}^{n-1}(\lambda - \mu_k)}{\prod_{k=1}^{n}(\lambda - \alpha_k)} = -e_n^* J(\lambda I_n - A)^{-1} e_n = -\sum_{k=1}^{n} \frac{|x_k|^2 \varepsilon_k}{\lambda - \alpha_k}. \tag{3}$$

It easily follows from (3) that the rational function $p(\lambda)$ has only simple poles and they clearly belong to the spectrum of A. Moreover, if λ_0 is a multiple eigenvalue of A with multiplicity k, then λ_0 is an eigenvalue of $A(n|n)$ with multiplicity at least $k-1$. Therefore, λ_0 is not a removable singularity if and only if λ_0 is an eigenvalue of $A(n|n)$ with multiplicity $k-1$. The residue of $p(\lambda)$ in a (simple) pole α_k is given by

$$\mathrm{Res}_{\alpha_k} p(\lambda) = \lim_{\lambda \to \alpha_k} \frac{\prod_{j=1}^{n-1}(\lambda - \mu_j)}{\prod_{1 \leqslant j \leqslant n, j \neq k}(\lambda - \alpha_j)}.$$

Since $\sigma_J^+(A)$ and $\sigma_J^-(A)$ are disjoint,

$$\mathrm{Res}_{\alpha_k} p(\lambda) < 0 \quad \text{if } k \in \{1, \ldots, r\}; \quad \mathrm{Res}_{\alpha_k} p(\lambda) > 0, \quad \text{if } k \in \{r+1, \ldots, n\}.$$

We prove (i) $\Rightarrow$ (ii). Under the hypothesis,

$$p(\lambda) = -\sum_{k=1}^{r} \frac{|x_k|^2}{\lambda - \alpha_k} + \sum_{k=r+1}^{n} \frac{|x_k|^2}{\lambda - \alpha_k}$$

for some $x_k \in \mathbb{C},\ k = 1, \ldots, n$, and

$$-\sum_{k=1}^{r} |x_k|^2 + \sum_{k=r+1}^{n} |x_k|^2 = \lim_{\lambda \to \infty} \lambda\, p(\lambda) = 1.$$

Let $U^\# \in U_{r,n-r}$ be a J-unitary matrix whose last column is the vector $[x_1 \cdots x_r\, x_{r+1} \cdots x_n]^T$. Consider the J-normal matrix $A = U\mathrm{diag}\,(\alpha_1, \ldots, \alpha_n)U^\#$. It is clear that for $D = \mathrm{diag}\,(\alpha_1, \ldots, \alpha_n)$, we have $\sigma_J^+(D) = (\alpha_1, \ldots, \alpha_r)$, and $\sigma_J^-(D) = (\alpha_{r+1}, \ldots, \alpha_n)$. By straightforward computations we get

$$\frac{\det(\lambda I_{n-1} - A(n|n))}{\det(\lambda I_n - A)} = -e_n^* J(\lambda I_n - A)^{-1} e_n$$
$$= -\sum_{k=1}^{r} \frac{|x_k|^2}{\lambda - \alpha_k} + \sum_{k=r+1}^{n} \frac{|x_k|^2}{\lambda - \alpha_k} = \frac{\prod_{k=1}^{n-1}(\lambda - \mu_k)}{\prod_{k=1}^{n}(\lambda - \alpha_k)}.$$

This implies that $\det(\lambda I_{n-1} - A(n|n)) = \prod_{k=1}^{n-1}(\lambda - \mu_k)$, and so $\sigma(A(n|n)) = (\mu_1, \ldots, \mu_{n-1})$. Since $\sigma_J^+(D) = \sigma_J^+(A)$ and $\sigma_J^-(D) = \sigma_J^-(A)$, the result follows. □

Remark 2.1. In the proof of the above theorem, we have shown that if A is J-normal and J-unitarily diagonalizable and ξ_k is an eigenvector associated with an eigenvalue α_k which is a pole of p, then $[\xi_k, \xi_k]$ and $\mathrm{Res}_{\alpha_k} p(\lambda)$ have opposite signs.

The next result follows from the proof of (i) ⇒ (ii) in Theorem 2.1.

Corollary 2.1. *Let $(\alpha_k)_1^n$ and $(\mu_j)_1^{n-1}$ be two sequences of complex numbers under the assumptions of Theorem 2.1. Assume that*

$$p(\lambda) := \frac{\prod_{k=1}^{n-1}(\lambda - \mu_k)}{\prod_{k=1}^{n}(\lambda - \alpha_k)} = -\sum_{k=1}^{r} \frac{|x_k|^2}{\lambda - \alpha_k} + \sum_{k=r+1}^{n} \frac{|x_k|^2}{\lambda - \alpha_k}$$

for some complex numbers x_k. Then for any J-unitary matrix $U^\# \in U_{r,n-r}$ whose last column is the vector $[x_1 \cdots x_r\, x_{r+1} \cdots x_n]^T$, the J-normal matrix $A = U\mathrm{diag}(\alpha_1, \ldots, \alpha_n)U^\#$ is such that $\sigma_J^+(A) = (\alpha_1, \ldots, \alpha_r)$, $\sigma_J^-(A) = (\alpha_{r+1}, \ldots, \alpha_n)$, and $\sigma(A(n|n)) = (\mu_1, \ldots, \mu_{n-1})$.

