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Tilting Functors and Stable Equivalences for Selfinjective Algebras*

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INTRODUCTION

Let A be a finite dimensional algebra over a basic field k and T a right A-module satisfying the following three conditions:

(1) $\mathrm{Ext}_A^1(T, T) = 0$, (2) $\mathrm{Ext}_A^2(T, -) = 0$, (3) There is a short exact sequence $0 \to A_A \to T'_A \to T''_A \to 0$, with T', T'' being direct sums of summands of T. Then putting $B = \mathrm{End}\, T_A$ we call (B, T, A) and $\mathrm{Hom}_A(T, -)$: mod-$A \to$ mod-B a tilting triple and a tilting functor, respectively.

Tilting functors have been introduced by Brenner and Butler [6] as a generalization of the Bernstein–Gelfand–Ponomarev's reflection functors [5]. They and Happel and Ringel [8] have proved that we have (usually nonhereditary) torsion theories $(\mathscr{T}, \mathscr{F})$ in mod-A and $(\mathscr{X}, \mathscr{Y})$ in mod-B, where $\mathscr{T} = \{X \in \text{mod-}A \mid \mathrm{Ext}_A^1(T, X) = 0\}$ and $\mathscr{X} = \{Y \in \text{mod-}B \mid Y \otimes_B T = 0\}$, and the tilting functor and $\mathrm{Ext}_A^1(T, -)$ give category-equivalences between $\mathscr{T}$ and $\mathscr{Y}$ and between $\mathscr{F}$ and $\mathscr{X}$, respectively.

These equivalences give us, however, no information about indecomposable A-modules and B-modules which do not belong to the above subcategories $\mathscr{T}$, $\mathscr{F}$, $\mathscr{X}$, and $\mathscr{Y}$. The purpose of this paper is to point out that there is a method to enlarge our view by which we can supply the lack of information.

Let us consider trivial extension algebras $R = A \ltimes DA$ and $S = B \ltimes DB$ of A and B by DA and DB respectively, where ${}_A DA_A = \mathrm{Hom}_k(A, k)$ and ${}_B DB_B = \mathrm{Hom}_k(B, k)$ are injective cogenerator bimodules. In this case R and S are selfinjective (more precisely symmetric) algebras and mod-A and mod-B are naturally embedded into projectively stable categories $\underline{\text{mod}}$-R and $\underline{\text{mod}}$-S. Then our main theorem states that for any tilting triple

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(A, T, B) there exists always a stable equivalence $\mathcal{S}$: $\underline{\text{mod}}\text{-}R \to \underline{\text{mod}}\text{-}S$ such that the restriction of $\mathcal{S}$ to the torsion class $\mathcal{T}$ coincides with the tilting functor $\text{Hom}_A(T, -)$.

In fact this stable equivalence $\mathcal{S}$ is a generalization of S_k^+ which was introduced by one of the authors [11] for a trivial extension of a path-algebra of an oriented tree Q and a reflection functor s_k^+ with respect to a sink vertex of Q. And it is to be noted that Assem and Iwanaga [1], and Wakamatsu [15] also proved the existence of such stable equivalence for the following special cases, respectively,

(1) $R = A \ltimes DA$ is of finite representation type,

(2) $\text{Hom}_A(T, -)$ is a partial coxeter functor in the sense of Auslander–Platzeck–Reiten [4].

Here it is not too much to say that our theorem is fairly general because it needs no restriction for the representation type of A and the torsion theories induced by T. Even in the case where A and hence R are of infinite representation type $\mathcal{S}$ teaches us concretely not only the correspondence between many connected components of Auslander–Reiten quivers of R and S but also the correspondence of (stable) homomorphisms between indecomposable modules which belong not necessarily to the same connected component of Auslander–Reiten quivers (cf. Examples in Sect. 3).

Our proof is also available to artin algebras provided we replace $D = \text{Hom}_k(-, k)$ by $\text{Hom}_C(-, E(C/\text{rad } C))$ where C are centers of algebras and $E(C/\text{rad } C)$ are injective envelopes of ${}_C C/\text{rad } C$.

In Section 1 we shall introduce the notion of torsion resolutions of A-modules and using them we shall define the stable functor $\mathcal{S}$. The proof for $\mathcal{S}$ to be stable equivalence is reduced to the proofs of the commutativities of great many diagrams and will be given in Section 2. In Section 3 we give remarks and examples in which Auslander–Reiten quivers of A, B, R and S, and correspondences defined by $\text{Hom}_A(T, -)$, $\text{Ext}_A(T, -)$ and $\mathcal{S}$ will be explicitly given.

Throughout this paper unless otherwise specified modules are unital finitely generated right modules, but homomorphisms operate from the left hand. $[X, Y]_A$ denotes $\text{Hom}_A(X, Y)$ for A-modules X and Y.

1. Torsion Resolutions and Stable Extensions

Let (B, T, A) be a tilting triple. For the definition see Introduction. Let $(\mathcal{T}, \mathcal{F})$, $(\mathcal{X}, \mathcal{Y})$ be the corresponding torsion theories in mod-A and mod-B. That is, T_A is a tilting module, $B = \text{End } T_A$ and $A = \text{End }_B T$;

$$\mathcal{T} = \{X \in \text{mod-}A \mid \text{Ext}_A^1(T, X) = 0\} = \text{Gen}(T_A)$$

and

$$\mathscr{F} = \{X \in \text{mod-}A \mid \text{Hom}_A(T, X) = 0\} = \text{Cog}(\tau_A(T)),$$

where $\tau_A = D\,\text{Tr}$ is the Auslander–Reiten translation, and $\text{Gen}(T_A)$ (resp. $\text{Cog}(\tau_A(T))$ is a subcategory of mod-A consist of all modules which are homomorphic images (resp. submodules) of direct sums (resp. products) of copies of T_A (resp. $\tau_A(T)$);

$$\mathscr{Y} = \{Y \in B\text{-mod} \mid \text{Tor}_1^B(Y, T) = 0\}$$

and

$$\mathscr{X} = \{Y \in B\text{-mod} \mid Y \otimes T = 0\}.$$

LEMMA 1.1. *For any A-module X there is an exact sequence*

$$0 \longrightarrow X \xrightarrow{\alpha_X} V(X) \xrightarrow{\beta_X} T(X) \longrightarrow 0$$

such that $V(X) \in \mathscr{T}$, $T(X) = P \otimes T \in \text{add-}T$, *where P is a projective cover of right B-module* $\text{Ext}_A^1(T, X)$.

Proof. For $X \in \mathscr{T}$ we can take $V(X) = X$ and $T(X) = 0$. Hence we divide the proof into the following two cases:

(i) Let X belong to $\mathscr{F}$. Take the projective cover $P \to^{\rho} \text{Ext}_A^1(T, X)$ of $\text{Ext}_A^1(T, X)$ and denote $\text{Ker}\,\rho$ by K.

Apply $(- \otimes_B T)$ to

$$0 \longrightarrow K \longrightarrow P \xrightarrow{\rho} \text{Ext}_A^1(T, X) \longrightarrow 0.$$

Then

$$0 \to \text{Tor}_1^B(\text{Ext}_A^1(T, X), T) \to K \otimes T$$
$$\to P \otimes T \to \text{Ext}_A^1(T, X) \otimes T \to 0 \qquad \text{is exact.}$$

However $\text{Tor}_1^B(\text{Ext}_A^1(T, X), T) \simeq X$ and

$$\text{Ext}_A^1(T, X) \otimes T = 0 \qquad \text{for } X \in \mathscr{F}.$$

So we can take $K \otimes T$ and $P \otimes T$ as $V(X)$ and $T(X)$, respectively.

(ii) Let X be an A-module which is not necessarily torsion free. Let us consider the exact sequence

$$0 \to t(x) \to X \to X/t(X) \to 0$$

as an element of $\mathrm{Ext}^1_A(X/t(X), t(X))$, where $t(X)$ is the torsion part of X with respect to $(\mathscr{T}, \mathscr{F})$.

By (i) there is an exact sequence

$$0 \to X/t(X) \to V(X/t(X)) \to T(X/t(X)) \to 0$$

with $V(X/t(X)) \in \mathscr{T}$ and $T(X/t(X)) \in \text{add-}T$.

Applying $\mathrm{Ext}^1_A(-, t(X))$ we have an isomorphism $\theta\colon \mathrm{Ext}^1_A(V(X/t(X)), t(X)) \to \mathrm{Ext}^1_A(X/t(X), t(X))$ because $\mathrm{Ext}^1_A(T(X/t(X)), t(X))=0$ and $\mathrm{Ext}^2_A(T(X/t(X)), t(X))=0$. Thus we have the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & t(X) & \simeq & t(X) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & X & \longrightarrow & V & \longrightarrow & T' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X/t(X) & \longrightarrow & V(X/t(X)) & \longrightarrow & T(X/t(X)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where the first column is considered as E and the second column is considered as $\theta^{-1}(E)$. Now from $t(X)$, $V(X/t(X)) \in \mathscr{T}$ it follows $V \in \mathscr{T}$ and $T' \simeq T(X/t(X)) \in \text{add-}T$. Because $T(X/t(X)) \cong P \otimes T$ where P is the projective cover of $\mathrm{Ext}^1_A(T, X/t(X))_B$ but $\mathrm{Ext}^1_A(T, X)_B \simeq \mathrm{Ext}^1_A(T, X/t(X))_B$. This completes the proof.

For any $X \in \text{mod-}A$ we shall call an exact sequence

$$0 \to X \to V_1 \to V_2 \to 0$$

a torsion resolution of X if

$$V_1 \in \mathscr{T} \qquad \text{and} \qquad V_2 \in \text{add-}T.$$

PROPOSITION 1.2. *For any torsion resolution of X,*

$$0 \to X \to V' \to T' \to 0$$

there is an isomorphic torsion resolution such that

$$0 \longrightarrow X \xrightarrow{\binom{\alpha_X}{0}} V(X) \oplus T_0 \xrightarrow{\begin{pmatrix}\beta_X & 0\\ 0 & 1\end{pmatrix}} T(X) \oplus T_0 \longrightarrow 0,$$

where $T_0 \in \text{add-}T$ *and* α_X, β_X, $V(X)$, $T(X)$ *are same as in Lemma* 1.1.

