A pair of spaces of upper semi-continuous maps and continuous maps✩

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Abstract

For a Tychonoff space X, we use ↓USC(X) and ↓C(X) to denote the families of the regions below all upper semi-continuous maps and of the regions below all continuous maps from X to I = [0, 1], respectively. In this paper, we consider the spaces ↓USC(X) and ↓C(X) topologized as subspaces of the hyperspace Cld(X × I) consisting of all non-empty closed sets in X × I endowed with the Vietoris topology. We shall prove that ↓USC(X) is homeomorphic (∼) to the Hilbert cube Q = [−1, 1]ω if and only if X is an infinite compact metric space. And we shall prove that (↓USC(X), ↓C(X)) ≈ (Q, c0), where c0 = {(x_n) ∈ Q: lim_n→∞ x_n = 0}, if and only if ↓C(X) ∼ c0 if and only if X is a compact metric space and the set of isolated points is not dense in X.

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1. Introduction and main results

For a Tychonoff space X, the hyperspace Cld(X) is the set consisting of all non-empty closed subsets of X endowed with the Vietoris topology which is generated by the sets of the form

\[ U^- = \{ A \in \text{Cld}(X): A \cap U \neq \emptyset \} \quad \text{and} \]
\[ U^+ = \{ A \in \text{Cld}(X): A \subset U \}, \]

where U is open in X. Thus \{(U_1, U_2, \ldots, U_n): U_i \text{ is open in } X \} is an open base for this topology, where

\[ (U_1, U_2, \ldots, U_n) = \bigcap_{i=1}^{n} U_i^- \cap \left( \bigcup_{i=1}^{n} U_i \right)^+. \]

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It is well known that Cl(X) with this topology is metrizable if and only if X is a compactum (i.e. a compact and metrizable space) [11, Theorem I.3.4]. For a compact metric space X = (X, d), the Vietoris topology of Cl(X) is induced by the Hausdorff metric $d_H$ defined as follows:

$$d_H(A, B) = \inf\{ r > 0 : A \subset B_d(B, r) \text{ and } B \subset B_d(A, r) \},$$

where $B_d(C, r) = \{ x \in X : d(x, c) < r \text{ for some } c \in C \}$ for $C \subset X$ and $r > 0$.

For a Tychonoff space X and a subset L of the set R of all real numbers, we consider the sets $C(X, L)$ and USC(X, L) which consist of all continuous maps and all upper semi-continuous maps from X to L, respectively. The Kadec–Anderson Theorem states that when X is an infinite compactum and $L = R$ the space $C_u(X, L)$ which is the set $C(X, L)$ endowed with the uniform convergence topology is homeomorphic to ($\approx$) $R^\omega \approx I^2$ [9,1]. The so-called Dobrowolski–Marciszewski–Mogilski Theorem [8] (cf. [13, Theorem 6.12.15]) asserts that if X is a nondiscrete, countable metrizable space and $L = R$ or $L = I = [0, 1]$ then the space $C_p(X, L)$, the set $C(X, L)$ with the topology of pointwise convergence, is homeomorphic to the subspace $c_0 = \{ (x_n) \in Q : \lim_{n \to \infty} x_n = 0 \}$ of the Hilbert cube $Q = [-1, 1]^\omega$.

In [18,19], noticing that a map $f \in \text{USC}(X, I)$ is carried by a bijection to its region below $\downarrow f = \{(x, \lambda) \in X \times I : \lambda \leq f(x)\}$ in the product space $X \times I$, we studied the space $\downarrow \text{USC}(X, I) = \{ \downarrow f : f \in \text{USC}(X, I) \}$ and its subspace $\downarrow C(X, I) = \{ \downarrow f : f \in C(X, I) \}$, where they are topologized as the subspaces of the hyperspace Cl(X × I). The family $\downarrow C(X, I)$ with this topology is usually different from the above two spaces $C_u(X, I)$ and $C_p(X, I)$ though there are bijections among them (see [19, Corollary 1]). It was proved in [19] that if X is an infinite locally connected compactum then there exists a homeomorphism $h : \downarrow \text{USC}(X, I) \to Q$ such that $h(\downarrow C(X, I)) \approx c_0$, that is, $(\downarrow \text{USC}(X, I), \downarrow C(X, I)) \approx (Q, c_0)$. But when X is a non-locally-connected compactum, $\downarrow \text{USC}(X, I)$ and especially $\downarrow C(X, I)$ become complicated. In [19], we pointed out that it is not necessary that $\downarrow C(X, I) \approx c_0$. In the following we shall be abbreviated $\downarrow \text{USC}(X, I)$ and $\downarrow C(X, I)$ as $\downarrow \text{USC}(X)$ and $\downarrow C(X)$, respectively. In the present paper, we shall prove the following theorems which generalize the above results.

**Theorem 1.** For an infinite Tychonoff space X, the following conditions are equivalent:

(a) $X$ is a compactum;
(b) $\downarrow C(X)$ is second countable;
(c) $\downarrow \text{USC}(X) \approx Q$.

**Remark 1.** Let us recall the celebrated Curtis–Schori–West Hyperspace Theorem which states that Cl(X) $\approx Q$ if and only if X is a non-degenerate Peano continuum (a connected locally connected compactum is called a Peano continuum) ([4,17]; cf. [12, Theorem 8.4.5]). Noticing that Cl(X) $\oplus$ $\{ 0 \}$ $\approx$ $\downarrow \text{USC}(X, \{ 0, 1 \})$ [19, Introduction], Theorem 1 is a deformed result in which $\{ 0, 1 \}$ is replaced by I.

A subset A of a space Y is called homotopy dense in Y if there exists a homotopy $h : Y \times I \to Y$ such that $h_0 = \text{id}_Y$ and $h_t(Y) \subset A$ for every $t > 0$. This concept is very important in ANR theory and infinite-dimensional topology (see [6,15]). We shall also prove the following theorem:

**Theorem 2.** Let X be an infinite compactum. Then both $\downarrow C(X)$ and $\downarrow \text{USC}(X) \setminus \downarrow C(X)$ are homotopy dense in $\downarrow \text{USC}(X)$.

**Corollary 1.** $\downarrow C(X)$ and $\downarrow \text{USC}(X) \setminus \downarrow C(X)$ are AR’s for each infinite compactum X.

**Proof.** It is well known that, for a metrizable space Y and its homotopy dense space A, Y is an AR if and only if so is A (cf. [15]). Corollary 1 follows from Theorems 1 and 2. □

As the main result we shall prove the following theorem:

**Theorem 3.** For a Tychonoff space X, the following conditions are equivalent:

(a) $X$ is a compactum;
(b) $\downarrow C(X)$ is second countable;
(c) $\downarrow \text{USC}(X) \approx Q$.
(a) $X$ is a compactum and the set of isolated points is not dense in $X$;
(b) $\downarrow C(X) \approx c_0$;
(c) $(\downarrow USC(X), \downarrow C(X)) \approx (Q, c_0)$.

2. Preliminaries

To show our theorems, let us first recall some necessary fundamental concepts and facts. For more information on them, please refer to [12,13].

