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Subclasses of meromorphically multivalent functions defined by a differential operator

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ABSTRACT

In this paper we introduce and study two new subclasses $\Sigma_{\lambda\mu mp}(\alpha,\beta)$ and $\Sigma^{+}_{\lambda\mu mp}(\alpha,\beta)$ of meromorphically multivalent functions which are defined by means of a new differential operator. Some results connected to subordination properties, coefficient estimates, convolution properties, integral representation and distortion theorems are obtained. We also extend the familiar concept of (n,δ)-neighborhoods of analytic functions to these subclasses of meromorphically multivalent functions.

1. Introduction

Let $\tilde{\mathcal{A}}$ be the class of analytic functions in the unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$.

Consider

$$\Omega=\{w\in\tilde{\mathcal{A}}:w(0)=0 \text{ and } |w(z)|<1,\ z\in\mathbb{U}\} \tag{1.1}$$

the class of Schwarz functions.

For $0\le\alpha<1$ let

$$\mathcal{P}(\alpha)=\{p\in\tilde{\mathcal{A}}:p(0)=1 \text{ and } \Re p(z)>\alpha,\ z\in\mathbb{U}\}. \tag{1.2}$$

Note that $\mathcal{P}=\mathcal{P}(0)$ is the well-known Carathéodory class of functions.

The classes of Schwarz and Carathéodory functions play an extremely important role in the theory of analytic functions and have been studied by many authors.

It is easy to see that

$$p\in\mathcal{P}(\alpha)\quad\text{if and only if}\quad\frac{p(z)-\alpha}{1-\alpha}\in\mathcal{P}. \tag{1.3}$$

Making use of the properties of functions in the class $\mathcal{P}$ and condition (1.3), the following properties of the functions in the class $\mathcal{P}(\alpha)$ can be obtained.

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Lemma 1.1. *Let $p \in \tilde{\mathcal{A}}$. Then $p \in \mathcal{P}(\alpha)$ if and only if there exists $w \in \Omega$ such that*

$$p(z) = \frac{1-(2\alpha-1)w(z)}{1-w(z)} \quad (z \in \mathbb{U}). \tag{1.4}$$

Lemma 1.2 (*Herglotz Formula*)**.** *A function $p \in \tilde{\mathcal{A}}$ belongs to the class $\mathcal{P}(\alpha)$ if and only if there exists a probability measure $\mu(x)$ on $\partial\mathbb{U}$ such that*

$$p(z) = \int_{|x|=1} \frac{1-(2\alpha-1)xz}{1-xz} \mathrm{d}\mu(x) \quad (z \in \mathbb{U}). \tag{1.5}$$

The correspondence between $\mathcal{P}(\alpha)$ and probability measure $\mu(x)$ on $\partial\mathbb{U}$, given by (1.5) is one-to-one.

If f and g are in $\tilde{\mathcal{A}}$, we say that f is subordinate to g, written $f \prec g$, if there exists a function $w \in \Omega$ such that $f(z) = g(w(z))$ $(z \in \mathbb{U})$. It is known that if $f \prec g$, then $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$. In particular, if g is univalent in $\mathbb{U}$ we have the following equivalence:

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \text{ if and only if } f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let Σ_p denote the class of all meromorphic functions f of the form

$$f(z) = z^{-p} + \sum_{k=1-p}^{\infty} a_k z^k \quad (p \in \mathbb{N} := \{1, 2, \ldots\}) \tag{1.6}$$

which are analytic and p-valent in the punctured unit disk $\mathbb{U}^* = \mathbb{U} \setminus \{0\}$.

Denote by Σ_p^+ the subclass of Σ_p consisting of functions of the form

$$f(z) = z^{-p} + \sum_{k=1-p}^{\infty} a_k z^k \quad a_k \geq 0 \ (z \in \mathbb{U}^*). \tag{1.7}$$

A function $f \in \Sigma_p$ is meromorphically multivalent starlike of order α $(0 \leq \alpha < 1)$ (see [1]) if

$$-\Re\left\{\frac{1}{p}\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in \mathbb{U}).$$

The class of all such functions is denoted by $\Sigma_p^*(\alpha)$.

If $f \in \Sigma_p$ is given by (1.6) and $g \in \Sigma_p$ is given by

$$g(z) = z^{-p} + \sum_{k=1-p}^{\infty} b_k z^k$$

then the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z^{-p} + \sum_{k=1-p}^{\infty} a_k b_k z^k = (g * f)(z) \quad (p \in \mathbb{N}\ z \in \mathbb{U}^*).$$

For a function $f \in \Sigma_p$, we define the differential operator $D_{\lambda\mu p}^m$ in the following way:

$$D_{\lambda\mu p}^0 f(z) = f(z)$$

$$D_{\lambda\mu p}^1 f(z) = D_{\lambda\mu p} f(z) = \lambda\mu \frac{[z^{p+1}f(z)]''}{z^{p-1}} + (\lambda-\mu)\frac{[z^{p+1}f(z)]'}{z^p} + (1-\lambda+\mu)f(z) \tag{1.8}$$

and, in general

$$D_{\lambda\mu p}^m f(z) = D_{\lambda\mu p}(D_{\lambda\mu p}^{m-1} f(z)), \tag{1.9}$$

where $0 \leq \mu \leq \lambda$ and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

If the function $f \in \Sigma_p$ is given by (1.6) then, from (1.8) and (1.9), we obtain

$$D_{\lambda\mu p}^m f(z) = z^{-p} + \sum_{k=1-p}^{\infty} \Phi_k(\lambda, \mu, m, p) a_k z^k \quad (m \in \mathbb{N},\ p \in \mathbb{N},\ z \in \mathbb{U}^*) \tag{1.10}$$

where

$$\Phi_k(\lambda, \mu, m, p) = [1 + (k+p)(\lambda - \mu + (k+p+1)\lambda\mu)]^m. \tag{1.11}$$

From (1.10) it follows that $D^m_{\lambda\mu p}f(z)$ can be written in terms of convolution as

$$D^m_{\lambda\mu p}f(z) = (f * h)(z) \tag{1.12}$$

where

$$h(z) = z^{-p} + \sum_{k=1-p}^{\infty} \Phi_k(\lambda, \mu, m, p)z^k. \tag{1.13}$$

Note that, the case $\lambda = 1$ and $\mu = 0$ of the differential operator $D^m_{\lambda\mu p}$ was introduced by Srivastava and Patel [2]. A special case of $D^m_{\lambda\mu p}$ for $p = 1$ was considered in [3].

Making use of the differential operator $D^m_{\lambda\mu p}$, we define a new subclass of the function class Σ_p as follows.