Proof. From V', $V(X) \in \mathcal{T}$ it follows $\mathrm{Ext}_A^1(T, V') = 0$ and $\mathrm{Ext}_A^1(T, V(X)) = 0$. Thus there are A-homomorphisms f, g, h, k such that

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{\alpha_X} & V(X) & \xrightarrow{\beta_X} & T(X) & \longrightarrow & 0\\
 & & \| & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle h} & & \\
0 & \longrightarrow & X & \longrightarrow & V' & \longrightarrow & T' & \longrightarrow & 0\\
 & & \| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle k} & & \\
0 & \longrightarrow & X & \xrightarrow{\alpha_X} & V(X) & \xrightarrow{\beta_X} & T(X) & \longrightarrow & 0
\end{array}$$

is commutative.

Here it is to be noted that

$$\begin{array}{ccc}
\mathrm{Hom}_A(T, T(X)) & \to \mathrm{Ext}_A^1(T, X) & \to 0\\
{\scriptstyle \mathrm{Hom}_A(T, kh)}\downarrow & \| & \\
\mathrm{Hom}_A(T, T(X)) & \to \mathrm{Ext}_A^1(T, X) & \to 0
\end{array}$$

is commutative and $\mathrm{Hom}_A(T, kh)$ is an isomorphism, for by Lemma 1.1 rows are the projective covers of $\mathrm{Ext}_A^1(T, X)$.

However $\mathrm{Hom}_A(T, -)$ gives an equivalence between $\mathcal{T}$ and $\mathcal{Y}$. Thus kh and gf are isomorphisms and we can conclude our proof by a routine calculation.

Dually for $Y \in B\text{-mod}$ we shall call an exact sequence

$$0 \to W' \to V' \to Y \to 0$$

a torsion free resolution if $V' \in \mathcal{Y}$ and $W' \in \text{add-}DT_B$

PROPOSITION 1.3. *For a right B-module Y*

(1) *there is a torsion free resolution*

$$0 \longrightarrow W(Y) \xrightarrow{\lambda_Y} U(Y) \xrightarrow{\gamma_Y} Y \longrightarrow 0$$

such that $0 \to \mathrm{Tor}_1^B(Y, T) \to W(Y) \otimes T$ *is the injective envelope of* $\mathrm{Tor}_1^B(Y, T)$.

(2) *for any torsion free resolution of Y there is an isomorphic torsion free resolution such that*

$$0 \longrightarrow W(Y) \oplus W_0 \xrightarrow{\begin{pmatrix} \lambda_Y & 0 \\ 0 & 1 \end{pmatrix}} U(Y) \oplus W_0 \xrightarrow{(\gamma_Y, 0)} Y \longrightarrow 0.$$

Let R and S be trivial extensions of A and B by injective cogenerators DA and DB, respectively. Then there are full embeddings of mod-$A \subset \underline{\text{mod}}$-$R$ and mod-$B \subset \underline{\text{mod}}$-$S$: Take A-homomorphism $f \in \text{Hom}_A(X, Y)$, $X, Y \in$ mod-A. Assume f is factored through a projective R-module $(P \oplus P \otimes DA, \begin{pmatrix} 0 & 0 \\ 1_{P \otimes DA} & 0 \end{pmatrix})$ such that

$$f = (X \xrightarrow{f_1} P \oplus P \otimes DA \xrightarrow{f_2} Y).$$

Then $\text{Im} f_1 \subset (0 \oplus P \otimes DA)$ and $\text{Ker} f_2 \supset (0 \oplus P \otimes DA)$, since X and Y are annihilated by ideal $(0, DA)$ of R, and hence f is the zero map. This implies mod-$A \subset \underline{\text{mod}}$-$R$ is a full embedding, and similarly mod-$B \subset \underline{\text{mod}}$-$S$ is a full embedding.

In the case where A and B are hereditary and $\text{Hom}_A(T, -)$ gives the Bernstein–Gelfand–Ponomarev's reflection functor, one of the authors [11] proved that there exists a stably equivalent functor between $\underline{\text{mod}}$-R and $\underline{\text{mod}}$-S which extends $\text{Hom}_A(T, -)$. It seems to us that the result is interesting because A and B are not only of finite representation type but also of infinite representation type (depending on neither tame nor wild type). The main purpose of this paper is to prove the following more general result:

THEOREM 1.4. *For any tilting triple (B, T, A)*

(1) *there is a stable functor $\mathscr{S}$ from $\underline{\text{mod}}$-R to $\underline{\text{mod}}$-S such that* $\mathscr{S} | \mathscr{T} = \text{Hom}(T, -)$.

(2) *$\mathscr{S}$ is always a stable equivalence.*

Now it needs to introduce several notations: For C-algebras E and F, and an E-F-bimodule U and an F-E-bimodul V we denote by η^U_W the map: mod-$E \ni W \ni w \mapsto (t \to w \otimes t) \in [U_F, \ W \otimes_E U_F] \in$ mod-E and by ε^V_Z the map: mod-$F \ni [V_E, Z_E] \otimes_F V_E \ni h \otimes t \mapsto h(t) \in Z \in$ mod-E. In the case of $V = {}_B T_A$ we abbreviate η^T_W and ε^T_Z to η_w and ε_Z, respectively. Denote by ι_U the adjunction, $[- \otimes_E U, -]_F \to [-, [U, -]_F]_E$.

We use also isomorphisms $\delta_A : {}_A DT \otimes_B T_A \to D\,\text{Hom}_B(T, T) = DA$ and $\delta_B : T \otimes_A DT \to D\,\text{Hom}_B(T, T) = DB$, and sometimes we identify $DT \otimes T$ (resp. $T \otimes DT$) with DA (resp. DB) by δ_A (resp. δ_B).

For a B-module W (resp. A-module Z) we can define an S-module

$(W \oplus W \otimes DB, \left(\begin{smallmatrix} 0 & 0 \\ 1_{W\otimes DB} & 0 \end{smallmatrix}\right) (W \oplus W \otimes DB) \otimes DB \to W \oplus W \otimes DB)$ (resp. an R-module $(Z \oplus Z \otimes DA, \left(\begin{smallmatrix} 0 & 0 \\ 1_{Z\otimes DA} & 0 \end{smallmatrix}\right))$ and we denote this module by

$$\begin{matrix} W \\ \wr\wr\wr\wr \\ W \otimes DB \end{matrix} \quad \left(\text{resp. } \begin{matrix} Z \\ \wr\wr\wr\wr \\ Z \otimes DA \end{matrix}\right).$$

PROPOSITION 1.5. *Let $(X, \phi: X \otimes DA_A \to X_A)$ be a right R-module and*

$$0 \longrightarrow X \xrightarrow{\alpha_X} V(X) \xrightarrow{\beta_X} T(X) \longrightarrow 0$$

the minimal torsion resolution of a right A-module X. Since $V(X) \in \mathscr{T}$, $\varepsilon_{V(X)}: [T, V(X)] \otimes T \to V(X)$ is an isomorphism. We can define a B-homomorphism

$$\Phi_X: X \otimes DT \to [T, V(X)] \oplus [T, V(X)] \otimes DB$$

by

$$\begin{pmatrix} [T, \alpha_X] \cdot [T, -\phi] \cdot [T, X \otimes \delta_A] \cdot \eta_{X \otimes DT} \\ [T, V(X)] \otimes \delta_B \cdot \varepsilon_{V(X)}^{-1} \otimes DT \cdot \alpha_X \otimes DT \end{pmatrix}$$

and denote Cok Φ_X *by $\mathscr{S}(X)$. Then it holds that*

(1) *$X \otimes DT$ is right S-module by*

$$-\phi \otimes DT \cdot X \otimes \delta_A \otimes DT \cdot X \otimes DT \otimes \delta_B^{-1}: \quad X \otimes DT \otimes DB \to X \otimes DT.$$

(2) *Φ_X can be considered as an S-homomorphism*:

$$X \otimes DT_S \to \begin{matrix} [T, V(X)] \\ \wr\wr\wr\wr\wr\wr\wr\wr \\ [T, V(X)] \otimes DB_S \end{matrix}.$$

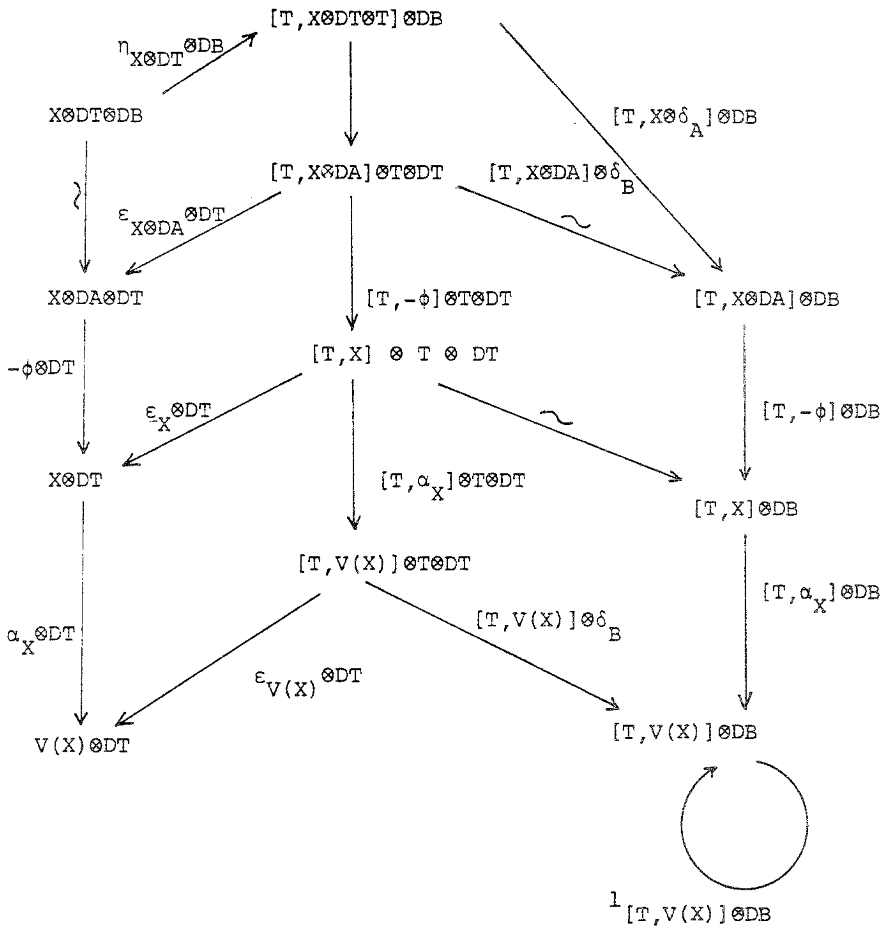
(3) *If X is a torsion right A-module, then $\mathscr{S}(X) \cong \operatorname{Hom}_A(T, X)$.*