A metrizable space $X$ is called an absolute retract (abbreviated AR) provided that for every metrizable space $Y$ containing $X$ as a closed subspace there exists a continuous map $r : Y \to X$ such that $r|_X = \text{id}_X$. We say that a space $X$ has the disjoint-cells property provided that for every natural number $n$, every continuous function $f : \mathbb{I}^n \times \{0, 1\} \to X$ can be approximated (arbitrarily closely) by continuous maps sending $\mathbb{I}^n \times \{0\}$ and $\mathbb{I}^n \times \{1\}$ to disjoint sets. A closed subset $A$ of $X$ is said to be a $Z$-set if there exist continuous maps $f : X \to X \setminus A$ arbitrarily close to the identity $\text{id}_X$. It is trivial that every $Z$-set is closed nowhere dense but the converse is not necessarily true. A $Z_\sigma$-set in a space is a countable union of $Z$-sets in the space. A $Z$-embedding is an embedding with a $Z$-set image. We use the following Toruńczyk’s Characterization Theorem to show Theorem 1.

Lemma 1 (Toruńczyk’s Characterization Theorem). (See [16]; cf. [12, Corollary 7.8.4].) A space $X$ is homeomorphic to the Hilbert cube $Q$ if and only if it is a compact AR with the disjoint-cells property.

Let $\mathcal{M}_0$ denote the class of compacta, and for a topological class $\mathcal{C}$ of separable metrizable spaces let $(\mathcal{M}_0, \mathcal{C})$ denote the class of all pairs $(Z, C)$ such that $Z \in \mathcal{M}_0$ and $C \in \mathcal{C}$ with $C \subset Z$.

Definition 1. Let $X$ be a copy of the Hilbert cube $Q$ and $\mathcal{C}$ a topological class of separable metrizable spaces. We say that a subspace $Y$ of $X$ is strongly $\mathcal{C}$-universal in $X$ provided for each $(M, C) \in (\mathcal{M}_0, \mathcal{C})$, every continuous map $f : M \to X$, each closed subset $K$ of $M$ such that $f|_K : K \to X$ is a $Z$-embedding and every $\varepsilon > 0$ there is a $Z$-embedding $g : M \to X$ such that $g|_K = f|_K$, $g^{-1}(Y) \setminus K = C \setminus K$ and $d(g(m), f(m)) < \varepsilon$ for each $m \in M$.

Definition 2. 1 For a class $\mathcal{C}$ of separable metrizable spaces and a copy $X$ of $Q$, we say that a subset $Y$ of $X$ is a $\mathcal{C}$-absorber in $X$ if

1. $Y \in \mathcal{C}$,
2. $Y$ is contained in a $Z_\sigma$-set of $X$, and
3. $Y$ is strongly $\mathcal{C}$-universal in $X$.

The following uniqueness theorem of absorbers is a key to prove our Theorem 3.

Lemma 2. (See [2, Theorem 8.2]; cf. [13, Theorem 5.5.2].) If $X$ and $Y$ are $\mathcal{C}$-absorbers in a copy $M$ of $Q$, then $(M, X) \approx (M, Y)$.

In this paper we are concerned with the class $\mathcal{F}_{\sigma \delta}$ of absolute $F_{\sigma \delta}$ spaces. It was proved in [7] that $c_0 = \{(x_n) \in Q : \lim_{n \to \infty} x_n = 0\}$ is an $\mathcal{F}_{\sigma \delta}$-absorber in $Q$.

In the following, $\mathbb{N}$ denotes the set of all natural numbers. If $(X, d)$ is a compact metric space, then $d((x, \lambda), (y, \mu)) = \max\{d(x, y), |\lambda - \mu|\}$ is an admissible metric on $X \times \mathbb{I}$ and so is $d_H$ on the hyperspace $\text{Cl}(X \times \mathbb{I})$. Put $B_d(a, \varepsilon) = B_d(|a|, \varepsilon)$. For $A \subset X$, $\overline{A}$ and $\text{diam}_d(A) = \sup\{d(x, y) : x, y \in A\}$ denote the closure and the diameter of $A$ in $(X, d)$, respectively.

Let $\phi : A \to B$ be a map from a set $A$ to a set $B$. If $A \subset \text{USC}(X)$ and/or $B \subset \text{USC}(Y)$ for spaces $X$ and $Y$.

We may define a corresponding map $\downarrow \phi : \downarrow A \to \downarrow B$ or $\downarrow \phi : A \to \downarrow B$ or $\downarrow \phi : \downarrow A \to B$ as $\downarrow \phi (\downarrow f) = \downarrow (\phi (f))$ or $\downarrow \phi (\downarrow f) = \phi (f)$, respectively.

1 There are different definitions of absorbers. Our definition is due to [2] and [13, p. 346]. In [3], the condition (a) in our definition is replaced by $Y \in \sigma \mathcal{C}$, where $\sigma \mathcal{C}$ is the class of spaces written as a countable union of closed subspaces which belong to $\mathcal{C}$.
For a space $X$ and $A \subseteq \text{USC}(X)$, define a map $\bigvee A : X \to I$ by $\bigvee A(x) = \sup \{ f(x) : f \in A \}$. If $A = \{ f, g \}$, we replace $\bigvee A$ by $f \vee g$. It is trivial that $f \vee g \in \text{USC}(X)(C(X))$ if $f, g \in \text{USC}(X)(C(X))$, respectively.

At last, we give the following lemma which is well known (see [10, Corollary 4.8]).

**Lemma 3.** Let $Y$ be a metric space and $X$ a compactum. Then a map $f : Y \to \text{Cld}(X)$ is continuous if and only if, for every sequence $(y_n)$ with the limit $y$ in $Y$, the following conditions hold:

1. If $a_n \in f(y_n)$ and $\lim_{n \to \infty} a_n = a$, then $a \in f(y)$, and
2. for every $a \in f(y)$, there exists a sequence $(a_n)$ in $X$ such that $a_n \in f(y_n)$ for each $n$ and $\lim_{n \to \infty} a_n = a$.