Definition 1.1. A function $f \in \Sigma_p$ is said to be in the class $\Sigma_{\lambda\mu mp}(\alpha, \beta)$ if it satisfies the condition

$$\left|\frac{1}{p}\frac{z(D^m_{\lambda\mu p}f(z))'}{D^m_{\lambda\mu p}f(z)} + 1\right| < \beta\left|\frac{1}{p}\frac{z(D^m_{\lambda\mu p}f(z))'}{D^m_{\lambda\mu p}f(z)} + 2\alpha - 1\right| \tag{1.14}$$

for some α $(0 \le \alpha < 1)$, β $(0 < \beta \le 1)$ and $z \in \mathbb{U}^*$.

Note that a special case of the class $\Sigma_{\lambda\mu mp}(\alpha, \beta)$ for $p = 1$ and $m = 0$ is the class of meromorphically starlike functions of order α and type β introduced earlier by Mogra et al. [4]. It is easy to check that for $m = 0$ and $\beta = 1$, the class $\Sigma_{\lambda\mu mp}(\alpha, \beta)$ reduces to the class $\Sigma^*_p(\alpha)$.

We consider another subclass of Σ_p given by

$$\Sigma^+_{\lambda\mu mp}(\alpha, \beta) := \Sigma^+_p \cap \Sigma_{\lambda\mu mp}(\alpha, \beta). \tag{1.15}$$

The main object of this paper is to present a systematic investigation of the classes $\Sigma_{\lambda\mu mp}(\alpha, \beta)$ and $\Sigma^+_{\lambda\mu mp}(\alpha, \beta)$.

2. Properties of the class $\Sigma_{\lambda\mu mp}(\alpha, \beta)$

We begin this section with a necessary and sufficient condition, in terms of subordination, for a function to be in the class $\Sigma_{\lambda\mu mp}(\alpha, \beta)$.

Theorem 2.1. *A function $f \in \Sigma_p$ is in the class $\Sigma_{\lambda\mu mp}(\alpha, \beta)$ if and only if*

$$\frac{z(D^m_{\lambda\mu p}f(z))'}{D^m_{\lambda\mu p}f(z)} \prec \frac{p(2\alpha - 1)\beta z - p}{1 - \beta z} \quad (z \in \mathbb{U}). \tag{2.1}$$

Proof. Let $f \in \Sigma_{\lambda\mu mp}(\alpha, \beta)$. Then, from (1.6), we have

$$\left|-\frac{1}{p}\frac{z(D^m_{\lambda\mu p}f(z))'}{D^m_{\lambda\mu p}f(z)} - 1\right|^2 < \beta^2\left|-\frac{1}{p}\frac{z(D^m_{\lambda\mu p}f(z))'}{D^m_{\lambda\mu p}f(z)} + 1 - 2\alpha\right|^2$$

or

$$(1-\beta^2)\left|-\frac{1}{p}\frac{z(D^m_{\lambda\mu p}f(z))'}{D^m_{\lambda\mu p}f(z)}\right|^2 - 2[1 + \beta^2(1-2\alpha)]\Re\left\{-\frac{1}{p}\frac{z(D^m_{\lambda\mu p}f(z))'}{D^m_{\lambda\mu p}f(z)}\right\} < \beta^2(1-2\alpha)^2 - 1.$$

If $\beta \ne 1$, we have

$$\left|-\frac{1}{p}\frac{z(D^m_{\lambda\mu p}f(z))'}{D^m_{\lambda\mu p}f(z)}\right|^2 - 2\frac{1 + \beta^2(1-2\alpha)}{1-\beta^2}\Re\left\{-\frac{1}{p}\frac{z(D^m_{\lambda\mu p}f(z))'}{D^m_{\lambda\mu p}f(z)}\right\} + \left[\frac{1 + \beta^2(1-2\alpha)}{1-\beta^2}\right]^2$$
$$< \frac{\beta^2(2\alpha-1)^2 - 1}{1-\beta^2} + \left[\frac{1 + \beta^2(1-2\alpha)}{1-\beta^2}\right]^2,$$

that is

$$\left|-\frac{1}{p}\frac{z(D^m_{\lambda\mu p}f(z))'}{D^m_{\lambda\mu p}f(z)} - \frac{1 - \beta^2(2\alpha - 1)}{1-\beta^2}\right| < \frac{2\beta(1-\alpha)}{1-\beta^2}. \tag{2.2}$$

The inequality (2.2) shows that the values region of $F(z) = -\frac{1}{p}\frac{z(D^m_{\lambda\mu p}f(z))'}{D^m_{\lambda\mu p}f(z)}$ is contained in the disk centered at $\frac{1-\beta^2(2\alpha-1)}{1-\beta^2}$ and radius $\frac{2\beta(1-\alpha)}{1-\beta^2}$. It is easy to check that the function $G(z) = \frac{1-(2\alpha-1)\beta z}{1-\beta z}$ maps the unit disk $\mathbb{U}$ onto the disk

$$\left|\omega - \frac{1-\beta^2(2\alpha-1)}{1-\beta^2}\right| < \frac{2\beta(1-\alpha)}{1-\beta^2}.$$

Since G is univalent and $F(0) = G(0)$, $F(\mathbb{U}) \subset G(\mathbb{U})$, we obtain that $F(z) \prec G(z)$, that is

$$-\frac{1}{p}\frac{z(D^m_{\lambda\mu p}f(z))'}{D^m_{\lambda\mu p}f(z)} \prec \frac{1-(2\alpha-1)\beta z}{1-\beta z}$$

or

$$\frac{z(D^m_{\lambda\mu p}f(z))'}{D^m_{\lambda\mu p}f(z)} \prec \frac{p(2\alpha-1)\beta z - p}{1-\beta z}.$$

Conversely, suppose that subordination (2.1) holds. Then

$$-\frac{1}{p}\frac{z(D^m_{\lambda\mu p}f(z))'}{D^m_{\lambda\mu p}f(z)} = \frac{1-(2\alpha-1)\beta w(z)}{1-\beta w(z)} \tag{2.3}$$

where $w \in \Omega$. After simple calculations, from (2.3), we obtain

$$\left|\frac{1}{p}\frac{z(D^m_{\lambda\mu p}f(z))'}{D^m_{\lambda\mu p}f(z)} + 1\right| < \beta\left|\frac{1}{p}\frac{z(D^m_{\lambda\mu p}f(z))'}{D^m_{\lambda\mu p}f(z)} + 2\alpha - 1\right|$$

which proves that $f \in \Sigma_{\lambda\mu mp}(\alpha, \beta)$.

