A collocation method with cubic B-splines for solving the MRLW equation

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Abstract

The modified regularized long wave (MRLW) equation is solved numerically by collocation method using cubic B-splines finite element. A linear stability analysis of the scheme is shown to be marginally stable. Three invariants of motion are evaluated to determine the conservation properties of the algorithm, also the numerical scheme leads to accurate and efficient results. Moreover, interaction of two and three solitary waves are studied through computer simulation and the development of the Maxwellian initial condition into solitary waves is also shown.

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1. Introduction

Solitary waves are wave packets or pulses, which propagate in nonlinear dispersive media. Due to dynamical balance between the nonlinear and dispersive effects these waves retain a stable waveform. The regularized long wave (RLW) equation of the form

\[ u_t + u_x + \delta uu_x - \mu u_{xx} = 0, \]

where \( \delta \) and \( \mu \) are positive constants, was originally introduced to describe the behavior of the undular bore by Peregrine [10]. This equation is very important in physics media since it describes phenomena with weak nonlinearity and dispersion waves, including nonlinear transverse waves in shallow water, ion-acoustic and magneto hydrodynamic waves in plasma and phonon packets in nonlinear crystals. The solutions of this equation are kinds of solitary waves named solitons whose shapes are not affected by a collision. The RLW equation was solved numerically by various forms of finite element methods [1–6,9,11,13,14] such as Galerkin method, least square method and collocation method with quadratic B-splines, cubic B-splines and recently septic splines. Indeed, the RLW equation is a special case of the...
generalized long wave (GRLW) equation which has the form

$$u_t + u_x + \delta u^p u_x - \mu u_{xx} = 0,$$  \(\text{(2)}\)

where \(\delta\) is a positive integer. The GRLW equation was studied by few authors, Zhang \cite{15} with finite difference method for a Cauchy problem and Kaya \cite{8} with Adomian decomposition method (ADM). In this paper, we consider another special case of the GRLW called the modified regularized long wave (MRLW) equation. This equation was considered by Gardner et al. \cite{7} using collocation method with quintic B-splines finite element. Here, the collocation method using cubic B-splines is shown to represent accurately the migration of single solitary wave. The interaction of solitary waves and other properties of the MRLW equation are also studied.

2. The governing equation and collocation solution

The MRLW equation as in the form

$$u_t + u_x + u_x^3 - \mu u_{xx} = 0,$$  \(\text{(3)}\)

where subscripts \(x\) and \(t\) denote differentiation, and is considered with the boundary conditions \(u \to 0\) as \(x \to \pm \infty\).

In this work, periodic boundary conditions on the region \(a \leq x \leq b\) are assumed in the form

$$u(a, t) = u(b, t) = 0,$$
$$u_x (a, t) = u_x (b, t) = 0.$$  \(\text{(4)}\)

The exact solution of our equation can be written as \cite{7}

$$u(x, t) = \sqrt{c} \sec (p(x - (c + 1)t - x_0)),$$  \(\text{(5)}\)

where \(p \equiv \frac{c}{\mu(c+1)},\) and \(x_0, c\) are arbitrary constants.

Also, the Eq. (3) has three invariants as in the form \cite{7}

$$I_1 = \int_a^b u \, dx,$$
$$I_2 = \int_a^b (u^2 + \mu u_x^2) \, dx,$$
$$I_3 = \int_a^b (u^4 - \mu u_x^4) \, dx.$$  \(\text{(6)}\)

Now, partition the interval \([a, b]\) as

$$a = x_0 < x_1 < \cdots < x_N = b, \quad h = x_{j+1} - x_j = \frac{b - a}{N}, \quad j = 0, \ldots, N.$$  

Let \(\{B_j\}_{j=-1}^{N+1}\) be the cubic B-splines at the knot points \(x_j\); the set of splines form a basis of functions defined over \([a, b]\).

A global approximation solution \(u^N(x, t)\) is expressed in terms of the cubic B-splines and unknown time dependent parameters as

$$u^N(x, t) = \sum_{j=-1}^{N+1} c_j(t) B_j(x),$$  \(\text{(7)}\)

where \(c_j\) are unknown time dependent quantities to be determined from boundary conditions and the collocation conditions. The values of \(B_j(x)\) and its derivatives \(B'_j(x), B''_j(x)\) at the knots are given in Table \textit{1}.