(4) *If X is a projective R-module, i.e., projective and injective R-module, then $\mathscr{S}(X)$ is a projective S-module.*

Proof. (1) It follows by the equality

$$\begin{aligned} &(-\phi \otimes DT \cdot X \otimes \delta_A \otimes DT \cdot X \otimes DT \otimes \delta_B^{-1}) \\ &\quad \cdot ((-\phi \otimes DT \cdot X \otimes \delta_A \otimes DT \cdot X \otimes DT \otimes \delta_B^{-1}) \otimes DB) \\ &= (-\phi \cdot -\phi \otimes DA \cdot X \otimes DA \otimes \delta_A) \otimes DT \cdot X \otimes \delta_A \otimes DT \otimes DB = 0. \end{aligned}$$

(2) It follows from the commutativity of the following diagram



(3) Since T is a tilting left B-module the torsion class $D\mathscr{Y} = \mathrm{Gen}\ {}_BT$ and ${}_BD(B) \in \mathrm{Gen}\ {}_BT$, $\mathrm{Tor}_1^A(T, DT) \simeq \mathrm{Tor}_1^A(T_A, [{}_BT, {}_BD(B)]) = 0$. It follows that

$$0 \longrightarrow X \otimes DT \xrightarrow[\alpha_X \otimes DT]{} V(X) \otimes DT \xrightarrow[\beta_X \otimes DT]{} T(X) \otimes DT \longrightarrow 0$$

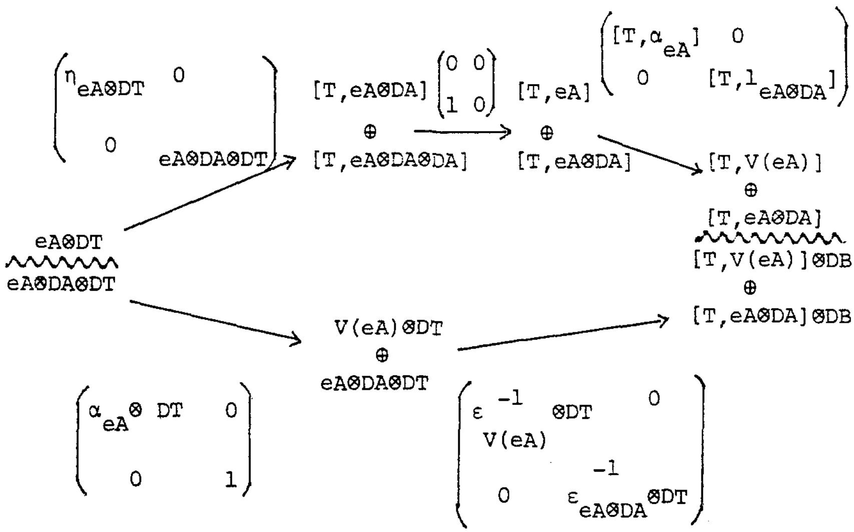
is exact. But from the assumption $X \in \mathscr{T}$ it follows $T(X) = 0$ and $\varepsilon_{V(X)}^{-1} \otimes DT \cdot \alpha_X \otimes DT$ is an isomorphism.

On the other hand, $[T, \alpha_X] \cdot [T, -\phi] \cdot [T, X \otimes \delta_A] \cdot \eta_{X \otimes DT} = 0$ since $\phi = 0$. Hence by the definition of $\mathscr{S}(X)$, $\mathscr{S}(X) \simeq \mathrm{Hom}_A(T, X)$.

(4) Let e be a primitive idempotent of A. Then e is also a primitive idempotent in R and

$$eR = \begin{array}{c} eA \\ \sim\!\sim\!\sim\!\sim\!\sim \\ eA \otimes DA \end{array}.$$

Let $0 \to eA \to^{\alpha_{eA}} V(eA) \to^{\beta_{eA}} T(eA) \to 0$ be the minimal torsion resolution of eA. Then $V(eA)$, $T(eA) \in \operatorname{add} T_A$. And Φ is given by



$$\Phi = \begin{pmatrix} 0 & 0 \\ [T, 1_{eA\otimes DA}] \cdot \eta_{eA\otimes DT} & 0 \\ \varepsilon^{-1}_{V(eA)}\otimes DT \cdot \alpha_{eA}\otimes DT & 0 \\ 0 & \varepsilon^{-1}_{eA\otimes DA}\otimes DT \cdot 1_{eA\otimes DA\otimes DT} \end{pmatrix}.$$

Here $\varepsilon^{-1}_{eA\otimes DA} \cdot 1_{eA\otimes DA\otimes DT}$: $eA\otimes DA\otimes DT \to [T, eA\otimes DA]\otimes DB$ is an isomorphism and

$$[T, 1_{eA\otimes DA}] \otimes \eta_{eA\otimes DT}\colon \quad eA\otimes DT \to [T, eA\otimes DA]$$

is also an isomorphism as a component of

$$\begin{pmatrix} [T, 1_{eA\otimes DA}] \otimes \eta_{eA\otimes DT} \\ \varepsilon^{-1}_{V(eA)} \otimes DT \cdot \alpha_{eA} \otimes DT \end{pmatrix}\colon \quad \begin{array}{c} eA\otimes DT \\ \sim\!\sim\!\sim\!\sim\!\sim\!\sim \\ eA\otimes DA\otimes DT \end{array} \to \begin{array}{c} [T, eA\otimes DA] \\ \oplus \\ [T, V(eA)]\otimes DB. \end{array}$$

Hence

$$\operatorname{Cok} \Phi \simeq \frac{[T, V(eA)]}{[T, V(eA)] \otimes DB},$$

but $[T, V(eA)]$ is a projective B-module and $\operatorname{Cok} \Phi$ is a projective S-module.

Proof of Theorem 1.4. (1) Let $(X, \phi: X \otimes DA \to X)$ and $(X_1, \phi_1: X_1 \otimes DA \to X_1)$ be R-modules and f an R-homomorphism of X to X'.

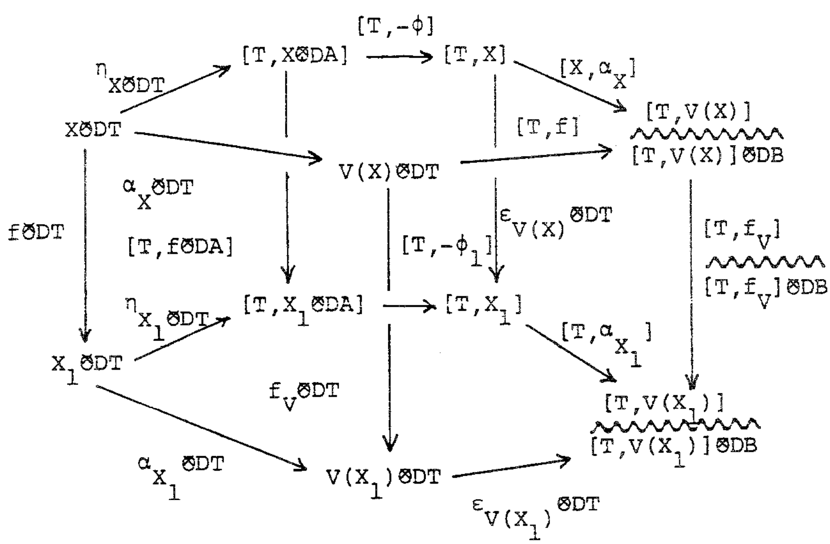
Then f is associated with

$$\begin{array}{ccc} X \otimes DA & \xrightarrow{\ \phi\ } & X \\ {\scriptstyle f \otimes DA}\downarrow & & \downarrow{\scriptstyle f} \\ X_1 \otimes DA & \xrightarrow[\ \phi_1\ ]{} & X_1 \end{array}$$

and by the property of torsion resolution there are A-homomorphisms f_V, f_T such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & V(X) & \longrightarrow & T(X) & \longrightarrow & 0 \\ & & {\scriptstyle f}\downarrow & & {\scriptstyle f_V}\downarrow & & {\scriptstyle f_T}\downarrow & & \\ 0 & \longrightarrow & X_1 & \longrightarrow & V(X_1) & \longrightarrow & T(X_1) & \longrightarrow & 0 \end{array}$$

is commutative. Therefore the commutativity holds for each square of the following diagram:



where we abbreviate isomorphisms

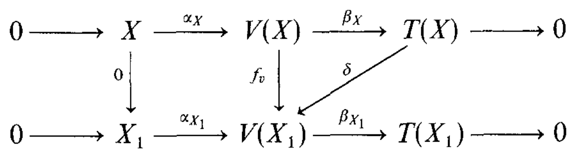
$$[T, X \otimes \delta_A], \quad [T, X_1 \otimes \delta_A], \quad [T, V(X)] \otimes \delta_B, \quad \text{and} \quad [T, V(X_1)] \otimes \delta_B.$$

Thus we have an S-homomorphism f^*:

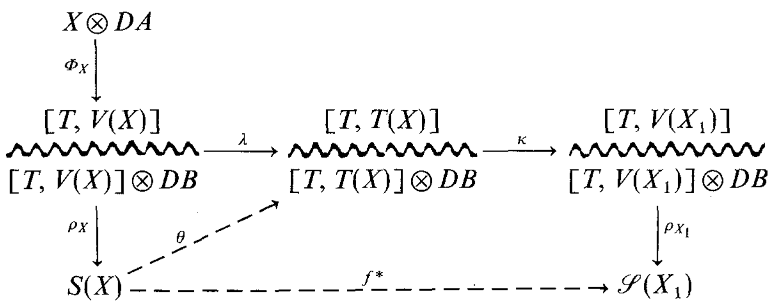
$$\mathscr{S}(X) \to \mathscr{S}(X_1), \qquad \text{though } f^* \text{ depends upon } f_V.$$

In the following we shall prove that f_V is uniquely determined modulo morphisms which factor through projective S-modules.