If $\beta = 1$, inequality (1.14) becomes

$$\left|-\frac{1}{p}\frac{z(D^m_{\lambda\mu p}f(z))'}{D^m_{\lambda\mu p}f(z)} - 1\right| < \left|-\frac{1}{p}\frac{z(D^m_{\lambda\mu p}f(z))'}{D^m_{\lambda\mu p}f(z)} + 1 - 2\alpha\right|. \tag{2.4}$$

From (2.4) we can easily obtain that

$$-\frac{1}{p}\frac{z(D^m_{\lambda\mu p}f(z))'}{D^m_{\lambda\mu p}f(z)} \prec \frac{1-(2\alpha-1)z}{1-z}$$

or

$$\frac{z(D^m_{\lambda\mu p}f(z))'}{D^m_{\lambda\mu p}f(z)} \prec \frac{(2\alpha-1)pz - p}{1-z}. \tag{2.5}$$

Thus, the proof of our theorem is completed. □

Remark 2.1. Since $\Re\frac{1-(2\alpha-1)\beta z}{1-\beta z} > \alpha$ it follows that

$$-\Re\left\{\frac{1}{p}\frac{z(D^m_{\lambda\mu p}f(z))'}{D^m_{\lambda\mu p}f(z)}\right\} > \alpha$$

which shows that $D^m_{\lambda\mu p}f(z) \in \Sigma^*_p(\alpha)$.

Making use of the subordination relationship for the class $\Sigma_{\lambda\mu mp}(\alpha, \beta)$, we derive a structural formula, first for the class $\Sigma_{\lambda\mu mp}(\alpha, 1)$ and then for the class $\Sigma_{\lambda\mu mp}(\alpha, \beta)$.

Theorem 2.2. *A function $f \in \Sigma_p$ belongs to the class $\Sigma_{\lambda\mu mp}(\alpha, 1)$ if and only if there exists a probability measure $\mu(x)$ on $\partial\mathbb{U}$ such that*

$$f(z) = \left[z^{-p} + \sum_{k=1-p}^{\infty}\frac{z^k}{\Phi_k(\lambda, \mu, m, p)}\right] * \left[z^{-p}\exp\int_{|x|=1} 2p(1-\alpha)\log(1-xz)\mathrm{d}\mu(x)\right] \quad z \in \mathbb{U}^*. \tag{2.6}$$

The correspondence between $\Sigma_{\lambda\mu mp}(\alpha, 1)$ and the probability measure $\mu(x)$ is one-to-one.

Proof. In view of the subordination condition (2.5), we have that $f \in \Sigma_{\lambda\mu mp}(\alpha, 1)$ if and only if

$$-\frac{1}{p}\frac{z(D^m_{\lambda\mu p}f(z))'}{D^m_{\lambda\mu p}f(z)} \in \mathcal{P}(\alpha).$$

From Lemma 1.2, we have

$$-\frac{1}{p}\frac{z(D^m_{\lambda\mu p}f(z))'}{D^m_{\lambda\mu p}f(z)} = \int_{|x|=1}\frac{1-(2\alpha-1)xz}{1-xz}\mathrm{d}\mu(x)$$

which is equivalent to

$$\frac{z(D^m_{\lambda\mu p}f(z))'}{D^m_{\lambda\mu p}f(z)} = \int_{|x|=1}\frac{p(2\alpha-1)xz-p}{1-xz}\mathrm{d}\mu(x).$$

Integrating this equality, we obtain

$$z^p D^m_{\lambda\mu p}f(z) = \exp\int_{|x|=1} 2p(1-\alpha)\log(1-xz)\mathrm{d}\mu(x)$$

or

$$D^m_{\lambda\mu p}f(z) = z^{-p}\exp\int_{|x|=1} 2p(1-\alpha)\log(1-xz)\mathrm{d}\mu(x). \tag{2.7}$$

Equality (2.6) now follows easily from (1.12), (1.13) and (2.7). □

Using a result of Goluzin [5] (see also [6] p. 50), we obtain the following result.

Theorem 2.3. *Let $f \in \Sigma_{\lambda\mu mp}(\alpha, 1)$. Then*

$$z^p D^m_{\lambda\mu p}f(z) \prec (1-z)^{2p(1-\alpha)} \quad (z \in \mathbb{U}).$$

Proof. Let $f \in \Sigma_{\lambda\mu mp}(\alpha, 1)$. Then by (2.5) we have

$$\frac{z(D^m_{\lambda\mu p}f(z))'}{D^m_{\lambda\mu p}f(z)} \prec \frac{p(2\alpha-1)z-p}{1-z}.$$

Since the function $\frac{p(2\alpha-1)z-p}{1-z}$ is univalent and convex in $\mathbb{U}$, in view of Goluzin's result, we obtain

$$\int_0^z \frac{(D^m_{\lambda\mu p}f(\zeta))'}{D^m_{\lambda\mu p}f(\zeta)}\mathrm{d}\zeta \prec \int_0^z \frac{p(2\alpha-1)\zeta-p}{\zeta(1-\zeta)}\mathrm{d}\zeta$$

or

$$\log(D^m_{\lambda\mu p}f(z)) \prec \log\frac{(1-z)^{2p(1-\alpha)}}{z^p}.$$

Thus, there exists a function $w \in \Omega$ such that

$$\log(D^m_{\lambda\mu p}f(z)) = \log\frac{(1-w(z))^{2p(1-\alpha)}}{w(z)^p}$$

which is equivalent to

$$z^p D^m_{\lambda\mu p}f(z) \prec (1-z)^{2p(1-\alpha)}.$$

Next we obtain a structural formula for the class $\Sigma_{\lambda\mu mp}(\alpha, \beta)$. □

Theorem 2.4. *Let $f \in \Sigma_{\lambda\mu mp}(\alpha, \beta)$. Then*

$$f(z) = \left[z^{-p} + \sum_{k=1-p}^{\infty}\frac{z^k}{\Phi_k(\lambda,\mu,m,p)}\right] * \left[z^{-p}\exp\left(2p(1-\alpha)\beta\int_0^z \frac{w(\zeta)}{\zeta(1-\beta w(\zeta))}\mathrm{d}\zeta\right)\right] \quad (z \in \mathbb{U}^*) \tag{2.8}$$

where $w \in \Omega$.

Proof. Assume $f \in \Sigma_{\lambda\mu mp}(\alpha,\beta)$. From (2.1) it follows

$$\frac{z(D^m_{\lambda\mu p}f(z))'}{D^m_{\lambda\mu p}f(z)} = \frac{p(2\alpha-1)\beta w(z)-p}{1-\beta w(z)} \quad (z\in\mathbb{U}). \tag{2.9}$$

In view of (2.9), we have

$$\frac{(D^m_{\lambda\mu p}f(z))'}{D^m_{\lambda\mu p}f(z)} + \frac{p}{z} = \frac{p(2\alpha-1)\beta w(z)}{z(1-\beta w(z))} \quad (z\in\mathbb{U}^*)$$

which upon integration, yields

$$\log(z^p D^m_{\lambda\mu p}f(z)) = 2p(1-\alpha)\beta\int_0^z \frac{w(\zeta)}{\zeta(1-\beta w(\zeta))}\mathrm{d}\zeta. \tag{2.10}$$

The assertion (2.8) of the theorem can be easily obtained from (1.12), (1.13) and (2.10). □

In the sequence a convolution property for the class $\Sigma_{\lambda\mu mp}(\alpha,\beta)$ is derived.