**3. Proofs of Theorems 1 and 2**

**Lemma 4.** For a Tychonoff space $X$, if $\downarrow C(X)$ is second countable, then $X$ is a compactum.

**Proof.** Step 1. We shall show that $X$ is second countable and hence it is metrizable. Suppose $\{ \{ U^n_1, U^n_2, \ldots, U^n_m(n) \} \} \cap \downarrow \text{C}(X) : n = 1, 2, \ldots \}$ is a countable open base for $\downarrow \text{C}(X)$ and $B$ is a countable open base for $I$. For every $n$, every $i \leq m(n)$ and $B \in B$, let

$$V(n, i, B) = \{ x \in X : H \times B \subseteq U^n_i \text{ for some open set } H \ni x \text{ in } I \}.$$

Then $V(n, i, B)$ is open in $X$ and $V(n, i, B) \times B \subseteq U^n_i$. Now let $\mathcal{C}$ be the family of all finite intersections of sets of the form $V(n, i, B)$. Then $\mathcal{C}$ is a countable open base for $X$. Trivially, it is a countable family of open sets in $X$. To show that $\mathcal{C}$ is an open base for $X$, take an open set $V$ in $X$ and $x \in V$. There exists $f \in \text{C}(X)$ such that $f(x) = 1$ and $f(y) = 0$ for all $y \in X \setminus V$. Let $U_1 = X \times \{ 0, \frac{1}{2} \}$ and $U_2 = V \times \{ 0, 1 \}$. Then $f \in \langle U_1, U_2 \rangle \cap \downarrow \text{C}(X)$. Thus there exists $n$ such that

$$\downarrow f \subseteq \langle U^n_1, U^n_2, \ldots, U^n_m(n) \rangle \cap \downarrow \text{C}(X) \subseteq \langle U_1, U_2 \rangle \cap \downarrow \text{C}(X).$$

Then for every $t \in I$ there exists $i(t) \leq m(n)$ such that $(x, t) \in U^n_{i(t)}$. It follows that there exist $B_t \subseteq B$ and an open set $H$ in $X$ such that $(x, t) \in H \times B_t \subseteq U^n_{i(t)}$. Choose a finite subcover $\{ B_t : j = 1, 2, \ldots, l \}$ of the open cover $\{ B_t : t \in I \}$ of $I$ and let $G = \bigcap_{j=1}^l V(n, i(t_j), B_{t_j})$ for each $z \in X \setminus G$. Let $h = f \vee g \in C(X)$. Then $\downarrow h \neq \langle U_1, U_2 \rangle$. On the other hand, $G \times I = \bigcap_{j=1}^l V(n, i(t_j), B_{t_j}) \times (U^n_{i(t_j)} \setminus B_{t_j}) \subseteq \bigcup_{j=1}^l U^n_{i(t_j)} \setminus B_{t_j}$. It follows that $\downarrow h = \downarrow f \cup \downarrow g \subseteq \downarrow f \cup \downarrow (G \times I) \subseteq \bigcup_{j=1}^l U^n_{i(t_j)} \setminus B_{t_j}$. Thus, $\downarrow h \subseteq \langle U^n_1, U^n_2, \ldots, U^n_m(n) \rangle \cap \downarrow \text{C}(X)$ since $\downarrow h \supset \downarrow f$ and $\downarrow f \cap U^n_i \neq \emptyset$ for every $i \leq m(n)$. A contradiction occurs.

**Step 2.** We shall show that $X$ is compact. Otherwise, since $X$ is metrizable, there exists a countable infinite discrete set $\mathcal{C} = \{ x_n : n = 1, 2, \ldots \}$ in $X$. Let $0 \in \text{C}(X)$ be the map with the constant value $0$ and $\{ \{ U^n_1, U^n_2, \ldots, U^n_m(n) \} \} \cap \downarrow \text{C}(X) : n = 1, 2, \ldots \}$ a countable neighborhood base at $0$ in $\downarrow \text{C}(X)$. For every $n$, choose open sets $G_n, H_n$ in $X$ and $t_n \in (0, 1]$ such that $x_n \in G_n \subseteq \overline{G_n} \subseteq H_n$, the family $\{ H_n : n = 1, 2, \ldots \}$ is discrete and $H_n \times [0, t_n] \subseteq U^n_i$ for some $i \leq m(n)$. There exists a continuous map $f_n : X \to [0, t_n]$ such that $f_n(x_n) = t_n$ and $f_n(x) = 0$ for every $x \in X \setminus G_n$.

Let

$$U = \bigcup_{n=1}^{\infty} \left( H_n \times [0, t_n) \right) \cup \left( X \setminus \bigcup_{n=1}^{\infty} \overline{G_n} \right) \times [0, 1) \bigcup [0, 1].$$

Then $\langle U \rangle$ is a neighborhood of $0$ but $\downarrow f_n \subseteq \downarrow \text{C}(X) \cap (\langle U^n_1, U^n_2, \ldots, U^n_m(n) \rangle \setminus \langle U \rangle)$ for every $n$. This contradicts that $\{ \{ U^n_1, U^n_2, \ldots, U^n_m(n) \} \} \cap \downarrow \text{C}(X) : n = 1, 2, \ldots \}$ is a neighborhood base at $0$ in $\downarrow \text{C}(X)$. \[\Box\]

**Lemma 5.** For every compactum $X$, $\downarrow \text{USC}(X)$ is a retract of $\text{Cld}(X \times I)$ and for every subfamily $A \subseteq \downarrow \text{USC}(X)$, we have that $\bigcup A \in \downarrow \text{USC}(X)$ if $\bigcup A \in \text{Cld}(X \times I)$.

**Proof.** For every $A \subseteq \text{Cld}(X \times I)$, define $\phi(A) : X \to I$ by

$$\phi(A)(x) = \sup \{ t \in I : (x, t) \in A \}.$$
Regarding that sup θ = 0 we have that φ(A) is well-defined. We shall now prove that φ(A) is upper semi-continuous. For every t₀ ∈ I and x₀ ∈ φ(A)^⁻¹(0, t₀), we have sup{t ∈ I: (x₀, t) ∈ A} < t₀. Then ((x₀) × [t₀, 0]) ∩ A = φ(A) and t₀ > 0. It follows from the closedness of A that (U × (t₁, 1)) ∩ A = φ(A) for some open set U ⊃ x₀ in X and some t₁ ∈ (0, t₀). Thus φ(A)(y) < t₀ for every y ∈ U. We have that φ(A) ∈ USC(X). Moreover, it is easy to verify that d_H(φ(A), φ(B)) ≤ d_H(A, B) for every pair A, B ∈ Cl(X × I). That shows that A → φ(A) is continuous. Trivially, A = φ(A) if A ∈ ↓ USC(X). We have proved that the first statement holds.

From the first statement it follows that ↓ USC(X) is compact. Thus, for every directed subfamily A ⊂ ↓ USC(X) with respect to inclusion relation, we have that ∪ A is the limit of the net A ⊃ A → A ∈ Cl(X × I) if ∪ A ∈ Cl(X × I). Hence ∪ A ∈ ↓ USC(X) if A ⊂ ↓ USC(X) is directed and ∪ A ∈ Cl(X × I). Moreover, we note that ↓ USC(X) is closed with respect to finite union. It follows that the second statement holds also. ■

By Lemma 5, we have the following important lemma.

**Lemma 6.** For every compactum X, ↓ USC(X) is a compact AR.

**Proof.** We at first prove that ↓ USC(X) is a Peano continuum. From Lemma 5 it follows that ↓ USC(X) is a compactum. To show the connectedness and the local connectedness of ↓ USC(X), for each f, g ∈ USC(X), let θ : I → USC(X) be a map defined by

θ(t)(x) = \begin{cases} 
(1 - 2t)f(x) + 2t(f ∨ g)(x), & \text{if } 0 ≤ t ≤ \frac{1}{2}; \\
(2 - 2t)(f ∨ g)(x) + (2t - 1)g(x), & \text{if } \frac{1}{2} ≤ t ≤ 1.
\end{cases}

Trivially, θ(t) ∈ USC(X) for each t ∈ I and θ(0) = f, θ(1) = g. Moreover, it is not difficult to verify that ↓ θ : I → ↓ USC(X) is continuous. To complete the proof of the fact that ↓ USC(X) is a Peano continuum, it suffices to show that d_H(↓ f, ↓ θ(t)) ≤ d_H(↓ f, ↓ g) for every t ∈ I. For each (x, λ) ∈ ↓ f and each t ∈ I, there exists (y, μ) ∈ ↓ g such that d((x, λ), (y, μ)) ≤ d_H(↓ f, ↓ g).