3. An indefinite version of Cauchy–Poincaré interlacing theorem

Next we present an analog of Cauchy–Poincaré interlacing theorem for J-Hermitian matrices.

Theorem 3.1. *Let $A \in \mathcal{H}_J$ with $\sigma_J^+(A) = (\alpha_1 \geqslant \cdots \geqslant \alpha_r)$, $\sigma_J^-(A) = (\alpha_{r+1} \geqslant \cdots \geqslant \alpha_n)$, $\alpha_r > \alpha_{r+1}$, and let $J' = J(n|n)$. Then $A(n|n)$ is J'-unitarily diagonalizable and its spectrum is separated. For $\sigma_{J'}^+(A(n|n)) = (\mu_1 \geqslant \cdots \geqslant \mu_r)$, $\sigma_{J'}^-(A(n|n)) = (\mu_{r+1} \geqslant \cdots \geqslant \mu_{n-1})$, the sequences $(\alpha_j)_1^n$, $(\mu_j)_1^{n-1}$ interlace each other:*

$$\mu_1 \geqslant \alpha_1 \geqslant \mu_2 \geqslant \alpha_2 \geqslant \cdots \geqslant \mu_r \geqslant \alpha_r > \alpha_{r+1} \geqslant \mu_{r+1} \geqslant \cdots \geqslant \mu_{n-2} \geqslant \alpha_{n-1} \geqslant \mu_{n-1} \geqslant \alpha_n. \quad (4)$$

The converse is also true, that is, for any two sequences of real numbers $(\alpha_j)_1^n$ and $(\mu_j)_1^{n-1}$ satisfying (4), there exists a (nonunique) J-Hermitian matrix $A \in \mathcal{H}_J$ such that $\sigma(A) = (\alpha_j)_1^n$ and $\sigma(A(n|n)) = (\mu_j)_1^{n-1}$, being $\sigma_J^+(A) = (\alpha_1 \geqslant \cdots \geqslant \alpha_r)$, $\sigma_J^-(A) = (\alpha_{r+1} \geqslant \cdots \geqslant \alpha_n)$, and $\sigma_{J'}^+(A(n|n)) = (\mu_1 \geqslant \cdots \geqslant \mu_r)$, $\sigma_{J'}^-(A(n|n)) = (\mu_{r+1} \geqslant \cdots \geqslant \mu_{n-1})$.

Proof. We prove the *necessity* part of the theorem. Consider the sets

$$W_J^{\pm}(A) := \{[Ax, x] : x \in \mathbb{C}^n, \ [x, x] = \pm 1\}$$

and

$$W_J(A) = W_J^+(A) \cup W_J^-(A).$$

Since $\sigma_J^+(A) = (\alpha_1 \geqslant \cdots \geqslant \alpha_r)$, $\sigma_J^-(A) = (\alpha_{r+1} \geqslant \cdots \geqslant \alpha_n)$, $\alpha_r > \alpha_{r+1}$, by Theorem 3.1 of [2] $W_J(A) = (-\infty, \alpha_{r+1}] \cup [\alpha_r, +\infty)$. Taking into account that $W_{J'}(A(n|n))$ is a subset of $W_J(A)$, we may easily conclude that it is a union of two half-rays. The matrix $A(n|n)$ can not have complex eigenvalues, contrarily by Theorem 2.1 of [2] $W_{J'}(A(n|n))$ would be the whole real line. By Theorem 2.3 of [8]

$A(n|n)$ can have at most one isotropic eigenvalue μ, being in this case $W_{J'}(A(n|n))$ the real line except eventually μ, which is impossible. Thus, $A(n|n)$ has a real and separated spectrum and is J-unitarily diagonalizable. Assume that $\sigma_{J'}^{+}(A(n|n)) = (\mu_1 \geqslant \cdots \geqslant \mu_r)$, $\sigma_{J'}^{-}(A(n|n)) = (\mu_{r+1} \geqslant \cdots \geqslant \mu_{n-1})$, with $\mu_r > \mu_{r+1}$. Let

$$p(\lambda) := -e_n^* J(\lambda I_n - A)^{-1} e_n.$$

Consider the J-orthonormal basis constituted by the vectors ξ_k, where ξ_k is an eigenvector of A associated with α_k. Writing $e_n = \sum_{k=1}^{n} x_k \xi_k$, $x_k \in \mathbb{C}$, and having in mind that $\xi_k^* J \xi_l = \varepsilon_k \delta_{kl}$, we obtain

$$p(\lambda) = -\sum_{k=1}^{n} \frac{|x_k|^2 \varepsilon_k}{\lambda - \alpha_k} = \frac{\prod_{k=1}^{n-1}(\lambda - \mu_k)}{\prod_{k=1}^{n}(\lambda - \alpha_k)}.$$