Theorem 2.5. *If $f\in\Sigma_p$ belongs to $\Sigma_{\lambda\mu mp}(\alpha,\beta)$, then*

$$D^m_{\lambda\mu p}f(z) * \left\{\frac{-pz^{-p}+(p+1)z^{-p+1}}{(1-z)^2}(1-\beta e^{i\theta}) + \frac{z^{-p}}{1-z}[p-p(2\alpha-1)\beta e^{i\theta}]\right\} \neq 0 \tag{2.11}$$

for $z\in\mathbb{U}^*$ and $\theta\in(0,2\pi)$.

Proof. Let $f\in\Sigma_{\lambda\mu mp}(\alpha,\beta)$. Then, from (2.1) it follows

$$-\frac{z(D^m_{\lambda\mu p}f(z))'}{D^m_{\lambda\mu p}f(z)} \neq \frac{p-p(2\alpha-1)\beta e^{i\theta}}{1-\beta e^{i\theta}} \quad (z\in\mathbb{U},\ 0<\theta<2\pi). \tag{2.12}$$

It is easy to see that condition (2.12) can be written as follows

$$(1-\beta e^{i\theta})z(D^m_{\lambda\mu p}f(z))' + [p-p(2\alpha-1)\beta e^{i\theta}]D^m_{\lambda\mu p}f(z)\neq 0. \tag{2.13}$$

Note that

$$D^m_{\lambda\mu p}f(z) = D^m_{\lambda\mu p}f(z) * \left(z^{-p}+z^{-p+1}+\cdots+\frac{1}{z}+1+\frac{z}{1-z}\right) = D^m_{\lambda\mu p}f(z) * \frac{z^{-p}}{1-z} \tag{2.14}$$

and

$$\begin{aligned} z(D^m_{\lambda\mu p}f(z))' &= D^m_{\lambda\mu p}f(z) * \left[-pz^{-p}-(p-1)z^{-p+1}-\cdots-\frac{1}{z}+\frac{z}{(1-z)^2}\right] \\ &= D^m_{\lambda\mu p}f(z) * \frac{-pz^{-p}+(p+1)z^{-p+1}}{(1-z)^2}. \end{aligned} \tag{2.15}$$

By virtue of (2.13), (2.14) and (2.15), the assertion (2.12) of the theorem follows. □

Coefficient estimates for functions in the class $\Sigma_{\lambda\mu mp}(\alpha,\beta)$ are given in the next theorem.

Theorem 2.6. *Let f of the form (1.6) be in the class $\Sigma_{\lambda\mu mp}(\alpha,\beta)$. Then, for $n\geq 3-p$*

$$|a_n| \leq \frac{2p\beta(1-\alpha)}{(n+p)\Phi_n(\lambda,\mu,m,p)} \tag{2.16}$$

where $\Phi_n(\lambda,\mu,m,p)$ is given by (1.11).

Proof. To prove the coefficient estimates (2.16) we use the method of Clunie and Koegh [7].

Let $f\in\Sigma_{\lambda\mu mp}(\alpha,\beta)$. From (1.14), we have

$$\frac{1}{p}\frac{z(D^m_{\lambda\mu p}f(z))'}{D^m_{\lambda\mu p}f(z)} + 1 = zw(z)\left[\frac{1}{p}\frac{z(D^m_{\lambda\mu p}f(z))'}{D^m_{\lambda\mu p}f(z)} + 2\alpha - 1\right]$$

where w is analytic in $\mathbb{U}$ and $|w(z)|\leq\beta$ for $z\in\mathbb{U}$. Then

$$z(D^m_{\lambda\mu p}f(z))' + pD^m_{\lambda\mu p}f(z) = zw(z)[z(D^m_{\lambda\mu p}f(z))' + p(2\alpha-1)D^m_{\lambda\mu p}f(z)]. \tag{2.17}$$

If $zw(z)=\sum_{k=1}^{\infty}w_kz^k$, making use of (1.10) and (2.17), we obtain

$$\sum_{k=1-p}^{\infty}(k+p)\Phi_k(\lambda,\mu,m,p)a_kz^{k+p}=\left[-2p(1-\alpha)\sum_{k=1-p}^{\infty}[k+p(2\alpha-1)]\Phi_k(\lambda,\mu,m,p)a_kz^{k+p}\right]\sum_{k=1}^{\infty}w_kz^k. \tag{2.18}$$

Equating the coefficients in (2.18), we have

$$n\Phi_{n-p}(\lambda,\mu,m,p)a_{n-p}=-2p(1-\alpha)w_n,\quad \text{for } n=1,2$$

and

$$n\Phi_{n-p}(\lambda,\mu,m,p)a_{n-p}=-2p(1-\alpha)w_n+\sum_{k=1-p}^{n-1-p}[k+p(2\alpha-1)]\Phi_k(\lambda,\mu,m,p)a_kw_{n-p-k} \tag{2.19}$$

for $n\geq 3$.

From (2.19), we obtain

$$\left[-2p(1-\alpha)+\sum_{k=1-p}^{n-1-p}[k+p(2\alpha-1)]\Phi_k(\lambda,\mu,m,p)a_kz^{k+p}\right]\sum_{k=1}^{\infty}w_kz^k$$
$$=\sum_{k=1-p}^{n-p}(k+p)\Phi_k(\lambda,\mu,m,p)a_kz^{k+p}+\sum_{k=n+1-p}^{\infty}c_kz^{k+p}. \tag{2.20}$$

It is known that, if $h(z)=\sum_{n=0}^{\infty}h_nz^n$ is analytic in $\mathbb{U}$, then for $0<r<1$

$$\sum_{n=0}^{\infty}|h_n|^2r^{2n}=\frac{1}{2\pi}\int_0^{2\pi}|h(re^{i\theta})|^2\mathrm{d}\theta. \tag{2.21}$$

Since $|\sum_{k=1}^{\infty}w_kz^k|\leq\beta|z|<\beta$, making use of (2.20) and (2.21), we have

$$\sum_{k=1-p}^{n-p}(k+p)^2\Phi_k(\lambda,\mu,m,p)^2|a_k|^2r^{2(k+p)}+\sum_{k=n+1-p}^{\infty}|c_k|^2r^{2(k+p)}$$
$$\leq\beta^2\left\{4p^2(1-\alpha)^2+\sum_{k=1-p}^{n-1-p}[k+p(2\alpha-1)]^2\Phi_k(\lambda,\mu,m,p)^2|a_k|^2r^{2(k+p)}\right\}.$$

Letting $r\to 1$, we obtain

$$\sum_{k=1-p}^{n-p}(k+p)^2\Phi_k(\lambda,\mu,m,p)^2|a_k|^2\leq 4p^2\beta^2(1-\alpha)^2+\sum_{k=1-p}^{n-1-p}\beta^2[k+p(2\alpha-1)]^2\Phi_k(\lambda,\mu,m,p)^2|a_k|^2.$$

The above inequality implies

$$n^2\Phi_{n-p}(\lambda,\mu,m,p)^2|a_{n-p}|^2\leq 4p^2\beta^2(1-\alpha)^2-\sum_{k=1-p}^{n-1-p}(k+p)^2\Phi_k(\lambda,\mu,m,p)^2|a_k|^2$$
$$+\sum_{k=1-p}^{n-1-p}\beta^2[k+p(2\alpha-1)]^2\Phi_k(\lambda,\mu,m,p)^2|a_k|^2\leq 4p^2\beta^2(1-\alpha)^2.$$

Finally, replacing $n-p$ by n, we have

$$|a_n|\leq\frac{2p\beta(1-\alpha)}{(n+p)\Phi_n(\lambda,\mu,m,p)}.$$

Thus, the proof of our theorem is completed. □

Theorem 2.6 enables us to obtain a distortion result for the class $\Sigma_{\lambda\mu mp}(\alpha,\beta)$.