**Case A.** g(x) ≥ f(x). Then θ(t)(x) ≥ f(x). It follows that (x, λ) ∈ ↓ θ(t). Thus d((x, λ), ↓ θ(t)) = 0.

**Case B.** g(x) ≤ f(x). Then (x, λ) ∈ ↓ θ(t) if t ∈ [0, \frac{1}{2}]. And, if t ∈ [\frac{1}{2}, 1], then θ(t)(y) ≥ g(y) and hence (y, μ) ∈ ↓ θ(t). It follows that d((x, λ), ↓ θ(t)) ≤ d_H(↓ f, ↓ g) for each t ∈ I.

On the other hand, it is easy to see that ↓ θ(t) ⊂ ↓ θ(\frac{1}{2}) ⊂ B_d(↓ f, d_H(↓ f, ↓ g) + ε) for every t ∈ I and ε > 0. Thus d_H(↓ f, ↓ θ(t)) ≤ d_H(↓ f, ↓ g) for each t ∈ I.

By the above fact and the Wojdysławski Theorem [17], Cl(↓ USC(X)) is an AR. From Lemma 5 and [12, Proposition 5.3.6] it follows that ∪ : Cl(↓ USC(X)) → ↓ USC(X) is a retraction. Thus ↓ USC(X) is a compact AR. ■

The following lemma is a key to show Theorem 2.

**Lemma 7.** Suppose (X, d) is an infinite compact metric space, A = C(X) or A = USC(X) \ C(X). For every map f : K(0) → A from the 0-skeleton K(0) of a locally finite simplicial complex K to A, there exists an extension h : |K| → A such that ↓ h : |K| → ↓ A is continuous and

\[ \text{diam}_{d_H} ↓ h(σ) ≤ 6 \text{diam}_{d_H} ↓ f(σ(0)) \]

for every σ ∈ K.

**Proof.** Choose a non-isolated point x₀ in X. For σ, τ ∈ K, τ < σ means that τ is a face of σ. Take the barycentric subdivision Sd(K) of K. The barycenter of σ ∈ K is denoted by b(σ). Now we fix σ ∈ K. For every τ < σ, define h_σ(b(τ)) : X → I as follows:

In the case A = C(X), let

\[ h_σ(b(τ)) = \sqrt{\{ f(v) : v ∈ τ(0) \}}. \]

In the case A = USC(X) \ C(X), let

\[ h_σ(b(τ))(x) = \begin{cases} 
(1 - d(τ)) \cdot \sup \{ f(v)(x) : v ∈ τ(0) \}, & \text{if } x ≠ x₀;

d(τ) + (1 - d(τ)) \cdot \sup \{ f(v)(x) : v ∈ τ(0) \}, & \text{if } x = x₀.
\end{cases} \]
where $d(\tau) = \min\{\text{diam}_{d_H} \downarrow f(\tau(0)), 1\}$.

It is trivial to verify that $h_\sigma(b(\tau)) \in \text{USC}(X)$. Moreover, it is in $A$. In fact, this fact is trivial in the case $A = \text{C}(X)$. We check that it is also true in another case. If $d(\tau) \neq 0$, it follows from the definition of $h_\sigma(b(\tau))$ that this map is not continuous at $x_0$. If $d(\tau) = 0$, then $f(v) = f(v')$ for any $v, v' \in \tau(0)$. It follows that $h_\sigma(b(\tau)) = f(v)$ is not continuous.

For every $z \in \sigma$, there exists a set $\{s_\tau(z): \tau \prec \sigma\}$ (the barycentric coordinates of $z$ in $\text{Sd}(\sigma)$) of real numbers such that

1. $s_\tau(z) \geq 0$ and $\sum_{\tau \prec \sigma} s_\tau(z) = 1$;
2. $z = \sum_{\tau \prec \sigma} s_\tau(z) b(\tau)$.

Let

$$h_\sigma(z) = \sum_{\tau \prec \sigma} s_\tau(z) h_\sigma(b(\tau)).$$

Trivially, $h_\sigma(z) \in A$ for every $z \in \sigma$. It is not hard to verify that, if $\lim_{n \to \infty} z_n = z$ in $\sigma$, then the sequence $(h_\sigma(z_n))$ of functions converges uniformly to the function $h_\sigma(z)$ in $X$. It follows that $(\downarrow h_\sigma(z_n))$ converges to $\downarrow h_\sigma(z)$ in $\downarrow \text{USC}(X)$. Thus $\downarrow h_\sigma: \sigma \to \downarrow A$ is continuous.

Trivially, $h_\sigma|\tau = h_\tau$ if $\tau \prec \sigma$ for $\sigma, \tau \in K$. Thus

$$h = \bigcup \{h_\sigma: \sigma \in K\}: |K| \to A$$

is well-defined and $\downarrow h: |K| \to \downarrow A$ is continuous. By the definitions of $h_\sigma$, it is easy to see that $h(v) = h(v) = f(v)$ for every $v \in \tau(0)$ in both $A = \text{C}(X)$ and $A = \text{USC}(X) \setminus \text{C}(X)$. It follows that $h$ is an extension of $f$.

It is remainder to show that $\text{diam}_{d_H} (\downarrow h_\sigma(z) \downarrow \sigma) \leq 6 \text{diam}_{d_H} (\downarrow f(\sigma(0)))$ for every $\sigma \in K$. To this end, suppose $z \in \sigma$. There exists an $m$-dimensional simplex $\sigma_1$ in $\text{Sd}(\sigma)$, where $m = \dim \sigma$, such that $z \in \sigma_1$. Suppose $\{v_1\} = \sigma(0) \cap \sigma_1(0)$. Then we have $(1 - d(\sigma)) f(v_1) \leq h_\sigma(b(\tau))$ for every $\tau \prec \sigma$ with $s_\tau(z) > 0$. It follows that

$$(1 - d(\sigma)) f(v_1) \leq h_\sigma(z).$$ (1)

On the other hand, $h_\sigma(b(\tau)) \leq d(\sigma) + \sqrt{\{f(v): \tau \in \sigma(0)\}}$ for every $\tau \prec \sigma$. It implies that $h_\sigma(z) \leq d(\sigma) + \sqrt{\{f(v): \tau \in \sigma(0)\}}$. (2)

Trivially, if $z = b(\sigma)$, then (1) and (2) also hold. Thus,

$$d_H (\downarrow h_\sigma(z), \downarrow h_\sigma(b(\sigma))) \leq d_H (\downarrow (1 - d(\sigma)) f(v_1), \downarrow (d(\sigma) + \sqrt{\{f(v): \tau \in \sigma(0)\}})).$$