The singularities of the rational function $p(\lambda)$ are $\alpha_1, \ldots, \alpha_n$. If α_k is a removable singularity with algebraic multiplicity s, there are at least s μ_j's with $\mu_j = \alpha_k$. So, without loss of generality we assume that α_k is not a removable singularity. Thus,

$$\mathrm{Res}_{\alpha_k} p(\lambda) < 0, \quad \text{if } k \in \{1, \ldots, r\}; \quad \mathrm{Res}_{\alpha_k} p(\lambda) > 0, \quad \text{if } k \in \{r+1, \ldots, n\}.$$

For $1 \leqslant j \leqslant r$, we obtain $\lim_{\lambda \to \alpha_j^+} p(\lambda) = -\infty$ and $\lim_{\lambda \to \alpha_j^-} p(\lambda) = +\infty$. For $r+1 \leqslant j \leqslant n$, we find that $\lim_{\lambda \to \alpha_j^+} p(\lambda) = +\infty$ and $\lim_{\lambda \to \alpha_j^-} p(\lambda) = -\infty$. Hence, the intermediate value theorem ensures that $p(\lambda)$ has one zero between two consecutive poles α_{j-1} and α_j for $j = 2, \ldots, r$, and there also exists one zero between α_j and α_{j+1} for $j = r+1, \ldots, n-1$. The rational function $p(\lambda)$ has a zero above α_1, this being justified by the fact that $\lim_{\lambda \to \alpha_1^+} p(\lambda) = -\infty$ and $\lim_{\lambda \to +\infty} \lambda p(\lambda) = 1$. Moreover, $p(\lambda)$ has no zeros between α_r and α_{r+1}. In fact, since $\lim_{\lambda \to \alpha_r^+} p(\lambda) = -\infty$ and $\lim_{\lambda \to \alpha_{r+1}^-} p(\lambda) = -\infty$, the existence of one zero would imply the existence of at least two zeros between α_r and α_{r+1}, which is impossible because we have just $n-1$ μ's. The zeros of $p(\lambda)$ are obviously the $n-1$ roots (counting multiplicities) $\mu_1, \ldots, \mu_{n-1}$ of the degree $n-1$ polynomial $\prod_{j=1}^{n-1}(\lambda - \mu_j)$. Thus, (4) follows.

We prove the *sufficiency* part of the theorem. It is enough to show that (i) in Theorem 2.1 holds. Consider

$$p(\lambda) = \frac{\prod_{k=1}^{n-1}(\lambda - \mu_k)}{\prod_{k=1}^{n}(\lambda - \alpha_k)}.$$

Suppose that (4) is fulfilled. Clearly, the singularities of p are $\alpha_1, \ldots, \alpha_n$. Having in mind (4), if λ_0 has multiplicity $s > 1$ in the list $\alpha_1, \ldots, \alpha_n$, then λ_0 belongs to $\{\mu_1, \ldots, \mu_{n-1}\}$ and has multiplicity at least $s-1$. Thus, either α_k is a removable singularity or α_k is a simple pole of p. For simplicity, in the latter case we assume that α_k has multiplicity one, otherwise we eliminate the common factors in the numerator and in the denominator of p in order to make its expression irreducible. Then,

$$\mathrm{Res}_{\alpha_k}\, p(\lambda) = \frac{(\alpha_k - \mu_1) \cdots (\alpha_k - \mu_{n-1})}{(\alpha_k - \alpha_1) \cdots (\alpha_k - \alpha_{k-1})(\alpha_k - \alpha_{k+1}) \cdots (\alpha_k - \alpha_n)}.$$

Counting the number of positive and negative numbers in the numerator and in the denominator, it follows that

$$\mathrm{Res}_{\alpha_k} p(\lambda) < 0, \quad \text{if } k \in \{1, \ldots, r\}; \quad \mathrm{Res}_{\alpha_k} p(\lambda) > 0, \quad \text{if } k \in \{r+1, \ldots, n\}.$$

By Theorem 2.1 (i) $\Rightarrow$ (ii), there exists a J-normal matrix A such that $\sigma_J^{+}(A) = (\alpha_1, \ldots, \alpha_r)$, $\sigma_J^{-}(A) = (\alpha_{r+1}, \ldots, \alpha_n)$ and $\sigma(A(n|n)) = (\mu_1, \ldots, \mu_{n-1})$. The matrix A is J-Hermitian because its eigenvalues are real, and so is $A(n|n)$. By an argument similar to the one given in the necessity part of the proof, we get $\sigma_{J'}^{+}(A(n|n)) = (\mu_1 \geqslant \cdots \geqslant \mu_r)$ and $\sigma_{J'}^{-}(A(n|n)) = (\mu_{r+1} \geqslant \cdots \geqslant \mu_{n-1})$ for $J' = J(n|n)$. $\square$

Given a J-Hermitian matrix A, the interlacing relation between its eigenvalues and the eigenvalues of $A(j|j)$ depends on whether $j \leqslant r$ or $j \geqslant r+1$, where j labels the row and the column of the matrix A which are deleted.