Then, the used nodal values \(u_j, u'_j\) and \(u''_j\) are given in terms of the parameter \(c_j\) by

$$u_j = c_{j-1} + 4c_j + c_{j+1},$$
$$u'_j = \frac{3}{h}(c_{j+1} - c_{j-1}),$$
$$u''_j = \frac{6}{h^2}(c_{j-1} - 2c_j + c_{j+1}).$$  \(\text{(8)}\)
Now, let us rewrite the Eq. (3) in the form
\[ u_{xx} + 6u^2 u_x = 0 \] (9)
and if the time derivative is discretized in a central finite difference fashion
\[ \frac{\partial u}{\partial t} = \frac{u^{n+1} - u^n}{2k}, \] where \( k = \Delta t = t_{n+1} - t_n \). Also if we consider \( u = u^n \), then Eq. (9) becomes as
\[ u^{n+1} - \mu(u_{xx})^{n+1} - u^{n-1} + \mu(u_{xx})^{n-1} + 2k((u_x)^n + 6(u^2 u_x)^n) = 0. \] (10)

Introducing Eqs. (8) into Eq. (10) yields
\[ a_j c_{j-1}^{n+1} + b_j c_j^{n+1} + a_j c_{j+1}^{n+1} = F_j, \quad j = 0, 1, \ldots, N. \] (11)

where
\[ a_j = 1 - \frac{6\mu}{h^2}, \quad b_j = 4 + \frac{12\mu}{h^2}, \quad F_j = L_3^j - \frac{6\mu}{h^2} L_4^j - 2k[6(\frac{L_1^j}{h^2})^2L_2^j + L_2^j], \]
\[ L_1^j = c_{j-1}^n + 4c_j^n + c_{j+1}^n, \quad L_2^j = \frac{3}{h}(c_{j+1}^n - c_{j-1}^n), \]
\[ L_3^j = c_{j-1}^{n-1} + 4c_j^{n-1} + c_{j+1}^{n-1}, \quad L_4^j = c_{j-1}^{n-1} - 2c_j^{n-1} + c_{j+1}^{n-1}, \quad j = 0, 1, \ldots, N. \]

Then this set of equations is a recurrence relationship of element parameters vector \( d^n = (c_{-1}, c_0, \ldots, c_{N+1}) \), so we have \((N+1)\) equations by \((N+3)\) unknown at each level time \( n \), and using the boundary conditions: \( u(a, t) = u(b, t) = 0 \) gives the following relations:
\[ \begin{align*}
  c_{-1} &= -4c_0 - c_1, \\
  c_{N+1} &= -4c_N - c_{N-1}.
\end{align*} \] (12)

Now, by using the Eqs. (12), we can eliminate the variables \( c_{-1} \) and \( c_{N+1} \), so the matrix system (11) is reduced to \( (N+1) \times (N+1) \) tridiagonal system as follows:
The above system requires two initial time levels at \( t = 0 \) and \( t = \Delta t = k \), so we use the exact solution (5) to determine the initial conditions \( d^0 \) and \( d^1 \) as follows:

Firstly, the initial parameter \( d^0 \) is determined from the initial and boundary conditions expressed as

\[
\begin{align*}
\left. \begin{array}{l}
\frac{3}{h}c_{-1}^{0} + \frac{3}{h}c_{1}^{0} = 0, \\
\frac{3}{h}c_{-1}^{0} + 3h_{n+1}^{0} = 0, \\
c_{i}^{0} - c_{i+1}^{0} + c_{i+1}^{0} = 0, & i = 0, 1, \ldots, N.
\end{array} \right\}
\end{align*}
\]

This is a tridiagonal system of equations which is solved with a variant of Thomas algorithms. Also, the parameter \( d^1 \) is determined by solving the following system in the same way:

\[
\begin{align*}
\left. \begin{array}{l}
\frac{3}{h}c_{-1}^{1} + \frac{3}{h}c_{1}^{1} = 0, \\
\frac{3}{h}c_{-1}^{1} + 3h_{n+1}^{1} = 0, \\
c_{i}^{1} - c_{i+1}^{1} + c_{i+1}^{1} = 0, & i = 0, 1, \ldots, N.
\end{array} \right\}
\end{align*}
\]

3. Stability analysis

Like other authors [4,5,7,9,14], our stability analysis will be based on Von Neumann theory in which the growth factor of a typical Fourier mode defined as [12]

\[ C_{j}^{n} = \zeta^{n} e^{jkh}, \]

where \( k \) is a mode number and \( h \) is the element size. The nonlinear term \( u^{2}u_{x} \) cannot be handled by the Fourier mode method, therefore we linearize it. To do this, assume that the quantity \( u \) is locally constant. Then substituting Eq. (15) into the system (11) gives

\[ \zeta^{n+1} = g \zeta^{n}, \]

where \( g \) is the growth factor.