Noting $\downarrow \sqrt{\{f(v): \tau \in \sigma(0)\}} = \bigcup\{\downarrow f(v): \tau \in \sigma(0)\}$ and $d(\sigma) \leq \text{diam}_{d_H} \downarrow f(\sigma(0))$, it is not hard to verify that

$$d_H (\downarrow (1 - d(\sigma)) f(v_1), \downarrow (d(\sigma) + \sqrt{\{f(v): \tau \in \sigma(0)\}})) \leq 3 \text{diam}_{d_H} \downarrow f(\sigma(0)).$$

Thus

$$d_H (\downarrow h_\sigma(z), \downarrow h_\sigma(b(\sigma))) \leq 3 \text{diam}_{d_H} \downarrow f(\sigma(0))$$

for every $z \in \sigma$. It follows that

$$\text{diam}_{d_H} \downarrow h_\sigma(\sigma) \leq 6 \text{diam}_{d_H} \downarrow f(\sigma(0)).$$

We now are in a position to prove Theorems 1, 2. At first, we show Theorem 2.

**Proof of Theorem 2.** 2 Trivially, both $\downarrow \text{C}(X)$ and $\downarrow \text{USC}(X) \setminus \downarrow \text{C}(X)$ are dense in $\downarrow \text{USC}(X)$ if $X$ is an infinite compactum. It follows from Lemmas 6, 7 and a result in [15] that $\downarrow \text{C}(X)$ and $\downarrow \text{USC}(X) \setminus \downarrow \text{C}(X)$ are homotopy dense in $\downarrow \text{USC}(X)$. □

---

2 The referee pointed out that by using [10, Theorem 5.1], the fact that $\downarrow \text{C}(X)$ is homotopy dense in $\downarrow \text{USC}(X)$ has a simple proof.
Proof of Theorem 1. (a) ⇒ (c): We assume that \((X, d)\) is an infinite compact metric space. By Lemma 6, \(\downarrow\text{USC}(X)\) is a compact AR. It follows directly from Theorem 2 that \(\downarrow\text{USC}(X)\) has the disjoint-cells property. Thus, Lemma 1 implies that (c) holds.

(c) ⇒ (b): This is trivial.

(b) ⇒ (a): It follows from Lemma 4. □

4. Proof of Theorem 3

Lemma 8. Let \(Y\) be a metric space and \(a, b : Y \rightarrow \mathbb{I}\) two continuous maps with \(a(y) < b(y)\) for each \(y \in Y\). And let \(M : Y \times \mathbb{I} \rightarrow \mathbb{I}\) be a map satisfying the following conditions:

1. \(M(y_0, t) : \mathbb{I} \rightarrow \mathbb{I}\) is increasing for each fixed \(y_0 \in Y\), and;
2. \(M(y, t_0) : Y \rightarrow \mathbb{I}\) is continuous for every fixed \(t_0 \in \mathbb{I}\).

Then \(s : Y \rightarrow \mathbb{I}\) defined by

\[
s(y) = \frac{1}{b(y) - a(y)} \int_{a(y)}^{b(y)} M(y, t) \, dt
\]

is continuous and \(M(y, a(y)) \leq s(y) \leq M(y, b(y))\) for every \(y \in Y\).

Proof. The continuity of \(s\) follows from Lebesgue’s Dominated Convergence Theorem (cf. [14, 1.34]) and the inequality is trivial. □

Lemma 9. For each compactum \(Y\) and its \(F_{\sigma\delta}\)-set \(C\), there exists a Z-embedding \(g : Y \rightarrow Q_u = \prod_{n=1}^{\infty} [0, 1]\) such that \(g^{-1}(c_1) = C\), where \(c_1 = \{(x_n) \in Q_u : \lim_{n \rightarrow \infty} x_n = 1\}\).

Proof. It is a reform of [19, Lemma 2]. □

Proposition 1. For each infinite compact metric space \((X, d)\), \(\downarrow C(X)\) is strongly \(F_{\sigma\delta}\)-universal in \(\cup\text{USC}(X) \approx Q\).

Proof. Choose a non-isolated point \(x_{\infty} \in X\). Without loss of generality, we may assume that there exists a sequence \((x_n)_{n=0}^{\infty}\) in \(X\) such that \(d(x_n, x_{\infty}) = 2^{-n}\) for every \(n\) and \(d(x_n, x_{\infty}) \leq 1\) for each \(x \in X\).

Let \(C, K\) be an \(F_{\sigma\delta}\)-subset and a closed subset of a compactum \(Y\), respectively. And let \(\Phi : Y \rightarrow \text{USC}(X)\) be a map such that \(\downarrow \Phi : Y \rightarrow \cup\text{USC}(X)\) is continuous and \(\downarrow \Phi |_K : K \rightarrow \cup\text{USC}(X)\) is a Z-embedding. By [5, Lemma 1.1] and Theorem 1, without loss of generality, we may assume that \(\downarrow \Phi(K) \cap \downarrow \Phi(Y \setminus K) = \emptyset\). For every \(\epsilon \in (0, 1)\), we shall define a map \(\Psi : Y \rightarrow \text{USC}(X)\) such that \(\downarrow \Psi : Y \rightarrow \cup\text{USC}(X)\) is a Z-embedding, \(\Psi|_K = \Phi|_K\), \(\Psi^{-1}(C(X)) \setminus K = C \setminus K\) and \(d_H(\downarrow \Psi(y), \downarrow \Phi(y)) < \epsilon\) for each \(y \in Y\), which shows that Proposition 1 holds.

Let \(\delta : Y \rightarrow [0, 1]\) be a map defined by

\[
\delta(y) = \frac{1}{3} \min \{\epsilon, d_H(\downarrow \Phi(y), \downarrow \Phi(K))\}.
\]

Then \(\delta\) is continuous and \(\delta(y) = 0\) if and only if \(y \in K\).

It follows from [13, Proposition 4.1.7] and Theorem 2 that there exists a homotopy \(H : \cup\text{USC}(X) \times \mathbb{I} \rightarrow \cup\text{USC}(X)\) such that

\[
H_0 = \text{id}_{\cup\text{USC}(X)}, \quad H_t(\downarrow \text{USC}(X)) \subset \downarrow C(X) \quad \text{and} \quad d_H(H_t(\downarrow f), \downarrow f) \leq t
\]

for each \(f \in \text{USC}(X)\) and each \(t \in (0, 1]\). For each \(y \in Y\) and \(t \in [0, 1]\), let

\[
\downarrow h(y) = H(\downarrow \Phi(y), \delta(y)), \quad \text{and} \quad M(y, t) = \sup \{h(y)(x) : d(x, x_{\infty}) < t\}.
\]
Then \( h(y) \in C(X) \) for each \( y \in Y \setminus K \) and \( \downarrow h |_{Y \setminus K} : Y \setminus K \to \downarrow C(X) \) is continuous. Moreover, \( d_H(\downarrow h(y), \downarrow \Phi(y)) \leq \delta(y) \) for every \( y \in Y \). It follows from the continuities of \( \delta \) and \( H \) that \( M : (Y \setminus K) \times I \to I \) satisfies the conditions (a) and (b) in Lemma 8. Thus

\[
s(y) = \frac{1}{\delta(y)} \int_{\delta(y)}^{2\delta(y)} M(y, t) \, dt
\]

is continuous on \( Y \setminus K \) and \( M(y, \delta(y)) \leq s(y) \leq M(y, 2\delta(y)) \) for every \( y \in Y \setminus K \). Let \( g : Y \to Q_n \) be a map satisfying the conditions in Lemma 9.