Remark 3.1. Let $A \in H_J$ and $J' = J(1|1)$. Let $\sigma_J^+(A) = (\alpha_1 \geqslant \cdots \geqslant \alpha_r)$, $\sigma_J^-(A) = (\alpha_{r+1} \geqslant \cdots \geqslant \alpha_n)$, $\alpha_r > \alpha_{r+1}$. Then $A(1|1) \in H_{J'}$ and for $\sigma_{J'}^+(A(1|1)) = (\mu_1 \geqslant \cdots \geqslant \mu_{r-1})$, $\sigma_{J'}^-(A(1|1)) = (\mu_r \geqslant \cdots \geqslant \mu_{n-1})$ an analog of Theorem 3.1 holds, being the interlacing relations (4) replaced by

$$\alpha_1 \geqslant \mu_1 \geqslant \alpha_2 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_{r-1} \geqslant \alpha_r > \alpha_{r+1} \geqslant \mu_r \geqslant \cdots \geqslant \alpha_{n-1} \geqslant \mu_{n-2} \geqslant \alpha_n \geqslant \mu_{n-1}. \tag{5}$$

Remark 3.2. Replacing in Theorem 3.1 the condition $\alpha_r > \alpha_{r+1}$ by $\alpha_n > \alpha_1$, then an analog of the theorem is valid, with (4) replaced by

$$\alpha_{r+1} \geqslant \mu_{r+1} \geqslant \alpha_{r+2} \geqslant \mu_{r+2} \geqslant \cdots \geqslant \mu_{n-1} \geqslant \alpha_n > \alpha_1 \geqslant \mu_1 \geqslant \cdots \geqslant \alpha_{r-1} \geqslant \mu_{r-1} \geqslant \alpha_r \geqslant \mu_r. \tag{6}$$

Remark 3.3. Replacing in Theorem 3.1, the submatrix $A(n|n)$ by $A(1|1)$ and the condition $\alpha_r > \alpha_{r+1}$ by $\alpha_n > \alpha_1$, then an analog of the theorem holds, with (4) replaced by

$$\mu_r \geqslant \alpha_{r+1} \geqslant \mu_{r+1} \geqslant \alpha_{r+2} \geqslant \cdots \geqslant \mu_{n-1} \geqslant \alpha_n > \alpha_1 \geqslant \mu_1 \geqslant \cdots \geqslant \mu_{r-2} \geqslant \alpha_{r-1} \geqslant \mu_{r-1} \geqslant \alpha_r. \tag{7}$$

The interlacing results (4)–(7) are easily generalized when the number of deleted rows and columns in the original matrix is $m = n - t > 1$, by inserting intermediary sequences of eigenvalues, such that two consecutive sequences so obtained interlace similarly to (4)–(7), respectively. Thus, by the result for $t = n - 1$, there exists a chain of J-Hermitian matrices, with sizes increasing by unity, such that each one is imbeddable in the next.

In the sequel, we adopt the following notation. Given $p_1 \leqslant r$, $p_2 \leqslant n - r$, $J'' = J[p_1 + 1, \ldots, n - p_2]$, and $B = A[p_1 + 1, \ldots, n - p_2]$, with $\sigma(B) = \sigma_{J''}^+(B) \cup \sigma_{J''}^-(B)$, we consider

$$\sigma_{J''}^+(B) = (\mu_1 \geqslant \cdots \geqslant \mu_{r-p_1}),\ \ \sigma_{J''}^-(B) = (\mu_{r-p_1+1} \geqslant \cdots \geqslant \mu_{n-p_1-p_2}).$$

When we write $j' \leqslant j \leqslant j''$ with $j' > j''$ we mean that the interval where j ranges is empty.

The next lemma is a simple consequence of Theorem 3.1 and of Remarks 3.1–3.3.

Lemma 3.1. *(I) Given $A \in \mathcal{H}_J$, consider its $(n-m) \times (n-m)$ principal submatrix $A[m+1, \ldots, n]$ with $m \leqslant r$. If $m \leqslant r - 1$, then an analog of Theorem 3.1 holds with the following interlacing conditions:*

$$\begin{aligned} &\alpha_j \geqslant \mu_j \geqslant \alpha_{j+m}, \quad 1 \leqslant j \leqslant \min\{r-m, n-2m\};\\ &\alpha_{j+m} \geqslant \mu_j \geqslant \alpha_{j+2m}, \quad r-m+1 \leqslant j \leqslant n-2m;\\ &\alpha_{j+m} \geqslant \mu_j, \quad n-2m+1 \leqslant j \leqslant n-m, \end{aligned} \tag{8}$$

where (8) applies only if $r < n - m$. If $m = r$, then

$$\begin{aligned} &\alpha_{j+r} \geqslant \mu_j \geqslant \alpha_{j+2r}, \quad 1 \leqslant j \leqslant n-2r;\\ &\alpha_{j+r} \geqslant \mu_j, \quad n-2r+1 \leqslant j \leqslant n-r. \end{aligned}$$