Corollary 2.1. *If $f\in\Sigma_{\lambda\mu mp}(\alpha,\beta)$ is given by (1.6), then for $0<|z|=r<1$*

$$|f(z)|\geq\frac{1}{r^p}-2p\beta(1-\alpha)r^{1-p}\sum_{k=1-p}^{\infty}\frac{1}{(k+p)\Phi_k(\lambda,\mu,m,p)}$$

$$|f(z)| \leq \frac{1}{r^p} + 2p\beta(1-\alpha)r^{1-p} \sum_{k=1-p}^{\infty} \frac{1}{(k+p)\Phi_k(\lambda,\mu,m,p)}$$

and

$$|f'(z)| \geq \frac{p}{r^{p+1}} - 2p\beta(1-\alpha)r^{2-p} \sum_{k=1-p}^{\infty} \frac{k}{(k+p)\Phi_k(\lambda,\mu,m,p)}$$

$$|f'(z)| \leq \frac{p}{r^{p+1}} + 2p\beta(1-\alpha)r^{2-p} \sum_{k=1-p}^{\infty} \frac{k}{(k+p)\Phi_k(\lambda,\mu,m,p)}.$$

In the sequence we give a sufficient condition for a function to belong to the class $\Sigma_{\lambda\mu mp}(\alpha,\beta)$.

Theorem 2.7. *Let $f \in \Sigma_p$ be given by (1.6). If for $0 \leq \alpha < 1$ and $0 < \beta \leq 1$*

$$\sum_{k=1-p}^{\infty} [k(\beta+1) + p(1+\beta(2\alpha-1))]\Phi_k(\lambda,\mu,m,p)|a_k| \leq 2p\beta(1-\alpha) \qquad (2.22)$$

then $f \in \Sigma_{\lambda\mu mp}(\alpha,\beta)$.

Proof. Assume that $f(z) = z^{-p} + \sum_{k=1-p}^{\infty} a_k z^k$. We have

$$\begin{aligned} M &= |z(D^m_{\lambda\mu p}f(z))' + pD^m_{\lambda\mu p}f(z)| - \beta|z(D^m_{\lambda\mu p}f(z))' + p(2\alpha-1)D^m_{\lambda\mu p}f(z)| \\ &= \left|\sum_{k=1-p}^{\infty}(k+p)\Phi_k(\lambda,\mu,m,p)a_k z^k\right| - \beta\left|\frac{-2p(1-\alpha)}{z^p} + \sum_{k=1-p}^{\infty}[k+p(2\alpha-1)]\Phi_k(\lambda,\mu,m,p)a_k z^k\right|. \end{aligned}$$

For $0 < |z| = r < 1$, we obtain

$$r^pM \leq \sum_{k=1-p}^{\infty}(k+p)\Phi_k(\lambda,\mu,m,p)|a_k|r^{p+k} - \beta\left[2p(1-\alpha) - \sum_{k=1-p}^{\infty}|k+p(2\alpha-1)|\Phi_k(\lambda,\mu,m,p)|a_k|r^{p+k}\right]$$

or

$$r^pM \leq \sum_{k=1-p}^{\infty}[k(\beta+1) + p(1+\beta(2\alpha-1))]\Phi_k(\lambda,\mu,m,p)|a_k|r^{p+k} - 2p\beta(1-\alpha).$$

Since the above inequality holds for all r $(0 < r < 1)$, letting $r \to 1$, we have

$$M \leq \sum_{k=1-p}^{\infty}[k(\beta+1) + p(1+\beta(2\alpha-1))]\Phi_k(\lambda,\mu,m,p)|a_k| - 2p\beta(1-\alpha).$$

Making use of (2.22), we obtain $M \leq 0$, that is

$$\left|\frac{1}{p}\frac{z(D^m_{\lambda\mu p}f(z))'}{D^m_{\lambda\mu p}f(z)} + 1\right| < \beta\left|\frac{1}{p}\frac{z(D^m_{\lambda\mu p}f(z))'}{D^m_{\lambda\mu p}f(z)} + 2\alpha - 1\right|.$$

Consequently, $f \in \Sigma_{\lambda\mu mp}(\alpha,\beta)$. □

3. Properties of the class $\Sigma^+_{\lambda\mu mp}(\alpha,\beta)$

We begin this section by proving that condition (2.22) is both necessary and sufficient for a function to be in the class $\Sigma^+_{\lambda\mu mp}(\alpha,\beta)$.

Theorem 3.1. *Let $f \in \Sigma^+_p$. Then f belongs to the class $\Sigma^+_{\lambda\mu mp}(\alpha,\beta)$ if and only if*

$$\sum_{k=1-p}^{\infty}[k(\beta+1) + p(1+\beta(2\alpha-1))]\Phi_k(\lambda,\mu,m,p)a_k \leq 2p\beta(1-\alpha).$$

Proof. In view of Theorem 2.7, we have to prove "only if" part. Assume that $f(z) = z^{-p} + \sum_{k=1-p}^{\infty} a_k z^k$, $a_k \geq 0$ is in the class $\Sigma^{+}_{\lambda\mu mp}(\alpha, \beta)$. Then

$$\left| \frac{\frac{1}{p}\frac{z(D^m_{\lambda\mu p}f(z))'}{D^m_{\lambda\mu p}f(z)} + 1}{\frac{1}{p}\frac{z(D^m_{\lambda\mu p}f(z))'}{D^m_{\lambda\mu p}f(z)} + 2\alpha - 1} \right| = \left| \frac{\sum_{k=1-p}^{\infty}(k+p)\Phi_k(\lambda,\mu,m,p)a_k z^k}{\frac{2p(1-\alpha)}{z^p} - \sum_{k=1-p}^{\infty}[k+p(2\alpha-1)]\Phi_k(\lambda,\mu,m,p)a_k z^k} \right| < \beta$$

for all $z \in \mathbb{U}$. Since $\Re z \leq |z|$ for all z, it follows that

$$\Re\left\{ \frac{\sum_{k=1-p}^{\infty}(k+p)\Phi_k(\lambda,\mu,m,p)a_k z^k}{\frac{2p(1-\alpha)}{z^p} - \sum_{k=1-p}^{\infty}[k+p(2\alpha-1)]\Phi_k(\lambda,\mu,m,p)a_k z^k} \right\} < \beta. \tag{3.1}$$