So, we have

\[ g^{2} + 2i \left( \frac{\gamma \sin \varphi}{2\pi \cos \varphi + \beta} \right) g - 1 = 0, \]

where \( \alpha = (1 - \frac{6\mu}{h^{2}}), \beta = (4 + \frac{12\mu}{h^{2}}), \gamma = 6(\frac{\Delta t}{h} + \frac{6u^{2}\Delta t}{h}), \) and \( \varphi = k h, i = \sqrt{-1}. \)

Lemma. For \( h, \Delta t, \varphi \) and \( u \) are defined as above

\[ \left| \frac{\gamma \sin \varphi}{2\pi \cos \varphi + \beta} \right| \leq 1, \quad \forall \varphi, \]

where \( \alpha = (1 - \frac{6\mu}{h^{2}}), \beta = (4 + \frac{12\mu}{h^{2}}), \gamma = 6(\frac{\Delta t}{h} + \frac{6u^{2}\Delta t}{h}). \)

Proof.

\[ \left| \frac{\gamma \sin \varphi}{2\pi \cos \varphi + \beta} \right| = \left| \frac{3(1 + 6u^{2}) \sin \varphi}{h[2 + \cos \varphi - (6\mu/h^{2})(\cos \varphi - 1)]} \right|. \]
Now, suppose $x = 1 - \cos \varphi$, then
\[
\sup_{\varphi} \left[ \frac{3 \Delta t (1 + 6 u^2) \sin \varphi}{h[2 + \cos \varphi - (6 \mu / h^2)(\cos \varphi - 1)]} \right]^2 = \sup_{0 \leq x \leq 2} \frac{9(\Delta t)^2 (1 + 6 u^2)^2 x (2 - x)}{h^2 [3 - (1 - (6 \mu / h^2)) x]^2}.
\]
Suppose that $3 - (1 - \frac{6 \mu}{h^2}) x = 0$, then $x = \frac{3}{1 - 6 \mu / h^2}$, and so $0 \leq \frac{3}{1 - 6 \mu / h^2} \leq 2$, and we have $1 - \frac{6 \mu}{h^2} \geq \frac{3}{2} \Rightarrow \frac{6 \mu}{h^2} \leq - \frac{1}{2}$. But $\mu$ is a positive constant, so we have a contradiction. Hence $3 - (1 - \frac{6 \mu}{h^2}) x \neq 0$. Let $f'(x) = \frac{2x - x^2}{(3 - rx)^2}$, where $r = 1 - \frac{6 \mu}{h^2}$, then $f(x)$ is continuous on $[0, 2]$. Now, we differentiate $f$ with respect to $x$, to find the maximum, then
\[
f'(x) = \frac{2(1 - x)(3 - rx) + 2r(2x - x^2)}{(3 - rx)^3} = 0,
\]
so $x = \frac{3}{3 - r}$. Now, we have
\[
\sup_{0 \leq x \leq 2} \frac{x(2 - x)}{[3 - rx]^2} = \frac{3(3 - r) (2 - 3/(3 - r))}{[3 - r(3/(3 - r))]^2} = \frac{3(3 - 2r)}{(9 - 6r)^2} = \frac{1}{3(3 - 2r)} = \frac{h^2}{3(h^2 + 12 \mu)},
\]
so we get
\[
\sup_{\varphi} \left[ \frac{3 \Delta t (1 + 6 u^2) \sin \varphi}{h[2 + \cos \varphi - (6 \mu / h^2)(\cos \varphi - 1)]} \right]^2 = \frac{3(\Delta t)^2 (1 + 6 u^2)^2}{h^2 + 12 \mu},
\] (19)