(II) Given $A \in \mathcal{H}_J$, consider its $(n-m) \times (n-m)$ principal submatrix $A[1, \ldots, n-m]$ with $m \leqslant n - r$. If $n - m \geqslant r + 1$, then an analog of Theorem 3.1 holds with the following interlacing conditions:

$$\begin{aligned} &\mu_j \geqslant \alpha_j, \quad 1 \leqslant j \leqslant m;\\ &\alpha_{j-m} \geqslant \mu_j \geqslant \alpha_j, \quad m+1 \leqslant j \leqslant r;\\ &\alpha_j \geqslant \mu_j \geqslant \alpha_{j+m}, \quad \max\{r+1, m+1\} \leqslant j \leqslant n-m. \end{aligned} \tag{9}$$

where (9) applies only if $m < r$. If $n - m = r$, then

$$\mu_j \geqslant \alpha_j, \quad 1 \leqslant j \leqslant n-r;$$
$$\alpha_{j-n+r} \geqslant \mu_j \geqslant \alpha_j, \quad n-r+1 \leqslant j \leqslant r.$$

Theorem 3.2. *Let* $A \in \mathcal{H}_J$ *and* $J' = J[p+1,\dots,n-m], p \leqslant r, m \leqslant n-r$. *Let* $\sigma_J^+(A) = (\alpha_1 \geqslant \cdots \geqslant \alpha_r), \sigma_J^-(A) = (\alpha_{r+1} \geqslant \cdots \geqslant \alpha_n), \alpha_r > \alpha_{r+1}$. *Then* $A[p+1,\dots,n-m] \in H_{J'}$. *For* $\sigma_{J'}^+(A[p+1,\dots,n-m]) = (\mu_1 \geqslant \cdots \geqslant \mu_{r-p}), \sigma_{J'}^-(A[p+1,\dots,n-m]) = (\mu_{r-p+1} \geqslant \cdots \geqslant \mu_{n-p-m})$, *the sequences* $(\alpha_j)_1^n, (\mu_j)_1^{n-m-p}$ *interlace each other as follows:*

If $n-m \geqslant r+1$ *and* $p \leqslant r-1$, *then*

$$\mu_j \geqslant \alpha_{j+p}, \quad 1 \leqslant j \leqslant \min\{m, r-p\};$$
$$\alpha_{j-m} \geqslant \mu_j \geqslant \alpha_{j+p}, \quad m+1 \leqslant j \leqslant r-p; \tag{10}$$
$$\alpha_{j+p} \geqslant \mu_j \geqslant \alpha_{j+m+2p}, \quad r-p+1 \leqslant j \leqslant n-m-2p; \tag{11}$$
$$\alpha_{j+p} \geqslant \mu_j, \quad \max\{n-m-2p+1, r-p+1\} \leqslant j \leqslant n-m-p,$$

where (10) *applies only if* $m+p < r$ *and* (11) *applies only if* $m+p < n-r$.

If $n-m = r$ *and* $p < r$ *then*

$$\mu_j \geqslant \alpha_{j+p}, \quad 1 \leqslant j \leqslant n-r;$$
$$\alpha_{j-n+r} \geqslant \mu_j \geqslant \alpha_{j+p}, \quad n-r+1 \leqslant j \leqslant r-p.$$

If $p = r$ *and* $m < n-r$, *then*

$$\alpha_{j+r} \geqslant \mu_j \geqslant \alpha_{j+m+2r}, \quad 1 \leqslant j \leqslant n-m-2r;$$
$$\alpha_{j+r} \geqslant \mu_j, \quad n-m-2r+1 \leqslant j \leqslant n-m-r.$$

The converse is also true, that is, for any two sequences of real numbers $(\alpha_j)_1^n$ *and* $(\mu_j)_1^{n-p-m}$ *satisfying the above inequalities, there exists a (nonunique) J-Hermitian matrix* $A \in H_J$ *such that* $\sigma(A) = (\alpha_j)_1^n$ *and* $\sigma(A[p+1,\dots,n-m]) = (\mu_j)_1^{n-p-m}$, *being* $\sigma_J^+(A) = (\alpha_1 \geqslant \cdots \geqslant \alpha_r)$, $\sigma_J^-(A) = (\alpha_{r+1} \geqslant \cdots \geqslant \alpha_n)$, *and* $\sigma_{J'}^+(A[p+1,\dots,n-m]) = (\mu_1 \geqslant \cdots \geqslant \mu_{r-p})$, $\sigma_{J'}^-(A[p+1,\dots,n-m]) = (\mu_{r-p+1} \geqslant \cdots \geqslant \mu_{n-p-m})$.

Proof. *Necessity*: Let $B = A[p+1,\dots,n]$ and $J'' = J[p+1,\dots,n]$. It can be easily seen that $B \in H_{J''}$. Let $\sigma_{J''}^+(B) = (\gamma_1 \geqslant \cdots \geqslant \gamma_{r-p})$ and $\sigma_{J''}^-(B) = (\gamma_{r-p+1} \geqslant \cdots \geqslant \gamma_{n-p})$. We have $\gamma_{r-p} > \gamma_{r-p+1}$. By Lemma 3.1 (II)

$$\mu_j \geqslant \gamma_j, \quad 1 \leqslant j \leqslant m; \tag{12}$$
$$\gamma_{j-m} \geqslant \mu_j \geqslant \gamma_j, \quad m+1 \leqslant j \leqslant r-p;$$
$$\gamma_j \geqslant \mu_j \geqslant \gamma_{j+m}, \quad r-p+1 \leqslant j \leqslant n-p-m.$$

By Lemma 3.1 (I)

$$\alpha_j \geqslant \gamma_j \geqslant \alpha_{j+p}, \quad 1 \leqslant j \leqslant r-p; \tag{13}$$
$$\alpha_{j+p} \geqslant \gamma_j \geqslant \alpha_{j+2p}, \quad r-p+1 \leqslant j \leqslant n-2p;$$
$$\alpha_{j+p} \geqslant \gamma_j, \quad n-2p+1 \leqslant j \leqslant n-p.$$

It is not hard to confirm that the previous inequalities imply the stated interlacing relations.