For a fixed \( k = 1, 2, \ldots \), let \( C_k = \{ y \in Y : 2^{-k} \leq \delta(y) \leq 2^{-k+1} \} \). Then \( \bigcup_{k=1}^{\infty} C_k = Y \setminus K \). For every \( m \in \mathbb{N} \), let \( S_m = \{ x \in X : 2^{-m} \leq d(x, x_\infty) \leq 2^{-m+1} \} \). Then \( x_{m-1}, x_m \in S_m \). Thus \( S_m \cap S_{m'} \neq \emptyset \) if and only if \( |m - m'| \leq 1 \). And \( \bigcup_{m=1}^{\infty} S_m = X \setminus \{ x_\infty \} \). Define continuous maps \( \varphi : C_k \to I \) by \( \varphi(y) = 2 - 2^k \delta(y) \) and \( \phi_m : S_m \to I \) by \( \phi_m(x) = 2^m(d(x, x_\infty) - 2^{-m}) \) for each \( m \in \mathbb{N} \). Then \( \phi_m(x_m) = 0 \) and \( \phi_m(x_{m-1}) = 1 \) for every \( m \geq 2 \). Now we define a sequence \( \{ f_m : C_k \to C(X) \}_{m} \) as follows:

\[
\begin{align*}
  f_1(y)(x) &= \delta(y)h(y)(x), \\
  f_2(y)(x) &= (1 - \varphi(y))s(y) + \varphi(y)h(y)(x), \\
  f_3(y)(x) &= h(y)(x)\varphi(y), \\
  f_4(y)(x) &= s(y), \\
  f_5(y)(x) &= 0, \\
  f_6(y)(x) &= (1 - \varphi(y))\delta(y) + \varphi(y)s(y), \\
  f_7(y)(x) &= (1 - \varphi(y))\delta(y)g(y)(1), \\
  f_m(y)(x) &= \delta(y), \quad \text{if } m \text{ is even and } m \geq 8, \\
  f_m(y)(x) &= \delta(y) \left( (1 - \varphi(y))g(y) \left( \frac{m + 1}{2} - 3 \right) + \varphi(y)g(y) \left( \frac{m + 1}{2} - 4 \right) \right), \quad \text{if } m \text{ is odd and } m \geq 9.
\end{align*}
\]

Then, \( X \times C_k \ni (x, y) \mapsto f_m(y)(x) \in I \) is obviously continuous for every \( m \geq 4 \). Using the above maps we define a map \( \Psi_k : C_k \to \text{USC}(X) \) as follows:

\[
\Psi_k(y)(x) = \begin{cases} 
  f_1(y)(x) = h(y)(x), & \text{if } x \in \bigcup_{i=1}^{2^k} S_i, \\
  \phi_{2k+i}(x)f_1(y)(x) + (1 - \phi_{2k+i}(x))f_i+1(y)(x), & \text{if } x \in S_{2k+i}, \\
  \delta(y), & \text{if } x = x_\infty.
\end{cases}
\]

**Fact 1.** For every \( y \in C_k \), \( \Psi_k(y) \) is well-defined and continuous on \( X \setminus \{ x_\infty \} \). And it is upper semi-continuous at \( x_\infty \). Moreover, it is continuous at \( x_\infty \) if and only if \( \lim_{n \to \infty} g(y)(n) = 1 \) if and only if \( y \in C \). Therefore, for every \( y \in C_k \), \( \Psi_k(y) \in C(X) \) if and only if \( y \in C \).

Trivially, we only need to verify that \( \Psi_k(y) \) is continuous at \( x_\infty \) if and only if \( \lim_{n \to \infty} g(y)(n) = 1 \). In fact, \( \Psi_k(y) \) is continuous at \( x_\infty \) if and only if \( \lim_{x \to x_\infty} \Psi_k(y)(x) = \Psi_k(y)(x_\infty) = \delta(y) \). It follows from \( \lim_{y \to x_\infty} x_{2k+i} = x_\infty \) that \( \lim_{y \to x_\infty} f_{i+1}(y)(x_{2k+i}) = \lim_{y \to x_\infty} \Psi_k(y)(x_{2k+i}) = \delta(y) \) if \( \Psi_k(y) \) is continuous at \( x_\infty \). In particular, considering that \( i + 1 \geq 9 \) is odd, we have that the continuity of \( \Psi_k(y) \) at \( x_\infty \) implies that \( \lim_{m \to \infty} g(y)(m) = 1 \) since \( \delta(y) \neq 0 \). Conversely, if \( \lim_{y \to x_\infty} g(y)(m) = 1 \), then \( \lim_{y \to x_\infty} f_m(y)(x) = \delta(y) \) uniformly holds for \( x \in X \). It follows from the definition of \( \Psi_k(y)(x) \) that \( \lim_{x \to x_\infty} \Psi_k(y)(x) = \delta(y) \), which means the continuity of \( \Psi_k(y) \) at \( x_\infty \). We are done.

To show the continuity of \( \downarrow \Psi_k : C_k \to \downarrow \text{USC}(X) \), we apply Lemma 3. Let \( (y_n) \) be a sequence in \( C_k \) with the limit \( y \). The conditions (a) and (b) in Lemma 3 are verified in Facts 4 and 3 below, respectively. In advance, we remark the following fact which follows from the definitions.

**Fact 2.** For each sequence \( (z_n) \) in \( X \) with the limit \( z \neq x_\infty \), if \( \lim_{n \to \infty} h(y_n)(z_n) = h(y)(z) \), then \( \lim_{n \to \infty} \Psi_k(y_n)(z_n) = \Psi_k(y)(z) \).
Fact 3. For each \((z, t) \in \downarrow \Psi_k(y)\), there exists a sequence \((z_n, t_n) \in \downarrow \Psi_k(y_n)\) such that \(\lim_{n \to \infty} (z_n, t_n) = (z, t)\).