Assume $f = 0$. Then there is an A-homomorphism δ such that the triangle in



is commutative. And in the following diagram it holds $\kappa\lambda = \begin{pmatrix} [T, f_v] & 0 \\ 0 & [T, f_v] \otimes DB \end{pmatrix}$ and $\lambda \Phi_X = 0$:



where $\lambda = \begin{pmatrix} [T, \beta_X] & 0 \\ 0 & [T, \beta_X] \otimes DB \end{pmatrix}$, $\kappa = \begin{pmatrix} [T, \delta] & 0 \\ 0 & [T, \delta] \otimes DB \end{pmatrix}$, and ρ_X and ρ_{X_1} are cokernels of Φ_X and Φ_{X_1}, respectively. Therefore we have an S-homomorphism θ such that $f^* \cdot \rho_X = \rho_{X_1} \kappa \theta \rho_X$. Hence $f^* = \rho_{X_1} \lambda \theta$, and

$$\begin{matrix} [T, T(X)] \\ \sim\!\sim\!\sim\!\sim\!\sim\!\sim \\ [T, T(X)] \oplus DB \end{matrix}$$

is a projective S-module.

Now for $f \in \mathrm{Hom}_R(X, X_1)$, $g \in \mathrm{Hom}_R(X_1, X_2)$ it is clear that $(g \cdot f)^* \equiv g^* \cdot f^*$ modulo S-homomorphisms which factor through projectives.

2. $\mathscr{S}$: $\underline{\text{mod}}\, A \ltimes DA \to \underline{\text{mod}}\, B \ltimes DB$ IS AN EQUIVALENCE

Torsion free resolution of $\mathscr{S}(X)$

Let $(Y, \psi\colon Y \to [DB, Y])$ be a right S-module and

$$0 \longrightarrow W(Y) \xrightarrow{\lambda_Y} U(Y) \xrightarrow{\gamma_Y} Y \longrightarrow 0$$

a minimal torsion free resolution of a right B-module Y.

Since $U(Y) \in \mathscr{Y}$ and $\eta_{U(Y)}$ is an isomorphism, we can define an A-homomorphism $\Psi_Y\colon [DA, U(Y) \otimes T] \oplus U(Y) \otimes T \to [DT, Y]$ by

$$([DT, \gamma_Y] \cdot [DT, \eta_{U(Y)}^{-1}] \cdot \iota_T \cdot [\delta_A, U(Y) \otimes T],$$
$$\varepsilon_{[DT, Y]} \cdot \iota_{DT} \cdot [\delta_B, Y]^{-1} \otimes T \cdot -\psi \otimes T \cdot \gamma_Y \otimes T).$$

Denote Ker Ψ_Y by $\mathscr{Q}(Y)$.

Further $[DT, Y]$ becomes a right R-module by

$$[DT, Y] \xrightarrow{[DT, -\psi]} [DT, [DB, Y]] \xrightarrow{[DT, [\delta_B, Y]]} [DT, [T \otimes DT, Y]]$$

$$\simeq [DT \otimes T, [DT, Y]] \xrightarrow{[\delta_A, [DT, Y]]} [DA, [DT, Y]]$$

and Ψ_Y is natural as an R-homomorphism:

$$\begin{matrix} [DA, U(Y) \otimes T] \\ \rightsquigarrow \\ U(Y) \otimes T \end{matrix} \to [DT, Y].$$

So $\mathscr{Q}(Y)$ is a right R-module and by a dual argument to the preceding section we have a stable functor $\mathscr{Q}\colon \underline{\text{mod}}\text{-}S \to \underline{\text{mod}}\text{-}R$.

Now let $(X, \phi\colon X \otimes DA \to X)$ be a right A-module and

$0 \longrightarrow X \xrightarrow{\alpha_X} V(X) \xrightarrow{\beta_X} T(X) \longrightarrow 0$ a minimal torsion resolution of X.

We shall seek a torsion free resolution of $\mathscr{S}(X)_B$.

Let $P_0 \to^{\rho_0} V(X) \to 0$ be a projective cover of $V(X)$. Then we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_1 & \xrightarrow{\alpha} & P_0 & \xrightarrow{\beta} & T(X) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \rho_1} & & \downarrow{\scriptstyle \rho_0} & & \downarrow{\scriptstyle \text{id}} & & \\ 0 & \longrightarrow & X & \xrightarrow{\alpha_X} & V(X) & \xrightarrow{\beta_X} & T(X) & \longrightarrow & 0, \end{array}$$

where $\beta = \beta_X \cdot \rho_0$ and P_1 is projective as proj. dim $T(X) \leqslant 1$.

It follows further the commutative diagram:

$$\begin{array}{ccccccc} 0 \longrightarrow & P_1 \otimes DT & \xrightarrow{(t_1, s_1)} & [T, V(X)] \oplus [T, P_0 \otimes DA] & & & \\ & \downarrow{\scriptstyle \rho_1 \otimes DT} & & \downarrow{\scriptstyle \begin{pmatrix} 1_{[T,V(X)]} & 0 \\ 0 & \kappa \end{pmatrix}} & & & \\ 0 \longrightarrow & X \otimes DT & \xrightarrow{(t_2, s_2)} & [T, V(X)] \oplus [T, V(X)] \otimes DB & \longrightarrow & \mathscr{S}(X) & \longrightarrow 0, \end{array} \quad (1)$$

where

$$t_1 = [T, \alpha_X][T, -\phi][T, X \otimes \delta_A]\, \eta_{X \otimes DT}\, \rho_1 \otimes DT,$$

$$t_2 = [T, \alpha_X][T, -\phi][T, X \otimes \delta_A]\, \eta_{X \otimes DT},$$

$$s_1 = [T, P_0 \otimes \delta_A]\, \eta_{P_0 \otimes DT} \otimes DT,$$

$$s_2 = [T, V(X)] \otimes \delta_B \varepsilon_{V(X)} \otimes DT\, \alpha_X \otimes DT,$$

and

$$\kappa = [T, V(X)] \otimes \delta_B (\varepsilon_{V(X)}^{-1} \otimes DT)\, \rho_0 \otimes DT(\eta_{P_0 \otimes DT})^{-1} [T, P \otimes \delta_A^{-1}],$$

because from $\mathrm{Tor}_1^A(T(X), DT) \simeq D\,\mathrm{Ext}_A^1(T(X), T) = 0$ it follows the exactness of two rows of the commutative diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & P_1 \otimes DT & \xrightarrow{\alpha \otimes DT} & P_0 \otimes DT & \xrightarrow{\beta \otimes DT} & T(X) \otimes DT & \longrightarrow 0 \\ & \downarrow{\scriptstyle \rho_1 \otimes DT} & & \downarrow{\scriptstyle \rho_0 \otimes DT} & & \| & \\ 0 \longrightarrow & X \otimes DT & \xrightarrow[\alpha_X \otimes DT]{} & V(X) \otimes DT & \xrightarrow[\beta_X \otimes DT]{} & T(X) \otimes DT & \longrightarrow 0 \end{array}$$

and then we have $\ker(\rho_1 \otimes DT) \simeq \mathrm{Ker}(\rho_0 \otimes DT)$. As $\mathrm{Ker}\begin{pmatrix} 1_{[T,V(X)]} & 0 \\ 0 & \kappa \end{pmatrix} \simeq \mathrm{Ker}\,\kappa \simeq \mathrm{Ker}(\rho_0 \otimes DT) \simeq \mathrm{Ker}(\rho_1 \otimes DT)$ we know $\mathrm{Coker}(t_1, s_1) \simeq \mathscr{S}(X)$ by the snake lemma.

Now from the fact that $P_1 \otimes DT \in \mathrm{add}\, DT_B$ and $[T, V(X)] \oplus [T, P_0 \otimes DA] \in \mathscr{Y}$ it follows that the upper row in (1) together with $\mathrm{Coker}(t_1, s_1)$ can be considered as a torsion free resolution of $\mathscr{S}(X)$.

Hereafter we shall denote by $(x, \hat{y})$ the pair of B-homomorphisms: $[T, V(X)] \oplus [T, P_0 \otimes DA] \to \mathscr{S}(X)$ in the above torsion free resolution. Then

$$\begin{array}{ccc} [T, P_0 \otimes DA] & \xrightarrow{\hat{y}} & \mathscr{S}(X) \\ \downarrow{\scriptstyle \kappa} & & \| \\ [T, V(X)] \otimes DB & \xrightarrow[y]{} & \mathscr{S}(X) \end{array}$$

is commutative.

Composition Length of $\mathcal{D}\mathcal{S}(X)$

Put an S-module

$$\frac{[DA, [T, V(X)]\otimes T]\oplus[DA, [T, P_0\otimes DA]\otimes T]}{[T, V(X)]\otimes T\oplus[T, P_0\otimes DA]\otimes T}$$

by L. Then by the torsion free resolution of $\mathcal{S}(X)$ which was obtained at (1)

$$0\to\mathcal{D}\mathcal{S}(X)\to\frac{[DA, \sigma(V(X))]\oplus[DA, \sigma(P_0\otimes DA)]}{\sigma(V(X))\oplus\sigma(P_0\otimes DA)}\to[DT, \mathcal{S}(X)]\to 0$$

is exact, where σ denotes the functor $[T, -]\otimes T$. Now $\sigma\, V(X)\cong V(X)$ and $\sigma(P_0\otimes DA)\cong P_0\otimes DA$ and hence

$$|\mathcal{D}\mathcal{S}(X)|=|[DA, V(X)]|+|P_0|+|V(X)|+|P_0\otimes DA|-|[DT, \mathcal{S}(X)]|.$$

However, we know

$$|[DT, \mathcal{S}(A)]|=|[DA, V(X)]|+|P_0|-|P_1|$$

and $|P_0|-|P_1|=|T(X)|=|V(X)|-|X|$, for the first equality follows from the exactness of

$$0\to[DT, P_1\otimes DT]\to[DT, [T, V(X)]]\oplus[DT, [T, P_0\otimes DA]]$$
$$\to[DT, \mathcal{S}(X)]\to 0.$$

Hence $|\mathcal{D}\mathcal{S}(X)|=|X\oplus P_0\otimes R|$.