Sufficiency: Let

$$\gamma_j = \min\{\mu_j, \alpha_j\}, \quad 1 \leqslant j \leqslant r - p;$$
$$\gamma_j = \max\{\alpha_{j+2p}, \mu_j\}, \quad r - p + 1 \leqslant j \leqslant \min\{n - p - m, n - 2p\}.$$

If $m \leqslant p$, let

$$\gamma_j = \mu_j, \quad n - 2p + 1 \leqslant j \leqslant n - p - m;$$
$$\gamma_j = \mu_{n-p-m}, \quad n - p - m + 1 \leqslant j \leqslant n - p.$$

If $m > p$, let

$$\gamma_j = \alpha_{j+2p}, \quad n - p - m + 1 \leqslant j \leqslant n - 2p;$$
$$\gamma_j = \alpha_n, \quad n - 2p + 1 \leqslant j \leqslant n - p.$$

Then (12) and (13) hold. By Lemma 3.1 (II), there exists a J''-Hermitian matrix B of size $n - p$ such that

$$\sigma^+_{J''}(B) = (\gamma_1, \ldots, \gamma_{r-p}), \quad \sigma^-_{J''}(B) = (\gamma_{r-p+1}, \ldots, \gamma_{n-p}),$$

with $J'' = J[p + 1, \ldots, n]$ and

$$\sigma^+_{J'}(B[1, \ldots, n - p - m]) = (\mu_1, \ldots, \mu_{r-p}),$$
$$\sigma^-_{J'}(B[1, \ldots, n - p - m]) = (\mu_{r-p+1}, \ldots, \mu_{n-p-m}).$$

By Lemma 3.1 (I), there exists a matrix $A \in H_J$ such that

$$\sigma^+_J(A) = (\alpha_1, \ldots, \alpha_r), \quad \sigma^-_J(A) = (\alpha_{r+1}, \ldots, \alpha_n),$$

and

$$\sigma^+_{J''}(A[p + 1, \ldots, n]) = (\gamma_1, \ldots, \gamma_{r-p}), \quad \sigma^-_{J''}(A[p + 1, \ldots, n]) = (\gamma_{r-p+1}, \ldots, \gamma_{n-p}).$$

Since B and $A[p + 1, \ldots, n]$ are J''-unitarily similar, the result follows. □

4. An indefinite version of Fan–Pall theorem

Fan–Pall interlacing theorem gives a necessary and sufficient condition for the tuples $(\alpha_k)_1^n$ and $(\mu_k)_1^{n-1}$ of complex numbers to be the spectrum of a normal matrix A and of its principal submatrix $A(n|n)$. In Theorem 3.1 we establish an analog of this result for J-normal matrices.

Contrarily to the J-Hermitian case, the next result is not generalizable to a principal submatrix B with size $m < n - 1$. In general, it is not true that there exists a chain of J-normal matrices beginning with B and ending with A, with orders increasing by unity and such that each one is imbeddable in the next one. The same situation occurs for normal matrices as shown by the following example in [5]. Consider $A = \mathrm{diag}\,(0, 1, i, 1 + i)$ and $B = 10^{-1}\mathrm{diag}\,(5 + 8i, 5 + 2i)$. Then B is imbeddable in A, but there does not exist a 3×3 normal matrix C such that B is imbeddable in C and C is imbeddable in A.

Theorem 4.1. *Let $A \in M_n$ be a J-normal matrix with $\sigma^+_J(A) = (\alpha_1, \ldots, \alpha_r)$, $\sigma^-_J(A) = (\alpha_{r+1}, \ldots, \alpha_n)$. Assume that $\sigma^+_J(A)$ is contained in the right half plane and $\sigma^-_J(A)$ is contained in the left half plane. For $J' = J(n|n)$, let B be a J'-normal matrix of order $n - 1$ with $\sigma^+_{J'}(B) = (\mu_1, \ldots, \mu_r)$, $\sigma^-_{J'}(B) = (\mu_{r+1}, \ldots, \mu_{n-1})$. Renumber the eigenvalues so that $\alpha_1 = \mu_1, \ldots, \alpha_s = \mu_s$, $s \leqslant r$, $\alpha_{n-t+1} = \mu_{n-t}, \ldots, \alpha_n = \mu_{n-1}$, $t \leqslant n - r - 1$, $\mu_{s+1}, \ldots, \mu_r$, are each distinct from $\alpha_{s+1}, \ldots, \alpha_r$ and $\mu_{r+1}, \ldots, \mu_{n-t-1}$ are each distinct from $\alpha_{r+1}, \ldots, \alpha_{n-t}$. Then a necessary and sufficient condition for B to be imbeddable in A is that the $2(n - s - t) - 1$ points $\alpha_{s+1}, \ldots, \alpha_{n-t}, \mu_{s+1}, \ldots, \mu_{n-t-1}$ shall be collinear, and may be ordered so that every line segment whose endpoints are α_l and α_{l+1} $(s + 1 \leqslant l \leqslant r - 1$ or $r + 1 \leqslant l \leqslant n - t - 1)$ shall contain one μ_j and μ_{s+1} belongs to the half-ray $\alpha_{s+1} + t(\alpha_{s+1} - \alpha_{r+1})$, $t \geqslant 0$.*