Trivially, we may only consider the case that \(\Psi_k(y)(z) \neq 0\). If \(z \neq x_\infty\), then it follows from the continuity of \(\downarrow h|_{C_k} : C_k \to \downarrow \text{USC}(X)\) and Lemma 3 that there exists a sequence \((z_n, t_n') \in \downarrow h(y_n)\) such that \(\lim_{n \to \infty} (z_n, t_n') = (z, h(y)(z))\). Thus \(\lim_{n \to \infty} h(y_n)(z_n) \geq h(y)(z)\). On the other hand, there exist \(n_1 < n_2 < \cdots\) such that \(\lim_{n \to \infty} h(y_n)(z_n) = \lim_{n \to \infty} h(y_n)(z_n) \leq h(y)(z)\) by Lemma 3. Hence \(\lim_{n \to \infty} h(y_n)(z_n) = h(y)(z)\). It follows from Fact 2 that \(\lim_{n \to \infty} (z_n, \Psi_k(y_n)(z_n)) = (z, \Psi_k(y)(z))\). Thus \((z_n, \frac{t_n}{\Psi_k(y_n)(z_n)}) \in \downarrow \Psi_k(y_n)\) and its limit is \((z, t)\). In the case \(z = x_\infty\), let \(z_n = z\) and \(t_n = \frac{t}{\delta(y_n)}\). Then \((z_n, t_n) \in \downarrow \Psi_k(y_n)\) and \(\lim_{n \to \infty} (z_n, t_n) = (z, t)\).

Fact 4. For every \((z_n, t_n) \in \downarrow \Psi_k(y_n)\), if \(\lim_{n \to \infty} (z_n, t_n) = (z, t)\), then \((z, t) \in \downarrow \Psi_k(y)\).

In the case \(z = x_\infty\), we may assume that \(z_n \in \bigcup_{i=1}^{\infty} S_{2k+i}\). Then \(t_n \leq \Psi_k(y_n)(z_n) \leq \delta(y_n)\). It follows from the continuity of \(\delta : Y \to I\) that \(t_\infty \leq \delta(y) = \Psi_k(y)(x_\infty) = \Psi_k(y)(z)\), that is, \((z, t) \in \downarrow \Psi_k(y)\). Now we consider the case \(z \neq x_\infty\). Since \(S_m\) is closed, we may assume, without loss of generality, that \(z_n, z \in S_m\) for some \(m\) and \(t_n, t > 0\) for all \(n\). We consider the following four cases:

Case A. \(m \leq 2k\). Then \(t_n \leq \Psi_k(y_n)(z_n) = h(y_n)(z_n)\). It follows from the continuity of \(\downarrow h : Y \setminus K \to \downarrow \text{USC}(X)\) and Lemma 3 that \(t \leq h(y)(z) = \Psi_k(y)(z)\), that is, \((z, t) \in \downarrow \Psi_k(y)\).

Case B. \(m = 2k + 1\). Then \(t_n \leq \Psi_k(y_n)(z_n) = \phi_{2k+1}(z_n)h(y_n)(z_n)\). Choose \(n_1 < n_2 < \cdots\) such that \(\lim_{n \to \infty} \frac{t_n}{\Psi_k(y_n)(z_n)} = a\) and \(\lim_{n \to \infty} h(y_n)(z_n) = b\). Then \(\lim_{n \to \infty} \Psi_k(y_n)(z_n)\) exists and \(t = \lim_{n \to \infty} t_n \leq \lim_{n \to \infty} \Psi_k(y_n)(z_n)\) because \(a \in I\). Since \(\downarrow h\) is continuous, we have that \(b \leq h(y)(z)\), hence \(t \leq \lim_{n \to \infty} \Psi_k(y_n)(z_n) \leq \Psi_k(y)(z)\). That is, \((z, t) \in \downarrow \Psi_k(y)\).

Case C. \(m = 2k + 2\) or \(2k + 3\). Proofs are similar to Case B.

Case D. \(m \geq 2k + 4\). Recall that \(X \times C_k \ni (x, y) \mapsto f_m(y)(x)\) is continuous for every \(m \geq 4\). Thus this fact is trivial.

It follows from the above facts that \(\downarrow \Psi_k : C_k \to \downarrow \text{USC}(X)\) is continuous for every \(k\). To define \(\Psi : Y \to \text{USC}(X)\), we need the following fact:

Fact 5. For every \(k\) and every \(y \in C_k \cap C_{k+1}\), we have \(\Psi_k(y) = \Psi_{k+1}(y)\).

At first, we note that the maps \(\phi : C_k \to I\) and \(f_m : C_k \to C(X)\) depend on \(k\). Thus, in the proof of Fact 5, we have to use different symbols to denote them. Let \(\phi : C_k \to I\) and \(f_m : C_k \to C(X)\) be these maps for the case \(k\) and let \(\phi' : C_{k+1} \to I\) and \(f_m' : C_{k+1} \to C(X)\) be these maps for the case \(k + 1\). For every \(y \in C_k \cap C_{k+1}\), we have \(\delta(y) = \frac{\phi(y)}{2}\). Thus, \(\phi(y) = 1\) and \(\phi'(y) = 0\). Using these facts, it is not hard to verify that \(f_1(y) = f_2(y) = f_3(y)\) and \(f_{m}'(y) = f_{m+2}(y)\) for every \(m\). It follows that

\[
\Psi_k(y)(x) = f_1(y)(x) = f_3(y)(x) = f_1'(y)(x) = \Psi_{k+1}(y)(x)
\]

for every \(x \in \bigcup_{i=1}^{2k+2} S_i\). For every \(i\) and \(x \in S_{2(k+1)+i} = S_{2k+(i+2)}\), we have

\[
\Psi_{k+1}(y)(x) = \phi_{2(k+1)+i}(x) f_1'(y)(x) + (1 - \phi_{2(k+1)+i}(x)) f_{i+1}'(x) = \phi_{2(k+2+i)}(x) f_{i+2}(y)(x) + (1 - \phi_{2(k+2+i)}(x)) f_{i+3}(y) = \Psi_k(y)(x).
\]

Trivially, \(\Psi_{k+1}(y)(x_\infty) = \delta(y) = \Psi_k(y)(x_\infty)\). Hence we complete the proof of the fact \(\Psi_k(y) = \Psi_{k+1}(y)\).

Thus we may define a map \(\Psi : Y \to \text{USC}(X)\) as follows:

\[
\Psi(y) = \begin{cases} \phi(y) = h(y) & \text{if } y \in K, \\ \Psi_k(y) & \text{if } y \in C_k. \end{cases}
\]

Then \(\Psi|_K = \Phi|_K\). Therefore, the following facts show that \(\Psi\) is as required.
Fact 6. For every $y \in Y$, 
\[ d_H(\downarrow \Phi(y), \downarrow \Phi(y)) \leq 4\delta(y) \leq \varepsilon. \]

It is trivial that this fact holds for $y \in K$. Now suppose $y \in C_k$ for some $k$. If \( x \in \bigcup_{i=1}^{2^k} S_i \), then $\Psi(y)(x) = \Psi_k(y)(x) = h(y)(x)$. Thus, 
\[ \{x\} \times [0, \Psi(y)(x)] = \{x\} \times [0, h(y)(x)]. \]

If $x \in S_{2k+i}$ for some $i$, then $d(x, x_\infty) \leq 2^{-2k-i+1} < 2^{-k} \leq \delta(y)$. It follows from the definition of $\Psi_k$ that 
\[ \Psi(y)(x) = \Psi_k(y)(x) \leq \sup\{ f_i(y)(x) : i \in \mathbb{N} \} \leq \max\{ h(y)(x), s(y), \delta(y) \} \leq \max\{ \delta(y), \sup\{ h(y)(x') : d(x', x_\infty) \leq 2\delta(y) \} \}. \]