We choose the values of z on the real axis such that $\frac{1}{p}\frac{z(D^m_{\lambda\mu p}f(z))'}{D^m_{\lambda\mu p}f(z)}$ is real. Upon clearing the denominator in (3.1) and letting $z \to 1$ through positive values, we obtain

$$\sum_{k=1-p}^{\infty}(k+p)\Phi_k(\lambda,\mu,m,p)a_k \leq 2p\beta(1-\alpha) - \sum_{k=1-p}^{\infty}\beta[k+p(2\alpha-1)]\Phi_k(\lambda,\mu,m,p)a_k$$

or

$$\sum_{k=1-p}^{\infty}[k(\beta+1)+p(1+\beta(2\alpha-1))]\Phi_k(\lambda,\mu,m,p)a_k \leq 2p\beta(1-\alpha). \quad \square$$

Hence, the result follows.

Corollary 3.1. *If $f \in \Sigma^{+}_{p}$, given by (1.7) is in the class $\Sigma^{+}_{\lambda\mu mp}(\alpha, \beta)$, then*

$$a_n \leq \frac{2p\beta(1-\alpha)}{[n(\beta+1)+p(1+\beta(2\alpha-1))]\Phi_n(\lambda,\mu,m,p)}, \quad n \geq 1-p \tag{3.2}$$

with equality for the functions of the form

$$f_n(z) = \frac{1}{z^p} + \frac{2p\beta(1-\alpha)}{[n(\beta+1)+p(1+\beta(2\alpha-1))]\Phi_n(\lambda,\mu,m,p)}z^n.$$

Coefficient estimates obtained in Corollary 3.1 enables us to give a distortion result for the class $\Sigma^{+}_{\lambda\mu mp}(\alpha, \beta)$.

Theorem 3.2. *If $f \in \Sigma^{+}_{\lambda\mu mp}(\alpha, \beta)$, then for $0 < |z| = r < 1$*

$$|f(z)| \geq \frac{1}{r^p} - \frac{2p\beta(1-\alpha)}{[\beta(1-p+(2\alpha-1)p)+1]\Phi_{1-p}(\lambda,\mu,m,p)}r^{1-p}$$

and

$$|f(z)| \leq \frac{1}{r^p} + \frac{2p\beta(1-\alpha)}{[\beta(1-p+(2\alpha-1)p)+1]\Phi_{1-p}(\lambda,\mu,m,p)}r^{1-p}$$

where equality holds for the function

$$f_p(z) = \frac{1}{z^p} + \frac{2p\beta(1-\alpha)}{[\beta(1-p+(2\alpha-1)p)+1]\Phi_{1-p}(\lambda,\mu,m,p)}z^{1-p}$$

at $z = ir, r$.

Proof. Suppose $f \in \Sigma^{+}_{\lambda\mu mp}(\alpha, \beta)$. Making use of inequality

$$\sum_{k=1-p}^{\infty} a_k \leq \frac{2p\beta(1-\alpha)}{[\beta(1-p+(2\alpha-1)p)+1]\Phi_{1-p}(\lambda,\mu,m,p)} \tag{3.3}$$

which follows easily from Theorem 3.1, the proof is trivial. $\square$

Now, we prove that the class $\Sigma^{+}_{\lambda\mu mp}(\alpha, \beta)$ is closed under convolution.

Theorem 3.3. *Let $h(z) = z^{-p} + \sum_{k=1-p}^{\infty} c_k z^k$ be analytic in $\mathbb{U}^*$ and $0 \le c_k \le 1$. If f given by (1.7) is in the class $\Sigma^+_{\lambda\mu mp}(\alpha, \beta)$, then $f * h$ is also in the class $\Sigma^+_{\lambda\mu mp}(\alpha, \beta)$.*

Proof. Since $f \in \Sigma^+_{\lambda\mu mp}(\alpha, \beta)$, then by Theorem 3.1, we have

$$\sum_{k=1-p}^{\infty} [k(\beta + 1) + p(1 + \beta(2\alpha - 1))]\Phi_k(\lambda, \mu, m, p)a_k \le 2p\beta(1 - \alpha).$$

In view of the above inequality and the fact that

$$(f * h)(z) = z^{-p} + \sum_{k=1-p}^{\infty} a_k c_k z^k$$

we obtain

$$\sum_{k=1-p}^{\infty} [k(\beta + 1) + p(1 + \beta(2\alpha - 1))]\Phi_k(\lambda, \mu, m, p)a_k c_k \le \sum_{k=1-p}^{\infty} [k(\beta + 1) + p(1 + \beta(2\alpha - 1))]\Phi_k(\lambda, \mu, m, p)a_k \le 2p\beta(1 - \alpha). \quad \square$$

Therefore, by Theorem 3.1, the result follows.

The next result involves an integral operator which was investigated in many papers [1,8,9].

Theorem 3.4. *If $f \in \Sigma^+_{\lambda\mu mp}(\alpha, \beta)$, then the integral operator*

$$F_{c,p}(z) = \frac{c}{z^{p+c}} \int_0^z t^{c+p-1} f(t)\mathrm{d}t, \quad c > 0$$

is also in the class $\Sigma^+_{\lambda\mu mp}(\alpha, \beta)$.

Proof. It is easy to check that

$$F_{c,p}(z) = f(z) * \left(z^{-p} + \sum_{k=1-p}^{\infty} \frac{c}{c + p + k} z^k\right).$$

Since $0 < \frac{c}{c+p+k} \le 1$, by Theorem 3.3, the proof is trivial. $\square$

4. Neighborhoods and partial sums

Following earlier investigations on the familiar concept of neighborhoods of analytic functions by Goodman [10], Ruschweyh [11] and more recently by Liu and Srivastava [12,13], Liu [14], Altintaş et al. [15], Orhan and Kamali [16], Srivastava and Orhan [17], Orhan [18], Deniz and Orhan [19] and Aouf [20], we define the (n, δ)-neighborhood of a function $f \in \Sigma_p$ of the form (1.6) as follows.

Definition 4.1. For $\delta > 0$ and a non-negative sequence $S = \{s_k\}_{k=1-p}^{\infty}$ where

$$s_k := \frac{[\beta(k + |2\alpha - 1|p + k + p)]\Phi_k(\lambda, \mu, m, p)}{2p\beta(1 - \alpha)} \quad (k \ge 1 - p,\ p \in \mathbb{N},\ 0 \le \alpha < 1,\ 0 < \beta \le 1) \tag{4.1}$$

the (n, δ)-neighborhood of a function $f \in \Sigma_p$ of the form (1.6) is defined by

$$\mathcal{N}_\delta(f) := \left\{ g \in \Sigma_p : g(z) = z^{-p} + \sum_{k=1-p}^{\infty} b_k z^k \text{ and } \sum_{k=1-p}^{\infty} s_k |b_k - a_k| \le \delta \right\}. \tag{4.2}$$

For $s_k = k$, Definition 4.1 corresponds to the (n, δ)-neighborhood considered by Ruscheweyh [11].