The space step length $h$ is usually small, and the time step length $\Delta t$ is also usually small. Also, $u$ represents single speed and is usually small. So, the following relation will be satisfied for any problem of practical significance:
\[
\frac{3(\Delta t)^2 (1 + 6 u^2)^2}{h^2 + 12 \mu} \leq 1.
\]

Hence,
\[
\left| \frac{\gamma \sin \varphi}{2 \pi \cos \varphi + \beta} \right| \leq 1, \ \forall \varphi.
\]
Next, from the above lemma, we can put
\[
\frac{\gamma \sin \varphi}{2 \pi \cos \varphi + \beta} = \sin \theta,
\] (20)
for any angle $\theta$.
So, Eq. (17) becomes
\[
g^2 + 2ig \sin \theta - 1 = 0,
\] (21)
Thus, $g_1 = e^{i \theta}$ and $g_2 = -e^{-i \theta}$. Therefore, we have
\[
|g_1| = 1 \quad \text{and} \quad |g_2| = 1.
\] (22)

Showing that our scheme is marginally stable. □

4. Numerical applications

In this section, we present some numerical experiments to find the numerical solutions of single solitary waves, and determining the solution of two and three solitons interaction at different time levels. We also show the development of the Maxwellian initial condition into solitary waves.
Table 2
Invariants and errors for single solitary wave with \( c = 1, h = 0.2, k = 0.025, 0 \leq x \leq 100 \)

<table>
<thead>
<tr>
<th>Time</th>
<th>( I_1 )</th>
<th>( I_2 )</th>
<th>( I_3 )</th>
<th>( L_2 \times 10^3 )</th>
<th>( L_\infty \times 10^3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4.44288</td>
<td>3.29982</td>
<td>1.41420</td>
<td>0</td>
<td>0</td>
</tr>
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</tr>
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<td>4.44288</td>
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</tr>
<tr>
<td>4</td>
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<td>1.41420</td>
<td>5.29405</td>
<td>3.22690</td>
</tr>
<tr>
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<td>4.44288</td>
<td>3.29983</td>
<td>1.41420</td>
<td>5.94584</td>
<td>3.57898</td>
</tr>
<tr>
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<td>4.44288</td>
<td>3.29983</td>
<td>1.41420</td>
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<td>3.93789</td>
</tr>
<tr>
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<td>3.29983</td>
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<td>3.29983</td>
<td>1.41420</td>
<td>9.30196</td>
<td>5.43718</td>
</tr>
</tbody>
</table>

Table 3
Errors and invariants for single solitary wave with \( c = 1, h = 0.2, k = 0.025, 0 \leq x \leq 100, t = 10 \)

<table>
<thead>
<tr>
<th>Schemes</th>
<th>( I_1 )</th>
<th>( I_2 )</th>
<th>( I_3 )</th>
<th>( L_2 \times 10^3 )</th>
<th>( L_\infty \times 10^3 )</th>
</tr>
</thead>
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<tr>
<td>Analytical</td>
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<td>3.29983</td>
<td>1.41420</td>
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<td>0</td>
</tr>
<tr>
<td>Our scheme</td>
<td>4.44288</td>
<td>3.29983</td>
<td>1.41420</td>
<td>9.30196</td>
<td>5.43718</td>
</tr>
</tbody>
</table>

4.1. The motion of single solitary waves

To examine the accuracy and the efficiency of our scheme, we consider two cases in our numerical work. Since \( L_\infty \)-norm and \( L_2 \)-norm are used to compare our numerical solutions with the exact solution (5), and the quantities \( I_1 \), \( I_2 \) and \( I_3 \) are evaluated to measure the conservation properties of the collocation scheme, the analytical values of these invariants can be found in [7] as

\[
\begin{align*}
I_1 &= \int_{-\infty}^{\infty} u(x, 0) \, dx = \frac{\pi \sqrt{c}}{p}, \\
I_2 &= \int_{-\infty}^{\infty} (u^2(x, 0) + \mu u_t^2(x, 0)) \, dx = \frac{2c}{p} + \frac{2\mu pc}{3}, \\
I_3 &= \int_{-\infty}^{\infty} (u^4(x, 0) - \mu u_t^4(x, 0)) \, dx = \frac{4c^2}{3p} - \frac{2\mu pc}{3}.
\end{align*}
\]  

(23)