Since $d(x, x') < 3\delta(y)$ for every $x' \in B_d(x_\infty, 2\delta(y))$, we have 
\[ \{x\} \times [0, \Psi(y)(x)] \subset B_d(\downarrow h(y), 3\delta(y)). \]

If $x = x_\infty$, it is trivial that 
\[ \{x\} \times [0, \Psi(y)(x)] \subset B_d(\downarrow h(y), \delta(y)). \]

Hence 
\[ \downarrow \Psi(y) = \bigcup_{x \in X} \{x\} \times [0, \Psi(y)(x)] \subset B_d(\downarrow h(y), 3\delta(y)). \]

Conversely, if $x \in \bigcup_{i=1}^\infty S_{2k+i} \cup \{ x_\infty \}$, then $d(x, x_\infty) < \delta(y)$ and hence $s(y) \geq M(y, \delta(y)) \geq h(y)(x)$. Thus 
\[ \{x\} \times [0, h(y)(x)] \subset B_d([x_{2k+3}] \times [0, s(y)], d(x, x_{2k+3} + \delta(y))). \]

Noting that $\Psi(y)(x_{2k+3}) = \Psi_k(y)(x_{2k+3}) = f_i(y)(x_{2k+3}) = s(y)$ and $d(x, x_{2k+3}) \leq 2\delta(y)$, we have 
\[ \{x\} \times [0, h(y)(x)] \subset B_d([x_{2k+3}] \times [0, \Psi(y)(x_{2k+3})], 3\delta(y)) \subset B_d(\downarrow \Psi(y), \delta(y)). \]

It follows that 
\[ \downarrow h(y) \subset B_d(\downarrow \Psi(y), 3\delta(y)). \]

Thus $d_H(\downarrow h(y), \downarrow \Psi(y)) \leq 3\delta(y)$. Hence, $d_H(\downarrow \Phi(y), \downarrow \Psi(y)) \leq d_H(\downarrow \Phi(y), \downarrow h(y)) + d_H(\downarrow h(y), \downarrow \Psi(y)) \leq \delta(y) + 3\delta(y) = 4\delta(y)$. 

Fact 7. $\downarrow \Psi : Y \rightarrow \downarrow \text{USC}(X)$ is a $Z$-embedding.

It follows from the continuity of $\downarrow \Psi : C_k \rightarrow \downarrow \text{USC}(X)$ and Facts 5 and 6 that $\downarrow \Psi : Y \rightarrow \downarrow \text{USC}(X)$ is continuous. Moreover, we shall show $\Psi(y_1) \neq \Psi(y_2)$ for any $y_1, y_2 \in Y$ with $y_1 \neq y_2$. By the symmetry, we may only consider the following three cases.

Case A. $y_1, y_2 \in K$. This fact is trivial.

Case B. $y_1 \in K$ and $y_2 \in Y \setminus K$. Then, by Fact 6, 
\[ d_H(\downarrow \Psi(y_2), \downarrow \Phi(y_2)) \leq 4\delta(y_2). \]

On the other hand, it follows from the definition of $\delta$ that 
\[ d_H(\downarrow \Phi(y_1), \downarrow \Phi(y_2)) \geq d_H(\downarrow \Phi(K), \downarrow \Phi(y_2)) \geq 5\delta(y_2) > 0. \]

We conclude $\Psi(y_1) = \Phi(y_1) \neq \Psi(y_2)$.

Case C. $y_1, y_2 \in Y \setminus K$. If $\Psi(y_1) = \Psi(y_2)$, then $\delta(y_1) = \Psi(y_1)(x_\infty) = \Psi(y_2)(x_\infty) = \delta(y_2) \neq 0$. Thus there exists $k$ such that $y_1, y_2 \in C_k$ and, for this $k$, $\varphi(y_1) = \varphi(y_2)$. On the other hand, for every $i \in \mathbb{N}$,
\[ f_{i+1}(y_1)(x_{2k+i}) = \Psi_k(y_1)(x_{2k+i}) = \Psi(y_1)(x_{2k+i}) = \Psi(y_2)(x_{2k+i}) = f_{i+1}(y_2)(x_{2k+i}). \]
If \( \varphi(y_1) = \varphi(y_2) = 1 \), then letting \( i \) be even and \( i \geq 8 \) in the above formula, by the definition of \( f_{i+1} \), we have \( g(y_1)(m) = g(y_2)(m) \) for every \( m \in \mathbb{N} \). If \( \varphi(y_1) = \varphi(y_2) \neq 1 \), then letting \( i = 6 \) in the above formula we have \( g(y_1)(1) = g(y_2)(1) \). Moreover, letting \( i = 8, 10, 12, \ldots \), respectively, in the above formula, we may inductively obtain \( g(y_1)(m) = g(y_2)(m) \) for every \( m \). Thus \( g(y_1) = g(y_2) \) in both cases. Since \( g: Y \to Q_\mathcal{U} \) is injective, we have that \( y_1 = y_2 \). A contradiction occurs.

Hence \( \downarrow \Psi: Y \to \downarrow \text{USC}(X) \) is an embedding. Noticing that, for every \( y \in C_k \), \( \Psi(y)(x_{2k+4}) = \Psi_k(y)(x_{2k+4}) = f_5(y)(x_{2k+4}) = 0 \), it follows from [19, Lemma 5] that \( \downarrow \Psi: Y \to \downarrow \text{USC}(X) \) is a \( Z \)-embedding.

**Fact 8.** \( \Psi^{-1}(C(X)) \setminus K = C \setminus K \).

In fact, for every \( y \in Y \setminus K \), there exists \( k \) such that \( y \in C_k \). By the definition of \( \Psi \) and Fact 1, we have that \( \Psi(y) \in C(X) \) if and only if \( \Psi_k(y) \in C(X) \) if and only if \( y \in C \). We are done. \( \square \)

**Proof of Theorem 3.** \( (a) \Rightarrow (c) \): Theorem 1 implies \( \downarrow \text{USC}(X) \approx Q \). By Theorem 2, the closure of every \( Z \)-set in \( \downarrow C(X) \) is a \( Z \)-set in \( \downarrow \text{USC}(X) \). It follows from [19, Lemma 6] that \( \downarrow C(X) \) is contained in a \( Z_{\sigma} \)-set in \( \downarrow \text{USC}(X) \). Then, combining this with Proposition 1 and [18, Proposition 1], \( \downarrow C(X) \) is an \( \mathcal{F}_{\sigma \delta} \)-absorber in \( \downarrow \text{USC}(X) \). Thus, using Lemma 2, we have the condition (c).

\( (c) \Rightarrow (b) \): This is trivial.

\( (b) \Rightarrow (a) \): By Lemma 4, (b) implies that \( X \) is a compactum. Note that \( c_0 \) is not Baire. It follows directly from [19, Theorem 2] that the set of isolated points is not dense in \( X \). \( \square \)

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**References**