Proof. We first prove the *necessity* part. Assume that there exists a J-unitary U such that

$$U^{\#}AU=\begin{bmatrix}\mu_1 & 0 & \cdots & 0 & 0 & \cdots & 0 & z_1\\ 0 & \mu_2 & \cdots & 0 & 0 & \cdots & 0 & z_2\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & \mu_r & 0 & \cdots & 0 & z_r\\ 0 & 0 & \cdots & 0 & \mu_{r+1} & \cdots & 0 & z_{r+1}\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & 0 & \cdots & \mu_{n-1} & z_{n-1}\\ w_1 & w_2 & \cdots & w_r & w_{r+1} & \cdots & w_{n-1} & \gamma\end{bmatrix}$$

for some complex numbers z_j, w_j and γ. The J-normality of A can be expressed by

$$|z_j|=|w_j|,\quad 1\leqslant j\leqslant n-1, \tag{14}$$

$$z_j\bar{z}_k=\bar{w}_jw_k\varepsilon_j\varepsilon_k,\quad 1\leqslant j,k\leqslant n-1, \tag{15}$$

$$(\mu_j-\gamma)\bar{w}_j=-\overline{(\mu_j-\gamma)}z_j\varepsilon_j,\quad 1\leqslant j\leqslant n-1, \tag{16}$$

where $\varepsilon_k=1, 1\leqslant k\leqslant r$, and $\varepsilon_k=-1, r+1\leqslant k\leqslant n$. We may assume that the vanishing z_j (if they exist) are $z_1,\ldots,z_s$ and $z_{n-1},\ldots,z_{n-t}$. If z_j is different from zero, then by (14) also $w_j\neq 0$. From (15) we get

$$(w_jz_j)(z_k\bar{z}_k)=(z_kw_k\varepsilon_k)(w_j\bar{w}_j\varepsilon_j),$$

which implies that all the nonvanishing numbers among the $(n-1)z_kw_k\varepsilon_k$ have the same argument

$$\arg(w_jz_j\varepsilon_j)=\arg(w_kz_k\varepsilon_k).$$

Denoting this argument by $2\theta+\pi$, with $-\pi/2<\theta\leqslant\pi/2$, from (14) it follows that $z_j=w_j=0$ or

$$z_j\varepsilon_j=-e^{i2\theta}\bar{w}_j.$$

Thus, in any case

$$z_je^{-i\theta}=-\overline{w_je^{-i\theta}}\varepsilon_j.$$

Having in mind (16) either $\mu_j-\gamma=0$ or $\arg(\mu_j-\gamma)=\theta\ (\mathrm{mod}\pi)$. In any case we may set $\mu_j=\gamma+e^{i\theta}b_j$ with b_j real. Then

$$(U^{\#}AU)[s+1,\ldots,n-t-1,n]=\gamma I_{n-s-t}+e^{i\theta}H,$$

where H is $\widetilde{J}$-Hermitian for $\widetilde{J}=J[s+1,\ldots,n-t-1,n]$.

Let us introduce the matrices

$$C=\mathrm{diag}\,(\mu_1,\ldots,\mu_s),\quad E=\mathrm{diag}\,(\mu_{n-t},\ldots,\mu_{n-1}),\quad P=I_{n-t-1}\oplus T,$$

where T is the circulant matrix

$$T=\begin{bmatrix}0 & 0 & \cdots & 0 & 1\\ 1 & 0 & \cdots & 0 & 0\\ 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & 0\end{bmatrix}\in M_{t+1}. \tag{17}$$

Hence

$$PU^{\#}AUP^{-1}=C\oplus D\oplus E,$$

where $D=\gamma I_{n-s-t}+\mathrm{e}^{i\theta}H$. Renumber the eigenvalues of D such that for $\alpha_j=\gamma+\mathrm{e}^{i\theta}a_j,\quad j=s+1,\ldots,n-t$, we have

$$\sigma_{\widehat{J}}^{+}(H)=(a_{s+1}\geqslant\cdots\geqslant a_r),\quad \sigma_{\widehat{J}}^{-}(H)=(a_{r+1}\geqslant\cdots\geqslant a_{n-t}).$$