In the first case, the parameters \( c = 1, h = 0.2, k = 0.025, \mu = 1 \) and \( x_0 = 40 \) with range \([0,100]\) are chosen to coincide with the cubic schemes [7], then solitary wave has amplitude 1, and the simulations are done up to \( t = 10 \). The analytical values for the invariants are \( I_1 = 4.44288 \), \( I_2 = 3.29983 \) and \( I_3 = 1.41421 \). The invariants \( I_2 \) and \( I_3 \) change from their initial values by less than \( 3 \times 10^{-5} \) and \( 5 \times 10^{-5} \) percent, respectively, during the time of running, whereas, the changes of invariant \( I_1 \) approach to zero throughout. Thus satisfactory quantities are obtained. Also the error norms \( L_\infty \) and \( L_2 \) are satisfactorily small at all times, where \( L_\infty \)-error = \( 5.43718 \times 10^{-5} \) and \( L_2 \)-error = \( 9.30196 \times 10^{-5} \) at \( t = 10 \). All results for the first case are documented in Table 2. Moreover, Table 3 represents the values of the invariants and error norms of the present method at time 10 against the recorded results of Gardner et al. [7]. We find that our scheme provides good results than others. The motion of solitary wave using our collocation scheme is plotted at different time levels in Fig. 1. In case II, we choose \( c = 0.3, h = 0.1, k = 0.01, \mu = 1 \) and \( x_0 = 40 \) with range \([0,100]\), then the amplitude is 0.54772. The simulations are done up to \( t = 20 \) and the analytical values of the invariants are \( I_1 = 3.58197 \), \( I_2 = 1.34508 \) and \( I_3 = 0.153723 \). The changes of the invariants \( I_1 \), \( I_2 \) and \( I_3 \) from the initial variants approach to zero throughout and agree with the analytical values for these invariants, which indicated that our scheme is...
Fig. 1. Single solitary wave with $c = 1, x_0 = 40, 0 \leq x \leq 100$ at level times: $t = 0, 2, 5$ and $10$.

Table 4

Errors and invariants for single solitary wave with $c = 0.3, h = 0.1, k = 0.01, 0 \leq x \leq 100$

<table>
<thead>
<tr>
<th>Time</th>
<th>$I_1$</th>
<th>$I_2$</th>
<th>$I_3$</th>
<th>$L_2 \times 10^4$</th>
<th>$L_\infty \times 10^4$</th>
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</thead>
<tbody>
<tr>
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<td>0.153723</td>
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</tr>
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Fig. 2. Single solitary wave with $c = 0.3, x_0 = 40, 0 \leq x \leq 100$ at level times: $t = 0, 5, 10$ and $20$.

satisfactorily conservative. Errors in $L_\infty$-norm and $L_2$-norm are satisfactorily small, since $L_\infty$-error = $2.96650 \times 10^{-4}$ and $L_2$-error = $6.06885 \times 10^{-4}$ at $t = 20$. The results for the second case are shown in Table 4. Fig. 2 illustrates the motion of the solitary wave for this case at different time levels.
Table 5
Invariants for interaction of two solitary waves with $c_1 = 4, c_2 = 1, x_1 = 25, x_2 = 55, h = 0.2, k = 0.025, 0 \leq x \leq 250$

<table>
<thead>
<tr>
<th>Time</th>
<th>$I_1$</th>
<th>$I_2$</th>
<th>$I_3$</th>
</tr>
</thead>
<tbody>
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<td>14.6295</td>
<td>22.8807</td>
</tr>
<tr>
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<td>14.6294</td>
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</tr>
<tr>
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<td>14.6293</td>
<td>22.8809</td>
</tr>
<tr>
<td>20</td>
<td>11.4677</td>
<td>14.6292</td>
<td>22.8809</td>
</tr>
</tbody>
</table>

4.2. Interaction of two solitary waves

In this section, we study the interaction of two MRLW solitary waves having different amplitudes and traveling in the same direction. We consider the MRLW equation with initial conditions given by the linear sum of two well separated solitary waves of various amplitudes

$$u(x, 0) = \sum_{i=1}^{2} A_i \sec h(p_i (x - x_i)), \quad (24)$$

where $A_i = \sqrt{c_i}, p_i = \sqrt{\frac{c_i}{\mu (c_i + 1)}}, i = 1, 2, x_i$ and $c_i$ are arbitrary constants. From Eqs. (23), the analytical values of the conservation laws of this case can be found as

$$I_1 = \sum_{i=1}^{2} \frac{\pi \sqrt{c_i}}{p_i},$$

$$I_2 = \sum_{i=1}^{2} \left( \frac{2c_i}{p_i} + \frac{2\mu p_i c_i}{3} \right),$$

$$I_3 = \sum_{i=1}^{2} \left( \frac{4c_i^2}{3p_i} - \frac{2\mu p_i c_i}{3} \right). \quad (25)$$