Consequently we can prove

$$\mathcal{D}\mathcal{S}(X)\simeq X\oplus\frac{P_0}{P_0\otimes DA}$$

as an R-module provided we can find an R-monomorphism Θ:

$$X\oplus P_0\otimes R\to L_R$$

such that

$$X\oplus P_0\otimes R\xrightarrow{\ \Theta\ }L_R\xrightarrow{\ \Delta\ }[DT, \mathcal{S}(X)]_R$$

is the zero map, where $\Delta=[DT, (x, \hat{y})]$.

Embedding $X \oplus P_0 \otimes R \to L$

Let Θ_1 and Θ_2 be A-homomorphisms defined by

$$\Theta_1 = ([DA, \varepsilon_{V(X)}^{-1}][DA, \alpha_X][DA, \phi]\, \eta_X^{DA}, \varepsilon_{V(X)}^{-1} \alpha_X):$$
$$X \to [DA, [T, V(X)] \otimes T] \oplus [T, V(X)] \otimes T \subset L$$

and

$$\Theta_2 = \begin{pmatrix} 0 & DT, \eta_{T, P_0 \otimes DA}]]\, \eta_{P_0}^{DA} & \varepsilon_{(X)}^{-1} \rho_0 & 0 \\ 0 & 0 & 0 & \varepsilon_{P_0}^{-1} \otimes DA \end{pmatrix}: \quad P_0 \oplus P_0 \otimes DA$$
$$\to [DA, T, (X)] \otimes T] \oplus [DA, [T, P_0 \otimes DA] \otimes T]$$
$$\oplus [T, V(X)] \otimes T \oplus T, P_0 \otimes DA] \oplus T.$$

Hereafter our main purpose is to show that the map (Θ_1, Θ_2): $X \oplus P \oplus P_0 \otimes DA \to L$ is the R-homomorphism Θ what we quoted before.

Each of Θ_1 and Θ_2 is monomorphism as $\varepsilon_{(X)}^{-1} \alpha_X$ and $\varepsilon_{P_0 \otimes DA}^{-1}$ are monomorphisms.

The next Lemma 2.1 is necessary to the proof:

LEMMA 2.1.

$$\begin{array}{ccc} [DT, [T, X]] & \xrightarrow{DT, \eta[T,X]]} & [DT, [T, [T, X] \otimes T]] \\ \downarrow & & \downarrow \\ [DA, X] & \xleftarrow{[DA, \varepsilon_X]} & [DA, [T, X] \otimes T] \end{array}$$

is a commutative diagram.

Proof. This follows immediately from the commutativity of

$$\begin{array}{ccc} [T, X] & \xrightarrow{\eta[T,X]} & [T, [T, X] \otimes T] \\ & \searrow\!\!\!\!= \quad \swarrow_{[T, \varepsilon_X]} & \\ & [T, X] & \end{array}$$

Now Lemma 2.1 induces the commutativity of

$$\begin{array}{ccc} [DT, [T, V(X)]] \otimes DA & \xrightarrow{[DT, \eta_{[T, V(X)]}] \otimes DA} & [DT, [T, T, V(X)] \otimes T]] \otimes DA \\ \| & & \| \\ [DA, V(X)] \otimes DA & \xleftarrow{[DA, \varepsilon_{V(X)}] \otimes DA} & [DA, [T, V(X)] \otimes T] \otimes DA \\ \downarrow{\scriptstyle \varepsilon_{V(X)}^{DA}} & & \downarrow{\scriptstyle \varepsilon_{[T, V(X)] \otimes T}^{DA}} \\ V(X) & \xleftarrow{\varepsilon_{V(X)}} & [T, V(X)] \otimes T. \end{array}$$

Further the last commutative diagram induces again the commutativity of

$$\begin{array}{ccccc}
[DA,X]\otimes DA & \xrightarrow{[DA,\alpha_X]\otimes DA} & [DA,V(X)]\otimes DA & \xrightarrow{[DT,\eta_{[T,V(X)]}]\otimes DA} & [DA,[T,V(X)]\otimes T]\otimes DA \\
\downarrow{\scriptstyle \varepsilon_X^{DA}} & & \downarrow{\scriptstyle \varepsilon_{V(X)}^{DA}} & & \downarrow{\scriptstyle \varepsilon_{[T,V(X)]\otimes T}^{DA}} \\
X & \xrightarrow[\alpha_X]{} & V(X) & \xleftarrow[\varepsilon_{V(X)}]{} & [T,V(X)]\otimes T.
\end{array}$$

Since

$$\varepsilon_X^{DA}[DA,\phi]\otimes DA\,\eta_X^{DA}\otimes DA=\phi,$$

it follows

$$\varepsilon_{V(X)}^{-1}\,\alpha_X\phi=\varepsilon_{[T,V(X)]}^{DA}\otimes T([DA,\varepsilon_{V(X)}^{-1}][DA,\alpha_X][DA,\phi]\,\eta_X^{DA})\otimes DA.$$

This implies that θ_1 is an R-homomorphism.

By Lemma 2.1 we have a commutative diagram

$$\begin{array}{ccc}
[DT,[T,P_0\otimes DA]]\otimes DA & \xrightarrow{[DT,\eta_{[T,P_0\otimes DA]}]\otimes DA} & [DT,[T,[T,P_0\otimes DA]\otimes T]]\otimes DA \\
\| & & \| \\
[DA,P_0\otimes DA]\otimes DA & \xleftarrow{[DA,\varepsilon_{P_0\otimes DA}]\otimes DA} & [DA,[T,P_0\otimes DA]\otimes T]\otimes DA \\
\downarrow{\scriptstyle \varepsilon_{P_0\otimes DA}^{DA}} & & \downarrow{\scriptstyle \varepsilon_{[T,P_0\otimes DA]}^{DA}} \\
P_0\otimes DA & \xleftarrow{\varepsilon_{P_0\otimes DA}} & [T,P_0\otimes DA]\otimes T
\end{array}$$

and similarly we obtain

$$(\varepsilon_{P_0\otimes DA})^{-1}\,\varepsilon_{P_0\otimes DA}^{DA}=\varepsilon_{[T,P_0\otimes DA]}^{DA}\,[DT,\eta_{[T,P_0\otimes DA]}]\otimes DA\,\eta_{P_0}^{DA}\otimes DA.$$

This shows Θ_2 is also an R-homomorphism.

Before proceeding to the proof of

$$\left(X\oplus \underset{P_0\otimes DA}{\overset{P_0}{\leadsto}}\xrightarrow{\ \Theta\ } L\xrightarrow{\ \Delta\ }[DT,\mathscr{S}(X)]\right)=0$$

it is necessary to prove

LEMMA 2.2.

$$\begin{array}{ccc}
X & \xrightarrow{\eta_X^{DA}} & [DA,X\otimes DA] \\
\downarrow{\scriptstyle \eta_X^{DT}} & & \uparrow{\scriptstyle \iota_T[\delta_A^{-1},X\otimes DA]}\ \wr \\
[DT,X\otimes DT] & \xrightarrow[{[DT,[T,X\otimes\delta_A]\eta_{X\otimes DT}]}]{} & [DT,[T,X\otimes DA]]
\end{array}$$

is commutative

Proof. It is obtained by a routine calculation.

LEMMA 2.3. *For a given homomorphism $Z\otimes D \to^{g} Y$,*

$$\begin{array}{ccc}
[DT, Z\otimes DB] & \xrightarrow{[DT,g]} & [DT, Y] \\
\uparrow{\scriptstyle \eta^{DT}_{Z\otimes T}} & & \uparrow{\scriptstyle \varepsilon_{[DT,Y]}} \\
Z\otimes T & & [T,[DT,Y]]\otimes T \\
\downarrow{\scriptstyle \eta^{DB}_{Z}\otimes T} & & \wr\uparrow{\scriptstyle (\iota_{DT}[\delta_B^{-1},Y])\otimes T} \\
[DB, Z\otimes DB]\otimes T & \xrightarrow[{[DB,g]\otimes T}]{} & [DB, Y]\otimes T
\end{array}$$

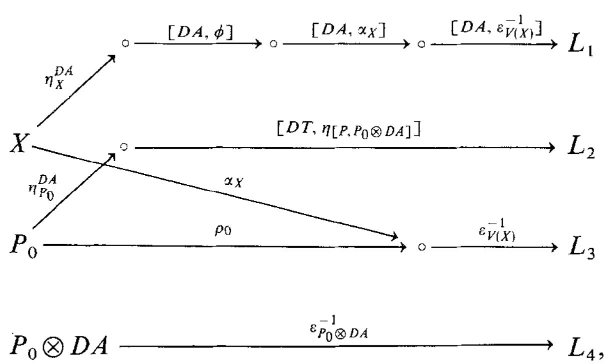
is commutative.

The proof is also a routine calculation.

Now we begin the proof of $(X \to^{\Theta_1} L \to^{\Delta} [DT, \mathscr{S}(X)]) = 0$, that is,