Since we are assuming that $\sigma_J^+(A)$ is contained in the right half plane and $\sigma_J^-(A)$ is contained in the left half plane, it follows that $a_r>a_{r+1}$. Renumber the

$$\mu_j=\gamma+\mathrm{e}^{i\theta}b_j,\quad j=s+1,\ldots,n-t-1,$$

so that

$$\sigma_{J''}^{+}(H(n-t|n-t))=(b_{s+1}\geqslant\cdots\geqslant b_r),\quad \sigma_{J''}^{-}(H(n-t|n-t))=(b_{r+1}\geqslant\cdots\geqslant b_{n-t-1}),$$

where $J''=J'(n-t|n-t)$. Then by Theorem 2.2

$$b_{s+1}\geqslant a_{s+1}\geqslant b_{s+2}\geqslant a_{s+2}\geqslant\cdots\geqslant b_r\geqslant a_r>a_{r+1}\geqslant b_{r+1}\geqslant\cdots\geqslant b_{n-t-1}\geqslant a_{n-t}.$$

We prove the *sufficiency*. Let $\alpha_j,\mu_j\in\mathbb{C}$ satisfy the conditions of the theorem. Since the distinct α_j,μ_j are collinear, there exist a complex number γ and real numbers

$$\theta,\ a_{s+1},\ldots,a_r,a_{r+1},\ldots,a_{n-t},b_{s+1},\ldots,b_r,b_{r+1},\ldots,b_{n-t-1},$$

with $-\pi/2<\theta\leqslant\pi/2$, such that $\alpha_j=\mathrm{e}^{i\theta}a_j+\gamma,\ s+1\leqslant j\leqslant n-t$, and $\mu_j=\mathrm{e}^{i\theta}b_j+\gamma, s+1\leqslant j\leqslant n-t$,

$$b_{s+1}\geqslant a_{s+1}\geqslant b_{s+2}\geqslant a_{s+2}\geqslant\cdots\geqslant b_r\geqslant a_r>a_{r+1}\geqslant b_{r+1}\geqslant\ldots\geqslant b_{n-t-1}\geqslant a_{n-t}.$$

From Theorem 2.2, there exists a $\widehat{J}$-Hermitian matrix H of size $n-s-t,\widehat{J}=J[s+1,\ldots,n-t]$, such that

$$\sigma_{\widehat{J}}^{+}(H)=(a_{s+1},\ldots,a_r),\quad \sigma_{\widehat{J}}^{-}(H)=(a_{r+1},\ldots,\alpha_{n-t})$$

and

$$\sigma_{J''}^{+}(H(n-t|n-t))=(b_{s+1},\ldots,b_r),\quad \sigma_{J''}^{-}(H(n-t|n-t))=(b_{r+1},\ldots,b_{n-t-1}),$$

where $J''=J'(n-t|n-t)$. Now we consider the J-normal matrix

$$D=\gamma I_{n-s-t}+\mathrm{e}^{i\theta}H.$$

It can be easily seen that $P^{-1}(C\oplus D\oplus E)P$, where $C=\mathrm{diag}\,(\mu_1,\ldots,\mu_s), E=\mathrm{diag}\,(\mu_{n-t},\ldots,\mu_{n-1})$ and $P=I_{n-t-1}\oplus T$, for T in (17), satisfy the asserted conditions. $\square$

The following analog to Theorem 3.1 holds.

Theorem 4.2. *Let $A\in M_n$ be a J-normal matrix with $\sigma_J^+(A)=(\alpha_1,\ldots,\alpha_r),\sigma_J^-(A)=(\alpha_{r+1},\ldots,\alpha_n)$. Assume that $\sigma_J^+(A)$ is contained in the right half plane and $\sigma_J^-(A)$ is contained in the left half plane. Let $J'=J(1|1)$, and $B\in M_{n-1}$ be a J'-normal matrix with $\sigma_{J'}^+(B)=(\mu_1,\ldots,\mu_{r-1}),\ \sigma_{J'}^-(B)=(\mu_r,\ldots,\mu_{n-1})$. Renumber the eigenvalues so that $\alpha_1=\mu_1,\ldots,\alpha_s=\mu_s, s\leqslant r-1,\alpha_{n-t+1}=\mu_{n-t},\ldots,\alpha_n=\mu_{n-1}$, $t\leqslant n-r$, $\mu_{s+1},\ldots,\mu_{r-1}$ are each distinct from $\alpha_{s+1},\ldots,\alpha_r$ and $\mu_r,\ldots,\mu_{n-t-1}$ are each distinct from $\alpha_{r+1},\ldots,\alpha_{n-t}$. Then a necessary and sufficient condition for B to be imbeddable in A is that the $2(n-s-t)-1$ points $\alpha_{s+1},\ldots,\alpha_{n-t},\mu_{s+1},\ldots,\mu_{n-t-1}$ shall be collinear, and may be ordered so that every line segment whose endpoints are α_l and α_{l+1} $(s+1\leqslant l\leqslant r-1$ or $r+1\leqslant l\leqslant n-t-1)$ shall contain one μ_j and μ_{n-t-1} belongs to the half-ray $\alpha_{n-t}+t(\alpha_{n-t}-\alpha_{r+1}), t\geqslant 0$.*

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