Making use of Definition 4.1, we prove the first result on (n, δ)-neighborhood of the class $\Sigma_{\lambda\mu mp}(\alpha, \beta)$.

Theorem 4.1. *Let $f \in \Sigma_{\lambda\mu mp}(\alpha, \beta)$ be given by (1.6). If f satisfies*

$$(f(z) + \epsilon z^p)(1 + \epsilon)^{-1} \in \Sigma_{\lambda\mu mp}(\alpha, \beta) \quad (\epsilon \in \mathbb{C},\ |\epsilon| < \delta,\ \delta > 0), \tag{4.3}$$

then

$$\mathcal{N}_\delta(f) \subset \Sigma_{\lambda\mu mp}(\alpha, \beta). \tag{4.4}$$

Proof. It is not difficult to see that a function f belongs to $\Sigma_{\lambda\mu mp}(\alpha,\beta)$ if and only if

$$\frac{z(D^m_{\lambda\mu p}f(z))'+pD^m_{\lambda\mu p}f(z)}{\beta z(D^m_{\lambda\mu p}f(z))'+\beta(2\alpha-1)pD^m_{\lambda\mu p}f(z)}\neq\sigma\quad(z\in\mathbb{U},\ \sigma\in\mathbb{C},\ |\sigma|=1)\tag{4.5}$$

which is equivalent to

$$\frac{(f*h)(z)}{z^{-p}}\neq 0\quad(z\in\mathbb{U}),\tag{4.6}$$

where for convenience,

$$\begin{aligned}h(z)&:=z^{-p}+\sum_{k=1-p}^{\infty}c_kz^k\\&=z^{-p}+\sum_{k=1-p}^{\infty}\frac{[\beta\sigma(k+(2\alpha-1)p)-(k+p)]\Phi_k(\lambda,\mu,m,p)}{2p\beta(1-\alpha)\sigma}z^k.\end{aligned}\tag{4.7}$$

From (4.7) it follows that

$$\begin{aligned}|c_k|&=\left|\frac{[\beta\sigma(k+(2\alpha-1)p)-(k+p)]\Phi_k(\lambda,\mu,m,p)}{2p\beta(1-\alpha)\sigma}\right|\\&\leq\frac{[\beta\sigma(k+|2\alpha-1|p)+k+p]\Phi_k(\lambda,\mu,m,p)}{2p\beta(1-\alpha)\sigma}\quad(k\geq 1-p,p\in\mathbb{N}).\end{aligned}\tag{4.8}$$

Furthermore, under the hypotheses (4.3), using (4.6) we obtain the following assertions:

$$\frac{((f(z)+\epsilon z^p)(1+\epsilon)^{-1})*h(z)}{z^{-p}}\neq 0\quad(z\in\mathbb{U})$$

or

$$\frac{(f*h)(z)}{z^{-p}}\neq\epsilon\quad(z\in\mathbb{U}),$$

which is equivalent to

$$\left|\frac{(f*h)(z)}{z^{-p}}\right|\geq\delta\quad(z\in\mathbb{U},\ \delta>0).\tag{4.9}$$

Now, if we let

$$g(z)=z^{-p}+\sum_{k=1-p}^{\infty}b_kz^k\in\mathcal{N}_\delta(f),$$

then we have

$$\begin{aligned}\left|\frac{(f(z)-g(z))*h(z)}{z^{-p}}\right|&=\left|\sum_{k=1-p}^{\infty}(a_k-b_k)z^{k+p}\right|\\&\leq\sum_{k=1-p}^{\infty}\frac{[\beta(k+|2\alpha-1|p)+k+p]\Phi_k(\lambda,\mu,m,p)}{2p\beta(1-\alpha)}|a_k-b_k||z|^{k+p}<\delta\quad(z\in\mathbb{U},\ \delta>0,\ k\geq 1-p,\ p\in\mathbb{N}).\end{aligned}$$

Thus, for any complex number σ with $|\sigma|=1$, we have

$$\frac{(g*h)(z)}{z^{-p}}\neq 0\quad(z\in\mathbb{U})$$

which implies $g\in\Sigma_{\lambda\mu mp}(\alpha,\beta)$. The proof of the theorem is completed. □

In the sequence we give the definition of (n,δ)-neighborhood of a function $f\in\Sigma_p^+$ of the form (1.7).

Definition 4.2. For $\delta>0$ and a non-negative sequence $S=\{s_k\}_{k=1-p}^{\infty}$ where

$$s_k:=\frac{[k(\beta+1)+p(1+\beta(2\alpha-1))]\Phi_k(\lambda,\mu,m,p)}{2p\beta(1-\alpha)}\quad(k\geq 1-p,\ p\in\mathbb{N},\ 0\leq\alpha<1,\ 0<\beta\leq 1)$$

the (n, δ)-neighborhood of a function $f \in \Sigma_p^+$ of the form (1.7) is defined by

$$\tilde{\mathcal{N}}_\delta(f) := \left\{ g \in \Sigma_p^+ : g(z) = z^{-p} + \sum_{k=1-p}^{\infty} b_k z^k \text{ and } \sum_{k=1-p}^{\infty} s_k |b_k - a_k| \le \delta \right\}. \tag{4.10}$$

We have the following result on (n, δ)-neighborhood of the class $\Sigma_{\lambda\mu mp}^+(\alpha, \beta)$.

Theorem 4.2. *If the function f given by (1.7) is in the class $\Sigma_{\lambda\mu mp}^+(\alpha, \beta)$, then*

$$\tilde{\mathcal{N}}_\delta(f) \subset \Sigma_{\lambda\mu mp}^+(\alpha, \beta), \tag{4.11}$$

where

$$\delta = \frac{2\lambda\mu + \lambda - \mu}{1 + 2\lambda\mu + \lambda - \mu}.$$

The result is the best possible in the sense that δ cannot be increased.