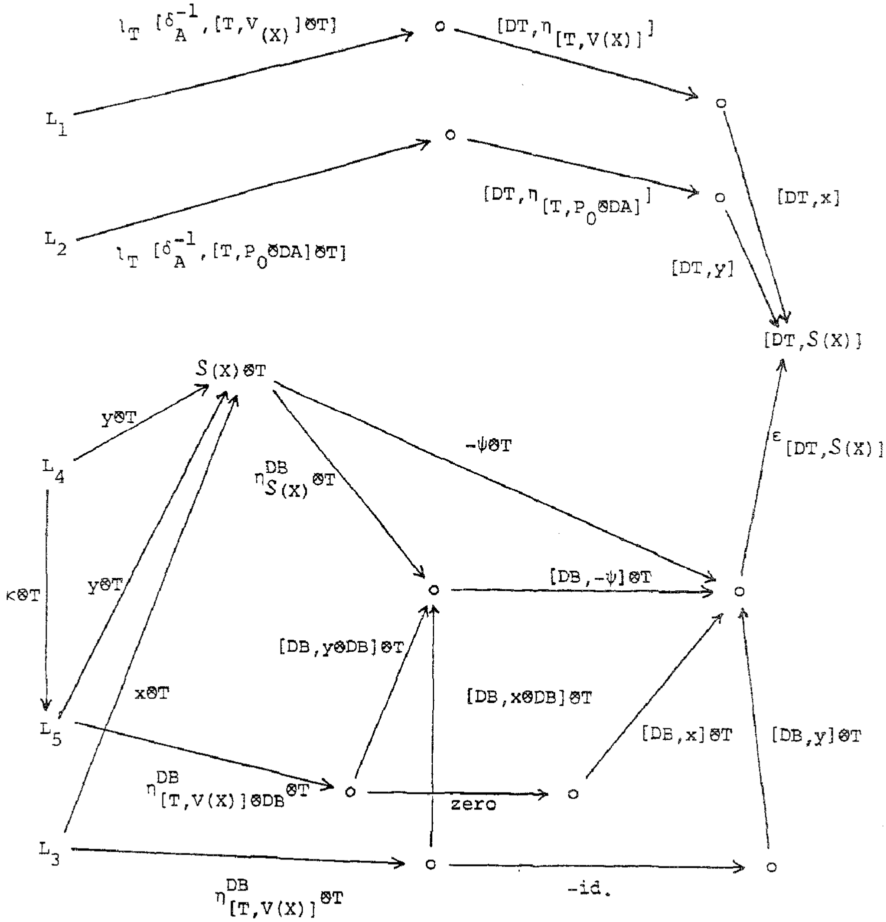
$$\varepsilon_{[DT,\mathscr{S}(X)]} - \psi\otimes T\, x\otimes T\, \varepsilon^{-1}_{V(X)}\, \alpha_X + [DT, x][DT, \eta_{[T,V(X)]}][DA, \varepsilon^{-1}_{V(X)}][DA, \alpha_x][DA, \phi]\, \eta^{DA}_X = 0.$$

At first we introduce the following diagrams (2) and (3) in order to confirm the definitions of $\Theta = (\Theta_1, \Theta_2)$ and Δ and further to see the squares and triangulars for which we need to prove the commutativity



(2)

where L_1, L_2, L_3, and L_4 denote $[DA, [T, V(X)]\otimes T]$, $[DA, [T, P_0\otimes DA]\otimes T]$, $[T, V(X)\otimes T]$ and $[T, P_0\otimes DA]\otimes T$, respectively, and each $\circ$ means the abbreviation of a corresponding module.



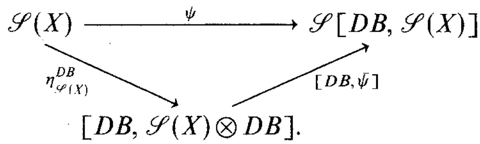
(3)

where L_5 denotes $[T, V(X)] \otimes DB \otimes T$ and we express $\mathscr{S}(X)$ by $(\mathscr{S}(X), \psi\colon \mathscr{S}(X) \to [DB, \mathscr{S}(X)])$, $\bar{\psi}\colon \mathscr{S}(X) \otimes DB \to \mathscr{S}(X)$ is a corresponding map of ψ in the adjoint relation $[\mathscr{S}(X) \otimes_B DB, \mathscr{S}(X)]_B \simeq [\mathscr{S}(X), [DB, \mathscr{S}(X)]_B]_B$. Each $\circ$ also means the abbreviation of a corresponding module.

By the definitions of x, y, and $\mathscr{S}(X)$ we know the commutativity of

$$\begin{array}{ccc} [T, V(X)] \otimes DB & \xrightarrow{x \otimes DB} & \mathscr{S}(X) \otimes DB \\ \big\downarrow \mathrm{id} & & \big\downarrow \bar{\psi} \\ [T, V(X)] \otimes DB & \xrightarrow{\ \ y\ \ } & \mathscr{S}(X) \end{array}$$

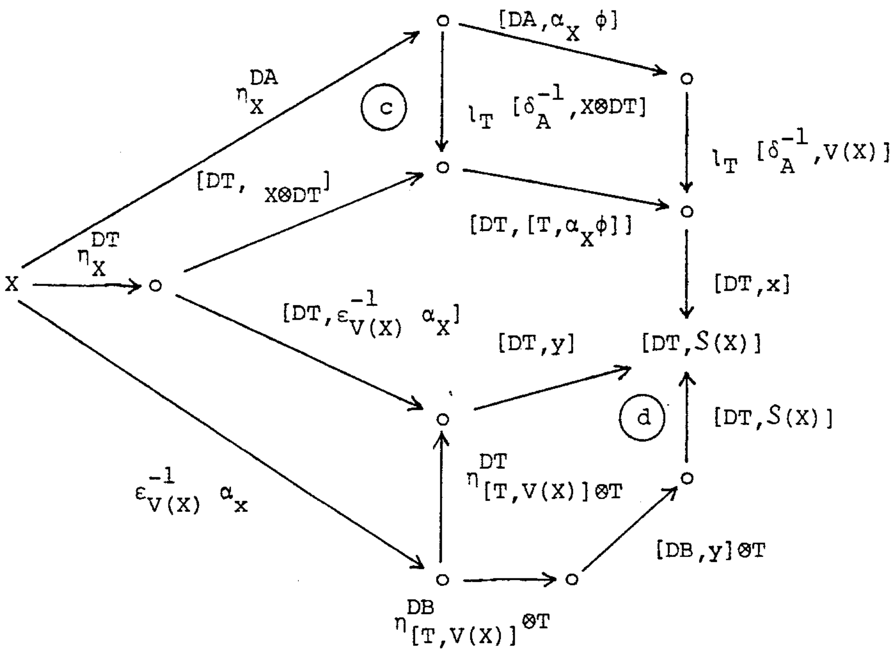
and



Further by Lemma 2.1 we have the commutativity

$$\iota_T[\delta_A^{-1}, V(X)] = [DT, \eta^{-1}_{[T, V(X)]}]\, \iota_T[\delta_A, [T, V(X)] \otimes T][DA, \varepsilon_X^{-1}].$$

Hence taking those commutativities into consideration for the diagrams (2) and (3) we know that it is enough to prove the commutativity of the outer polygon of the following diagram:



By Lemma 2.2 and 2.3, the quadrilateral ⓒ and pentagon ⓓ are commutative and by the naturality other quadrilaterals are commutative. And the inner pentagon is also commutative by the definition of $\mathscr{S}(X)$.

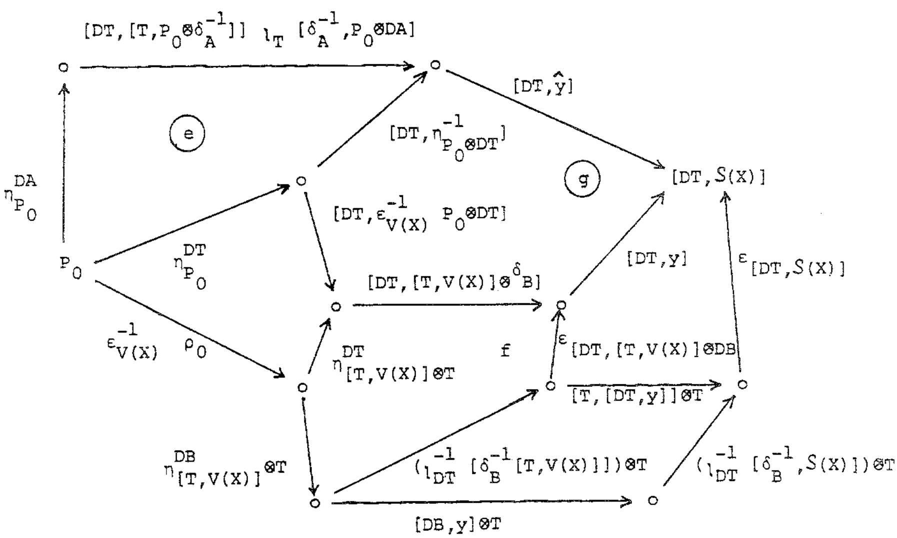
This completes the proof of $(X \to^{\Theta_1} L \to^{\Delta} [DT, \mathscr{S}(X)]) = 0$.

Since

$$\begin{matrix}[T, V(X)] \\ \rightsquigarrow \\ [T, V(X)]\otimes B\end{matrix} \xrightarrow{\binom{x}{y}} \mathscr{S}(X)$$

is an S-homomorphism, $\psi y=0$. Then from the diagrams (2) and (3) it follows that $(P_0\otimes DA \to^{\Theta_2} L \to^{\Delta} [DT, \mathscr{S}(X)])=0$.

Now it remains to prove that $(P_0 \to^{\Theta_2} L \to^{\Delta} [DT, \mathscr{S}(X)])=0$. Looking at diagrams (2) and (3) we know that it is enough to prove the commutativity of the outer polygon of the folowing diagram:



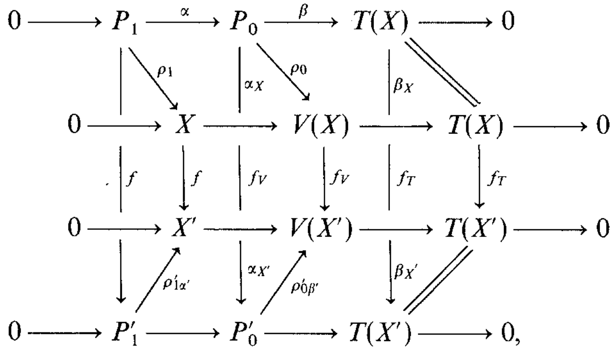
Since $\kappa=[T, V(X)]\otimes\delta_B(\varepsilon_{V(X)}^{-1}\rho_0)\otimes DT\,\eta_{P_0\otimes DT}^{-1}[T, P_0\otimes\delta_A^{-1}]$ the pentagon ⓖ is commutative.

By Lemma 2.2 the quadrilateral ⓔ is also commutative. Further by Lemma 2.3 the pentagon ⓕ is commutative.

Then the naturality of morphism in other squares induces the conclusion that $(P_0 \to^{\Theta_2} L \to^{\Delta} [DT, \mathscr{S}(X)])=0$.

Naturality of $1_{\underline{\text{mod}}\text{-}R} \to \mathscr{Q}\mathscr{S}$

Let X and X' be right A-modules and f and A-homomorphism. Then we have the following commutative diagram



where the exact sequences in the second and the third rows are torsion resolutions of X and X', respectively, and in the first and the fourth rows are projective resolutions of $T(X)$ such that P_0 and P'_0 are the projective covers of $V(X)$ and $V(X')$, respectively.