In our computational work, we choose $c_1 = 4, c_2 = 1, x_1 = 25, x_2 = 55, \mu = 1, h = 0.2, k = 0.025$ with interval [0,250], then the amplitudes are in ratio 2:1, where $A_1 = 2A_2$. The analytical values of the invariants for this case are $I_1 = 11.467698, I_2 = 14.629243$ and $I_3 = 22.880466$, and the changes in $I_2$ and $I_3$ are less than $3.32 \times 10^{-2}$ and $4.59 \times 10^{-2}$ percent, respectively, whereas, the changes in $I_1$ approach to zero throughout, as shown in Table 5. Fig. 3 shows the computer plot of the interaction of these solitary waves at different time levels, where the simulation is done up to $t = 20$.

4.3. Interaction of three solitary waves

The interaction of three MRLW solitary waves having different amplitudes and traveling in the same direction is illustrated. We consider the MRLW equation with initial conditions given by the linear sum of three well separated solitary waves of various amplitudes

$$u(x, 0) = \sum_{i=1}^{3} A_i \sec h(p_i (x - x_i)), \quad (26)$$

where $A_i = \sqrt{c_i}, p_i = \sqrt{\frac{c_i}{\mu (c_i + 1)}}, i = 1, 2, 3, x_i$ and $c_i$ are arbitrary constants.
And from Eqs. (23), the analytical values of the conservation laws of this case can be found as

\[
\begin{align*}
I_1 &= \sum_{i=1}^{3} \frac{\pi \sqrt{c_i}}{p_i}, \\
I_2 &= \sum_{i=1}^{3} \left(\frac{2c_i}{p_i} + \frac{2\mu p_i c_i}{3}\right), \\
I_3 &= \sum_{i=1}^{3} \left(\frac{4c_i^2}{3p_i} - \frac{2\mu p_i c_i}{3}\right).
\end{align*}
\]

(27)

In our computational work, we choose \(c_1 = 4, c_2 = 1, c_3 = 0.25, x_1 = 15, x_2 = 45, x_3 = 60, \mu = 1, h = 0.2, k = 0.025\) with interval \([0, 250]\), then the amplitudes are in ratio 4:2:1, where \(A_1 = 2A_2 = 4A_3\). The analytical values of the invariants...
Table 6
Invariants for interaction of three solitary waves with $c_1 = 4$, $c_2 = 1$, $c_3 = 0.25$, $x_1 = 15$, $x_2 = 45$, $x_3 = 60$, $h = 0.2$, $k = 0.025$, $0 \leq x \leq 250$

<table>
<thead>
<tr>
<th>Time</th>
<th>$I_1$</th>
<th>$I_2$</th>
<th>$I_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>13.6891</td>
<td>15.4549</td>
<td>22.8816</td>
</tr>
<tr>
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<td>22.9347</td>
</tr>
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</tr>
<tr>
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<tr>
<td>45</td>
<td>13.7043</td>
<td>15.6563</td>
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</table>

Fig. 4. Interaction of three solitary waves at: (a) $t = 0$, (b) $t = 5$, (c) $t = 15$, (d) $t = 40$. 
for this case are $I_1 = 14.9801$, $I_2 = 15.8218$ and $I_3 = 22.9923$. We find from our numerical scheme, that the invariants $I_1$, $I_2$ and $I_3$ for the interaction of these solitary waves are sensible constants, despite their big amplitudes. The changes are $1.02 \times 10^{-2}$, $3 \times 10^{-1}$ and $2.41 \times 10^{-1}$ percent, respectively, for the computer run and the results are recorded in Table 6. Fig. 4 shows the details of interaction of these solitary waves at different time levels, and the simulation is done up to $t = 45$.

