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On the structure of maximum 2-part Sperner families

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Abstract

Color the elements of a finite set S with two colors. A collection of subsets of S is called a 2-*part Sperner family* if whenever for two distinct sets A and B in this collection we have $A \subset B$ then $B - A$ has elements of S of both colors. All 2-part Sperner families of maximum size were characterized in Erdős and Katona (1986). In this paper we provide a different, and quite elementary proof of the structure and number of all maximum 2-part Sperner families, using only some elementary properties of symmetric chain decompositions of the poset of all subsets of a finite set.

1. Introduction

The celebrated Sperner's theorem [15] or [1, Theorem 1.2.1] states that the maximum size of an antichain in the poset of all subsets of a finite set S of size n is equal to $\binom{n}{\lfloor n/2 \rfloor}$. A complement to this theorem [1, Theorem 1.2.2] states that the only antichains of this maximum size are either all the subsets of S of size $\lfloor n/2 \rfloor$ or all the subsets of S of size $\lceil n/2 \rceil$. Thus if n is even, there is only one maximum antichain, and if n is odd, there are two such antichains.

One generalization (there are many others, see [1, 8]) was obtained independently by Katona [11] and Kleitman [13]. Color the elements of S with two colors. Call a collection of subsets of S a 2-*part Sperner family* if whenever for two distinct sets A and B in this collection we $A \subset B$ then $B - A$ has elements of S of both colors. Thus no matter what the coloring, an antichain (which could be called a 1-part Sperner family) is always a 2-part Sperner family. Rather surprisingly this weakening of the antichain condition does not yield larger 2-part Sperner families. The Katona–Kleitman theorem [1, Theorem 11.2.1] asserts that the maximum size of a 2-part Sperner family of S is still $\binom{n}{\lfloor n/2 \rfloor}$, regardless of the coloring!

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The next obvious question is: What are the 2-part Sperner families of maximum size? Here, the answer is not as straightforward as in the original Sperner theorem, however, much simple structure still remains. The structure and number of all 2-part Sperner families of maximum size was discovered and proved in [5] (the author became aware of this work after the present paper was finished). In their proof, Peter Erdős and Gyula Katona [5] use an LYM inequality and results on extreme points of profile matrices [6]. Their method is powerful and can be used in many other settings (for example see [4, 14]. In [10], Mark Huber uses this method to extend their result to the poset of the submultisets of certain kinds of multisets.) In this note we provide a different, and quite elementary proof of the structure and number of all maximum 2-part Sperner families, using only some elementary properties of symmetric chain decompositions of the poset of all subsets of a finite set. This same method, which (like Sperner's original proof) does not appeal to the LYM inequality, will also give a new proof of the characterization of the maximum antichains of this poset.

Two excellent sources for more background, and interesting related material are the survey article of Greene and Kleitman [8] and the book of Anderson [1]. A more recent survey can be found in [7].

2. The maximum 2-part Sperner families

We introduce the following notation: Let $S = R \cup W$ be a finite set with n elements, with $R \cap W = \emptyset$, in other words elements of S are colored using two colors W (hite) and R (ed). For k a nonnegative integer, $\binom{S}{k}$ will denote the collection of k-subsets of S, and $\binom{R}{k}\binom{W}{l}$ will denote the collection of all subsets of S with exactly k elements from R and l elements from W. Thus the only maximum antichains in the poset of all subsets of S is $\binom{S}{\lfloor n/2 \rfloor}$ or $\binom{S}{\lfloor n/2 \rfloor}$. We are now ready to write down exactly all the 2-part Sperner families. Just as with one color, the case n even will be different from n odd.

Theorem 1. *Let $S = R \cup W$ be a finite set with $R \cap W = \emptyset$. Write $n = |S|$, $w = |W|$, and $r = |R|$. Assume $n = w + r$ is even, and $r \geqslant w$. For $i = 0, \ldots, \lfloor w/2 \rfloor$, let*

$$X_i = \binom{R}{\lfloor r/2 \rfloor - i}\binom{W}{\lfloor w/2 \rfloor - i} \cup \binom{R}{\lceil r/2 \rceil + i}\binom{W}{\lceil w/2 \rceil + i},$$

$$Y_i = \binom{R}{\lceil r/2 \rceil + i}\binom{W}{\lfloor w/2 \rfloor - i} \cup \binom{R}{\lfloor r/2 \rfloor - i}\binom{W}{\lceil w/2 \rceil + i}.$$

Let $\mathscr{F}$ be a collection of subsets of S. Then $\mathscr{F}$ is a 2-part Sperner family of maximum possible size if and only if

$$\mathscr{F} = Z_0 \cup Z_1 \cup Z_2 \cup \cdots \cup Z_{\lfloor w/2 \rfloor},$$

where Z_i is either X_i or Y_i, for $i = 0, \ldots, \lfloor w/2 \rfloor$.

Corollary 2. *Let $S = R \cup W$ be a finite set with $R \cap W = \emptyset$, and $|R| \geqslant |W|$. Assume $|S|$ is even and write $w = |W|$. Then the number of 2-part Sperner families of maximum size in the poset of all subsets of S is $2^{\lceil w/2 \rceil}$.*

Proof. This is immediate from Theorem 1. Just note that in the case when w, and hence r, is odd, we have two choices for each of the Zs. Whereas, in the case when w is even, we have $X_0 = Y_0$ and thus Z_0 is fixed. □