Then there are $\hat{f}$ and $\hat{f}_V$ such that

$$f\rho_1 = \rho'_1 \hat{f}, \; f_V \rho_0 = \rho'_0 \hat{f}_V$$

and it holds

$$\rho'_0(\hat{f}_V \alpha - \alpha' \hat{f}) = 0$$

because

$$\begin{aligned} \rho'_0 f_V \alpha &= f_v \rho_0 \alpha = f_v \alpha_X \rho_1 = \alpha_{X'} f \rho_1 \\ &= \alpha_{X'} \rho'_1 \hat{f} = \rho'_0 \alpha' \hat{f}. \end{aligned}$$

But by the snake lemma there is $\delta \in \mathrm{Hom}_A(P_1, \mathrm{Ker}\, \rho'_1)$ such that

$$f_v \alpha - \alpha' \hat{f} = \alpha_{X'} \ker \rho'_1 \, \delta$$

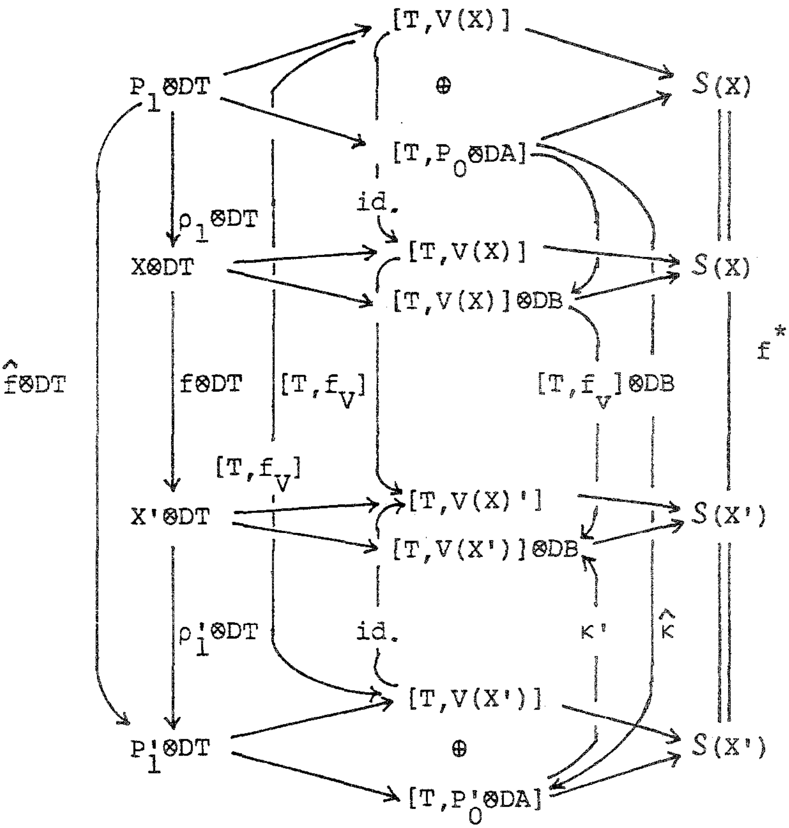
and hence

$$\rho'_1 (f + \ker \rho'_1 \, \delta) = \rho'_1 f = f \rho_1.$$

So we can use $\hat{f} + \ker \rho'_1 \, \delta$ in place of $\hat{f}$. Thus we may assume at the beginning that it holds

$$\rho'_1 \hat{f} = f \rho_1, \qquad \hat{f}_v \alpha_X = \alpha'_{X'} \hat{f}.$$

Then we can check commutativity for each square of the following diagram



where $\hat{\kappa} = (\eta_{P_0 \otimes DT})\hat{f} \otimes DT(\eta_{P_0 \otimes DT})^{-1}$. So by diagram (2) we know that in order to prove the naturality $X \simeq \mathscr{Q}\mathscr{S}(X)$ it is enough to check the commutativity of

$$\begin{array}{ccccc} X & \xrightarrow{[DA,\, \alpha_X \phi]\eta_X^{DA}} & [DA, X] & \xrightarrow{[DA,\, \varepsilon_{V(X)}]^{-1}} & [DA, [T, V(X)] \otimes T] \\ {\scriptstyle f}\downarrow & & & & \downarrow{\scriptstyle [DA, [T, f_V] \otimes T]} \\ X' & \xrightarrow{[DA,\, \alpha_{X'} \phi']\eta_{X'}^{DA}} & [DA, X'] & \xrightarrow{[DA,\, \varepsilon_{V(X')}]^{-1}} & [DA, [T, V(X')] \otimes T] \end{array}$$

and

$$\begin{array}{ccccc} X & \xrightarrow{\alpha_X} & V(X) & \xrightarrow{\varepsilon_{V(X)}^{-1}} & [T, V(X)] \otimes T \\ {\scriptstyle f}\downarrow & & & & \downarrow{\scriptstyle [T, f_V] \otimes T} \\ X' & \xrightarrow{\alpha_{X'}} & V(X') & \xrightarrow{\varepsilon_{V(X')}^{-1}} & [T, V(X')] \otimes T. \end{array}$$

However, it follows from the property of the minimal torsion resolutions and the definition of f_V. This completes the proof of Theorem 1.4.

3. Remarks and Examples

There are several applications of Theorem 1.4 for which we can refer to [13] and [15].

In connection with Nakayama's conjecture on dominant dimension of algebras [10] one of the author [11] proposed a conjecture on self-extensions that for a right module M over a selfinjective algebra R, M is projective if $\mathrm{Ext}^n_R(M, M) = 0$ for all positive integers n. Recently Hoshino [9] proved the conjecture is true for modules over trivial extensions $A \ltimes DA$ of hereditary algebras A.

On the other hand as a consequence of Theorem 1.4 we have

Proposition 3.1. *Assume that the conjecture on self-extensions is true for a trivial extension $A \ltimes DA$ of an algebra A. Then the conjecture is true for a trivial extension $B \ltimes DB$ if there is a chain of algebras $A = A_0, A_1, \ldots, A_t = B$ such that (A_1, T_1, A_0) $(A_2, T_2, A_1), \ldots, (A_t, T_t, A_{t-1})$ are tilting triples.*

Proof. It is enough to prove for the case $t = 1$. As in Theorem 1.4 denote $A \ltimes DA$, $B \ltimes DB$ and a stable equivalence: $\underline{\mathrm{mod}}\text{-}B \ltimes DB \to \underline{\mathrm{mod}}\text{-}A \ltimes DA$ by R, S, and $\mathcal{Q}$, respectively. Then for a nonprojective right S-module M it follows by Theorem 1.4 that $\mathrm{Ext}^{n+1}_S(M, M) \simeq \mathrm{Ext}^1_S(\Omega^n_S M, M) \simeq D\,\underline{\mathrm{Hom}}_S(\Omega^n_S M, \tau_S^{-1} M) \simeq D\,\underline{\mathrm{Hom}}_S(\mathcal{Q}\Omega^n_S M, \mathcal{Q}\tau_S^{-1} M) \simeq D\,\underline{\mathrm{Hom}}_R(\Omega^n_R \mathcal{Q}M, \tau_R^{-1} \mathcal{Q}M) \simeq \mathrm{Ext}^{n+1}_R \mathcal{Q}M, \mathcal{Q}M)$ for $n = 0, 1, 2, \ldots$, because by Auslander–Reiten's result [2] any stable equivalence commutes with loop functors of Heller for symmetric algebras. Now the conclusion is evident.

Now by Hoshino's result we have

Corollary 3.2. *The conjecture on self-extensions is true for a trivial extension $B \ltimes DB$ of an algebra B which is obtained from a hereditary algebra by applying repeatedly tilting processes.*

In order to show some examples, it is necessary to explain our convension concerning the expression of modules. Let k be a field and A an algebra over k defined by a quiver Q and an ideal I of the path algebra kQ, i.e., $A = kQ/I$. We denote by Q_0 and Q_1 the sets of vertices and arrows of the quiver Q respectively. Let e_i be the primitive idempotent of A corresponding to a vertex $i \in Q_0$. A right A-module M is given by attaching vector spaces $M(i)$ to every vertices $i \in Q_0$ and linear maps $M(\alpha)$: $M(i) \to M(j)$ to every arrow $\alpha: j \to i$ of Q_1 such that $M(\alpha)$'s satisfy all relations induced from I.

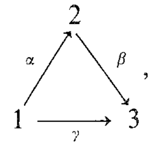
In the case where each vector space $M(i)$ can be decomposed into a direct sum of one-dimensional subspaces $M(i) = \bigoplus_s kv^{(i)}_s$ such that $M(\alpha)(kv^{(i)}_s) = kv^{(j)}_t$ or 0 for each linear map $M(\alpha)$: $M(i) \to M(j)$, we will express the module-structure of M by the following diagram $\Theta(M)$:

(i) The vertices of $\Theta(M)$ are the k-basis $v_s^{(i)}$ in the above decompositions of $M(i)$'s,

(ii) There is an arrow labeled by α from $v_s^{(i)}$ to $v_t^{(j)}$ if and only if $M(\alpha)(kv_s^{(i)})=kv_t^{(j)}$ for a linear map $M(\alpha)$: $M(i)\to M(j)$.

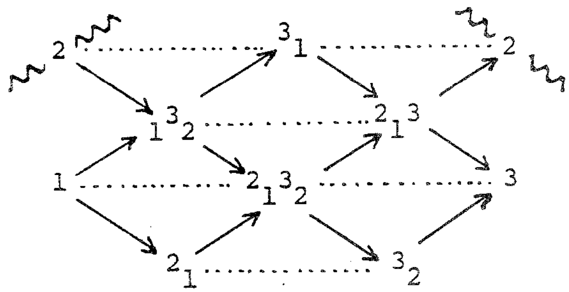
In practice, we simply denote the vertex $v_s^{(i)}$ by i and in the case where there is only one arrow from $v_s^{(i)}$ to $v_t^{(j)}$, we usually omit the arrow and write i over j in order to point out the existence of the arrow from $v_s^{(i)}$ to $v_t^{(j)}$.

EXAMPLE 1. Let A be an algebra defined by the following quiver and relation over k;

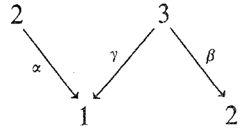
(quiver) 

, (relation) $\beta\alpha=0$.