Proof. For a function $f \in \Sigma_{\lambda\mu mp}^+(\alpha, \beta)$ of the form (1.7), Theorem 3.1 immediately yields

$$\sum_{k=1-p}^{\infty} \frac{[k(\beta+1) + p(1 + \beta(2\alpha - 1))]\Phi_k(\lambda, \mu, m, p)}{2p\beta(1-\alpha)} a_k \le \frac{1}{\Phi_{1-p}(\lambda, \mu, 1, p)}. \tag{4.12}$$

Let

$$g(z) = z^{-p} + \sum_{k=1-p}^{\infty} b_k z^k \in \tilde{\mathcal{N}}_\delta(f)$$

with

$$\delta = \frac{2\lambda\mu + \lambda - \mu}{1 + 2\lambda\mu + \lambda - \mu} > 0.$$

From condition (4.10) we find that

$$\sum_{k=1-p}^{\infty} s_k |b_k - a_k| \le \delta. \tag{4.13}$$

Using (4.12) and (4.13), we obtain

$$\begin{aligned} \sum_{k=1-p}^{\infty} s_k b_k &\le \sum_{k=1-p}^{\infty} s_k a_k + \sum_{k=1-p}^{\infty} s_k |b_k - a_k| \\ &\le \frac{1}{\Phi_{1-p}(\lambda, \mu, 1, p)} + \delta = 1. \end{aligned}$$

Thus, in view of Theorem 3.1, we get $g \in \Sigma_{\lambda\mu mp}^+(\alpha, \beta)$.

To prove the sharpness of the assertion of the theorem, we consider the functions $f \in \Sigma_{\lambda\mu mp}^+(\alpha, \beta)$ and $g \in \Sigma_p^+$ given by

$$f(z) = z^{-p} + \frac{2p\beta(1-\alpha)}{[\beta(1 - p + (2\alpha - 1)p) + 1]\Phi_{1-p}(\lambda, \mu, m, p)} z^{1-p} \tag{4.14}$$

and

$$\begin{aligned} g(z) &= z^{-p} + \left[\frac{2p\beta(1-\alpha)}{[\beta(1 - p + (2\alpha - 1)p) + 1]\Phi_{1-p}(\lambda, \mu, m, p)} \right. \\ &\quad \left. + \frac{2p\beta(1-\alpha)}{[\beta(1 - p + (2\alpha - 1)p) + 1]\Phi_{1-p}(\lambda, \mu, m, p)} \delta^* \right] z^{1-p} \end{aligned} \tag{4.15}$$

where $\delta^* > \delta$.

Clearly, the function g belongs to $\tilde{\mathcal{N}}_\delta(f)$ but according to Theorem 3.1, $g \notin \Sigma_{\lambda\mu mp}^+(\alpha, \beta)$. Consequently, the proof of our theorem is completed. □

Next, we investigate the ratio of real parts of functions of the form (1.6) and their sequences of partial sums defined by

$$k_m(z) = \begin{cases} z^{-p}, & m = 1, 2, \ldots, -p \\ z^{-p} + \sum_{k=1-p}^{m-1} a_k z^k, & m = 1-p, 2-p, \ldots. \end{cases} \tag{4.16}$$

We also determine sharp lower bounds for $\Re\{\frac{f(z)}{k_m(z)}\}$ and $\Re\{\frac{k_m(z)}{f(z)}\}$.

Theorem 4.3. *Let $f \in \Sigma_p$ be given by (1.6) and let $k_m(z)$ be given by (4.16). Suppose that*

$$\sum_{k=1-p}^{\infty} \theta_k |a_k| \leq 1 \tag{4.17}$$

where

$$\theta_k = \frac{[k(\beta+1) + p(1+\beta(2\alpha-1))]\Phi_k(\lambda, \mu, m, p)}{2p\beta(1-\alpha)}.$$

Then, for $m \geq 1-p$, we have

$$\Re\left\{\frac{f(z)}{k_m(z)}\right\} > 1 - \frac{1}{\theta_m} \tag{4.18}$$

and

$$\Re\left\{\frac{k_m(z)}{f(z)}\right\} > \frac{\theta_m}{1+\theta_m}. \tag{4.19}$$

The results are sharp for each $m \geq 1-p$ with the extremal function given by

$$f(z) = z^{-p} - \frac{1}{\theta_m} z^m. \tag{4.20}$$

Proof. Under the hypotheses of the theorem, we can see from (4.17) that

$$\theta_{k+1} > \theta_k > 1 \quad (k \geq 1-p).$$

Therefore, by using hypotheses (4.17) again, we have

$$\sum_{k=1-p}^{m-1} |a_k| + \theta_m \sum_{k=m}^{\infty} |a_k| \leq \sum_{k=1-p}^{\infty} \theta_k |a_k| \leq 1. \tag{4.21}$$

Let

$$\omega(z) = \theta_m \left[\frac{f(z)}{k_m(z)} - \left(1 - \frac{1}{\theta_m}\right)\right] = 1 + \frac{\theta_m \sum_{k=m}^{\infty} a_k z^{k+p}}{1 + \sum_{k=1-p}^{m-1} a_k z^{k+p}}. \tag{4.22}$$

Applying (4.21) and (4.22), we find

$$\left|\frac{\omega(z)-1}{\omega(z)+1}\right| = \left|\frac{\theta_m \sum_{k=m}^{\infty} a_k z^{k+p}}{2 + 2\sum_{k=1-p}^{m-1} a_k z^{k+p} + \theta_m \sum_{k=m}^{\infty} a_k z^{k+p}}\right| \leq \frac{\theta_m \sum_{k=m}^{\infty} |a_k|}{2 - 2\sum_{k=1-p}^{m-1} |a_k| - \theta_m \sum_{k=m}^{\infty} |a_k|} \leq 1 \quad (z \in \mathbb{U}) \tag{4.23}$$

which shows that $\Re\omega(z) > 0$ $(z \in \mathbb{U})$. From (4.22), we immediately obtain (4.18).

To prove that the function f defined by (4.20) gives sharp result, we can see that for $z \to 1^-$

$$\frac{f(z)}{k_m(z)} = 1 - \frac{1}{\theta_m} z^m \to 1 - \frac{1}{\theta_m}$$

which shows that the bound in (4.18) is the best possible.

Similarly, if we let

$$\phi(z) = (1+\theta_m)\left[\frac{k_m(z)}{f(z)} - \frac{\theta_m}{1+\theta_m}\right] = 1 - \frac{(1+\theta_m)\sum_{k=m}^{\infty} a_k z^{k+p}}{1+\sum_{k=1-p}^{\infty} a_k z^{k+p}} \tag{4.24}$$

and making use of (4.21), we find that

$$\left|\frac{\phi(z)-1}{\phi(z)+1}\right| = \left|\frac{-(1+\theta_m)\sum_{k=m}^{\infty} a_k z^{k+p}}{2+2\sum_{k=1-p}^{\infty} a_k z^{k+p} - (1+\theta_m)\sum_{k=m}^{\infty} a_k z^{k+p}}\right|$$

$$\leq \frac{(1+\theta_m)\sum_{k=m}^{\infty} |a_k|}{2-2\sum_{k=1-p}^{\infty} |a_k| + (1+\theta_m)\sum_{k=m}^{\infty} |a_k|} \leq 1 \quad (z \in \mathbb{U}) \tag{4.25}$$

which leads immediately to the assertion (4.19) of the theorem.

The bound in (4.19) is sharp for each $m \geq 1-p$, with the extremal function f given by (4.20). The proof of the theorem is now completed. □

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