4.4. The Maxwellian initial condition

In final series of numerical experiments, the development of the Maxwellian initial condition

$$u(x, 0) = \exp(-(x - 40)^2),$$

(28)
Table 7
Invariants of MRLW equation using the Maxwellian initial condition

<table>
<thead>
<tr>
<th>( \mu )</th>
<th>Time</th>
<th>( I_1 )</th>
<th>( I_2 )</th>
<th>( I_3 )</th>
</tr>
</thead>
<tbody>
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<td>0.762183</td>
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<tr>
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<td>1.26704</td>
<td>0.884998</td>
</tr>
<tr>
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<td>15</td>
<td>1.77365</td>
<td>1.26076</td>
<td>0.906078</td>
</tr>
</tbody>
</table>

Table 8
Solitary wave generated from a Maxwellian initial condition [7]

<table>
<thead>
<tr>
<th>( \mu )</th>
<th>Number of solitary waves</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>owpa</td>
</tr>
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<td>0.1</td>
<td>1</td>
</tr>
<tr>
<td>0.05</td>
<td>1</td>
</tr>
<tr>
<td>0.04</td>
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<td>2</td>
</tr>
<tr>
<td>0.02</td>
<td>2</td>
</tr>
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<td>3</td>
</tr>
<tr>
<td>0.01</td>
<td>4</td>
</tr>
<tr>
<td>0.005</td>
<td>4</td>
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</tbody>
</table>

*owp: oscillating wave packet.

into a train of solitary waves is examined. We apply it to the problem for different cases: (I) \( \mu = 0.1 \), (II) \( \mu = 0.04 \), (III) \( \mu = 0.015 \) and (IV) \( \mu = 0.01 \). When \( \mu \) is reduced, more and more solitary waves are formed. For case (I), only single soliton is generated as shown in Fig. 5a, and for case (II), the Maxwellian pulse breaks up into a train of at least two solitary waves as shown in Fig. 5b. Also for case (III), three stable solitons are generated as shown in Fig. 5c. Finally, for the fourth case, the Maxwellian initial condition has decayed into four stable solitary waves as shown in Fig. 5d. The peaks of the well developed wave lie on a straight line so that their velocities are linearly dependent on their amplitudes and we observe a small oscillating tail appearing behind the last wave in all Maxwellian figures. All states at \( t = 14.5 \).

Moreover, the total number of the solitary waves which are generated from the Maxwellian initial condition according to the results obtained from the numerical scheme in test problem like as in Table 8, can be found in [7]. The values of the quantities \( I_1 \), \( I_2 \) and \( I_3 \) for the cases: \( \mu = 0.1, 0.04, 0.015 \) and 0.01 are given in Table 7, since for \( \mu = 0.1 \), the variation of invariants \( I_2 \) and \( I_3 \) from initial variants changes less than \( 5 \times 10^{-5} \) and \( 6.3 \times 10^{-5} \) percent, respectively, whereas, the changes in invariant \( I_1 \) approach to zero throughout, and for \( \mu = 0.04 \), the invariants are changed by less than \( 1 \times 10^{-5}, 2 \times 10^{-5} \) and \( 2.8 \times 10^{-5} \) percent, respectively (see Table 8). For the case \( \mu = 0.015 \), the invariants \( I_1 \),
\( I_2 \) and \( I_3 \) are changed by less than \( 1 \times 10^{-5}, 4 \times 10^{-5} \) and \( 2.48 \times 10^{-4} \) percent, respectively. Finally, for \( \mu = 0.01 \), we find that the changes in these invariants are less than \( 4 \times 10^{-5} \) and \( 4 \times 10^{-4} \) and \( 1.722 \times 10^{-3} \) percent, respectively, during the time of running, and the simulations are done up to \( t = 15 \).

5. Conclusion

In this paper, the collocation method using cubic B-splines was applied to study the solitary waves of the MRLW equation, and it is shown that our scheme is marginally stable, and more accurate than other cubic collocation schemes in [7]. We tested our scheme through single solitary wave in which the analytic solution is known, and then extended it to study the interaction of solitons where no analytic solution is known during the interaction. The Maxwellian initial condition was also used. Moreover, despite the fact that the wave does not change, results show that the interaction results in a tail of small amplitude in two and clearly three soliton interactions, and the conservation laws were reasonably satisfied. The appearance of such a tail can be beneficial in further study.

References