Then, the Auslander–Reiten quiver Γ_A is the form

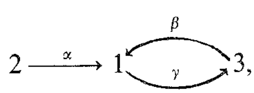


where dotted lines show τ_A-orbits and ripple marks indicate the vertices of Γ_A which should be identified to each other. Especially ${}^{2}{}_{1}{}^{3}2$ denotes an abbreviation of



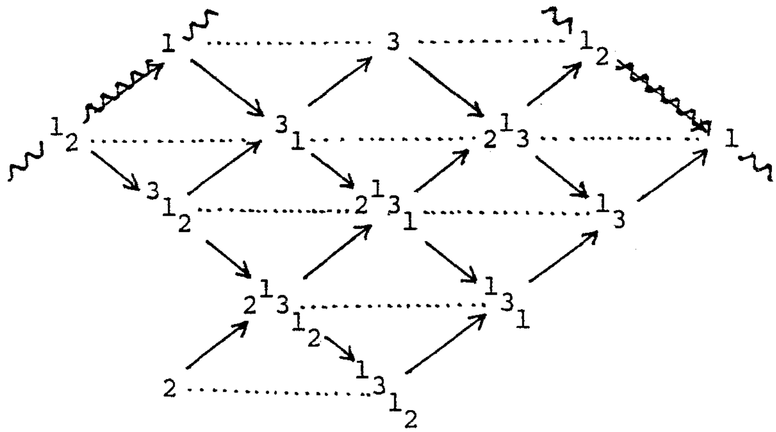
.

EXAMPLE 2. Let B be an algebra defined by the following quiver and relation;

(quiver) 

, (relation) $\beta\gamma=0$.

Then the following is the Auslander–Reiten quiver Γ_B:



For the above algebras and their Auslander–Reiten quivers we now have the following tilting module $T_A = \tau_A^{-1}(e_1 A) \oplus e_2 A \oplus e_3 A = {}^{2}{}_{1}{}^{3}2 \oplus {}^{2}1 \oplus 1^{3}2$ and the tilting triple $(B, {}_B T_A, A)$, where $B \cong \mathrm{End}_A(T)$. Then the distribution charts of $\mathscr{T}$, $\mathscr{F}$, $\mathscr{X}$, and $\mathscr{Y}$ and maps defined by $\mathrm{Hom}_A(T, -)$ and $\mathrm{Ext}_A(T, -)$ as follows:

$$\mathscr{T} = (\Gamma_A)_0 - \{1\}, \qquad \mathscr{F} = \{1\},$$

$$\mathscr{Y} = (\Gamma_B)_0 - \{1, {}^{3}1, 2^{1}3_{1}, {}^{1}3_{1}\}, \ \mathscr{X} = \{1\},$$

and

$\mathrm{Hom}_A(T, -)$

$\mathscr{T}$:	2,	$1^{3}2$,	${}^{2}1$,	${}^{3}1$,	${}^{2}{}_{1}{}^{3}2$,	${}^{2}{}_{1}{}^{3}$,	${}^{3}2$,	3
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\mathscr{Y}$:	${}^{1}2$,	${}^{3}1_{2}$,	2,	3,	$2^{1}3_{1_{2}}$,	$2^{1}3$,	${}^{1}3_{1_{2}}$,	${}^{1}3$

$\mathrm{Ext}_A(T, -)$

$\mathscr{F}$:	1
$\downarrow$	$\downarrow$
$\mathscr{X}$:	1

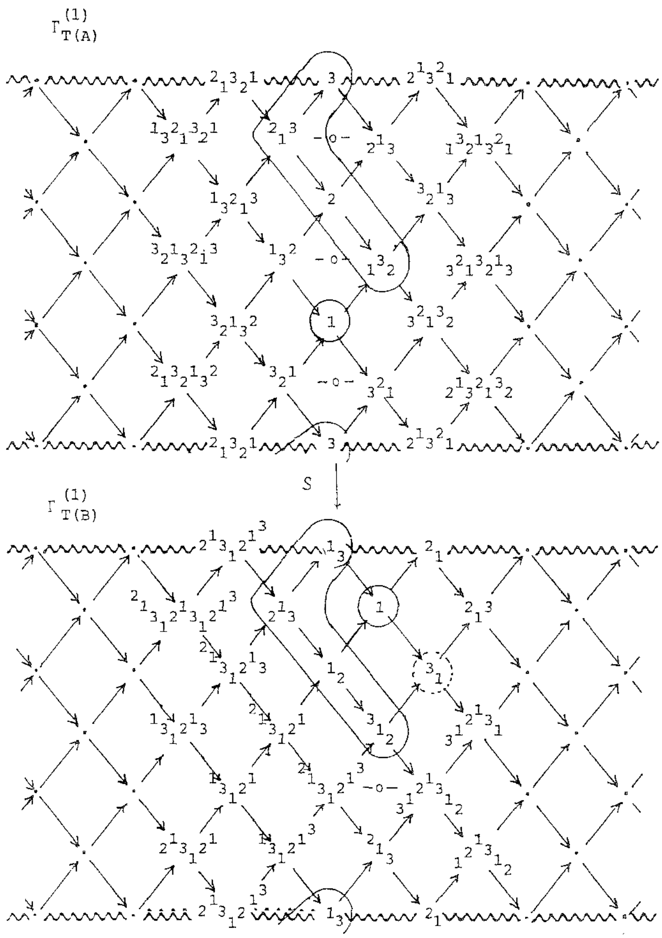


FIGURE 1

Now by Yamagata's theorem [17] the trivial extension $T(B) = B \ltimes DB$ is of infinite representation type since $Q(B)$ contains an oriented cycle $1 \underset{\gamma}{\overset{\beta}{\rightleftarrows}} 3$. Hence so is $T(A) = A \ltimes DA$ and there exist many connected components of $\Gamma_{T(A)}$ and $\Gamma_{T(B)}$. Therefore we show only connected components in which indecomposable A or B-modules appear as their vertices. In Fig. 1 by $\Gamma^{(1)}_{T(A)}$ we denote a connected component of $\Gamma_{T(A)}$ which contains a simple torsion free A-module 1 and we can check $\mathscr{S}(1) \cong \Omega_{T(B)} \operatorname{Ext}^1_A(T, -)$, where $\Omega_{T(B)}$ is the loop space functor of Heller (cf. [15]).

In Fig. 1, (-∘-) indicates the positions where projective $T(A)$- (resp.

$T(B)$-) modules appear. Further subquivers and vertices encircled by closed curves indicate ones consist of torsion or torsion free A- or B-modules. On the other hand subquivers and vertices encircled by dotted closed curves indicate ones consist of B-modules which are neither torsion nor torsion free. The vertical correspondence from the top to the bottom indicates $\mathscr{S}$ and we know that $\mathscr{S}$ preserves the correspondence defined by the tilting functor $\mathrm{Hom}_A(T, -)$.

In Fig. 2 we show other two connected components of $\Gamma_{T(A)}$ and $\Gamma_{T(B)}$ which contain the remaining A and B-modules. Of course it holds that $\underline{\mathrm{Hom}}_{T(A)}(M_1, M_2) \simeq \underline{\mathrm{Hom}}_{T(B)}(\mathscr{S}(M_1), \mathscr{S}(M_2))$ for $M_1, M_2 \in$ mod-$T(A)$. For example we can check that $\underline{\mathrm{Hom}}_{T(A)}({}^{2}3^{1}2, {}^{3}2^{1}3) \simeq \underline{\mathrm{Hom}}_{T(B)}({}^{1}2_{1}, {}^{1}3_{1}) \neq 0$ and $\underline{\mathrm{Hom}}_{T(A)}({}^{3}1, {}^{1}3) \cong \underline{\mathrm{Hom}}_{T(B)}(3, {}^{2}1_{3_{1}}) = 0$, where ${}^{2}3^{1}2, {}^{3}1 \in \Gamma^{(2)}_{T(A)}$ and ${}^{3}2^{1}3, {}^{1}3 \in \Gamma^{(3)}_{T(A)}$.

It is to be noted that to our example all indecomposable projective $T(A)$- and $T(B)$-modules appear in the connected components of Figs. 1 and 2. So we may propose a problem; For a tilting triple $(B, {}_BT_A, A)$ determine

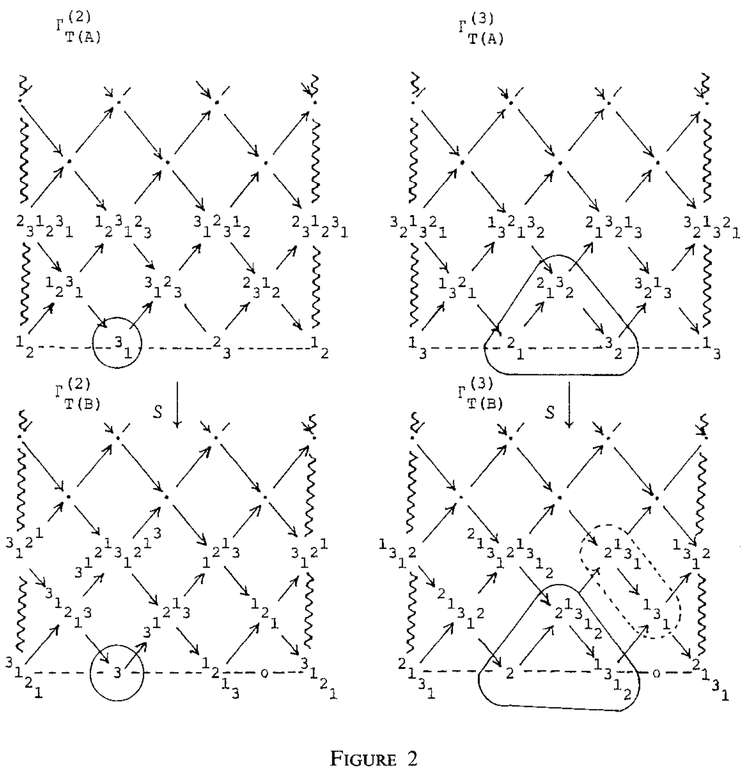


FIGURE 2

all connected components of $\Gamma_{T(A)}$ and $\Gamma_{T(B)}$ such that in each of them at least an indecomposable projective $T(A)$ or $T(B)$-module appears as a vertex.

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