Theorem 3. *Let $S = R \cup W$ be a finite set with $R \cap W = \emptyset$. Write $n = |S|$, $w = |W|$, and $r = |R|$. Assume $n = w + r$ is odd, and $r \geqslant w$. For $i = 0, \ldots, w$, let*

$$a_i = \left\lfloor \frac{r}{2} \right\rfloor + (-1)^{i+1} \left\lfloor \frac{i+1}{2} \right\rfloor,$$

$$b_i = \left\lceil \frac{r}{2} \right\rceil + (-1)^{i} \left\lfloor \frac{i+1}{2} \right\rfloor,$$

$$c_i = \left\lfloor \frac{w}{2} \right\rfloor + (-1)^{i+1} \left\lfloor \frac{i+1}{2} \right\rfloor,$$

$$d_i = \left\lceil \frac{w}{2} \right\rceil + (-1)^{i} \left\lfloor \frac{i+1}{2} \right\rfloor.$$

Let $\mathscr{F}$ be a collection of subsets of S. Then $\mathscr{F}$ is a 2-part Sperner family of maximum possible size if and only if

$$\mathscr{F} = \binom{R}{e_0}\binom{W}{f_0} \cup \binom{R}{e_1}\binom{W}{f_1} \cup \cdots \cup \binom{R}{e_w}\binom{W}{f_w},$$

where for $i = 0, \ldots, w$,

$$e_i = a_i \quad \text{or} \quad b_i, \qquad f_i = c_i \quad \text{or} \quad d_i,$$

and for $i = 0, \ldots, w-1$,

$$e_{i+1} \neq e_i, \qquad f_{i+1} \neq f_i.$$

Corollary 4. *Let $S = R \cup W$ be a finite set with $R \cap W = \emptyset$, and $|R| \geqslant |W|$. Assume $|S|$ is odd and write $w = |W|$. Then the number of 2-part Sperner families of maximum size in the poset of all subsets of S is 2^{w+1}.*

Proof. Continuing with the notation of Theorem 3, let $Z_i = \binom{R}{e_i}\binom{W}{f_i}$. Now by Theorem 3, a maximum 2-part Sperner family $\mathscr{F}$ has the form $Z_0 \cup Z_1 \cup \cdots \cup Z_w$. Now the corollary follows from the fact that based on our choice for $Z_0, \ldots, Z_i$, we have exactly two choices for each Z_{i+1}. To see this consider the case when r is even and w is odd. In this case we have $a_0 = b_0$, and $c_0 \neq d_0$. Thus we have 2 choices for Z_0. Also for i even, the only equalities among the relevant numbers is $c_i = d_{i+1}$, and $d_i = c_{i+1}$. So given our choice for e_i and f_i, the choice for f_{i+1} is determined, and we have two choices for e_{i+1}. For i odd, the relevant equalities are $a_i = b_{i+1}$ and $b_i = a_{i+1}$, which means that given our choice of e_i and f_i, there is only one choice for e_{i+1} but two choices for f_{i+1}. In any case there is always exactly two choices for Z_{i+1}. The case when r is odd, and w is even is quite similar. □

The most striking feature of the above theorems is the fact that maximum sized 2-part Sperner families are the union of collection of sets of form $\binom{R}{k}\binom{W}{l}$. In other words, when constructing a maximum 2-part Sperner family, if you pick one subset of S with k red elements and l white elements, then you have to pick all such subsets! Thus in finding maximum 2-part Sperner families, one can consider a much smaller partially ordered set: Partially order the collection

$$\left\{\binom{R}{k}\binom{W}{l} | k = 0, \ldots, |R|, l = 0, \ldots, |W|\right\},$$

by declaring $\binom{R}{k_1}\binom{W}{l_1} \leqslant \binom{R}{k_2}\binom{W}{l_2}$ if and only if $k_1 \leqslant k_2$, and $l_1 \leqslant l_2$ (This is a subposet of the poset of antichains defined by Dilworth [3], see [1, chapter 13]). Now according to the theorems, a maximum 2-part Sperner family is the union of elements of a 2-part Sperner family of this new (and smaller) poset. For example, consider the case when S has ten elements with 6 of them red and 4 of them white. We make a 7×5 table, with its rows indexed by $0, \ldots, 6 = |R|$, and its columns indexed by $0, \ldots, 4 = |W|$. The (i, j) entry of this table will be the size of $\binom{R}{i}\binom{W}{j}$:

		W				
		0	1	2	3	4
	0	1	4	6	4	1
	1	6	24	36	24	6
	2	15	60	90	60	15
R	3	20	80	120	80	20
	4	15	60	90	60	15
	5	6	24	36	24	6
	6	1	4	6	4	1

To find a maximum 2-part Sperner family we cannot pick two collections from any row or any column. Now Theorem 1 says that we have to pick the 120 elements each with 3 red and 2 white elements, and then we have to choose two of the 60 element collections, either entries (2, 1) and (4, 3), or entries (4, 1) and (2, 3). After these choices we have to pick two of the 6 element collections, either entries (1, 0) and (5, 4) or entries (5, 0) and (1, 4). Thus there are 4 maximum 2-part Sperner families, and each does have $120 + 60 + 60 + 6 + 6 = 252 = \binom{10}{\lfloor 10/2 \rfloor}$ elements. Note that the above table is much smaller than the Hasse diagram for the poset of all subsets of a set with 10 elements, and for all cases we can draw similar tables with similar patterns (the pattern will be slightly different for $|S|$ odd, since the 2 choices at each stage depend on the choices that came before).

For the proof of the theorems we need some easy facts about symmetric chain decompositions which we will discuss next.

3. Preliminaries on symmetric chains

Let S be a set with n elements, and let $\mathscr{P}(S)$ denote the set of all subsets of S. A chain of elements of $\mathscr{P}(S)$,

$$A_1 \subset A_2 \subset \cdots \subset A_h$$

form a *symmetric chain* if for $i = 1, \ldots, h-1$, A_{i+1} has one more element than A_i, and $|A_1| + |A_h| = n$ [1, chapter 3]. De Bruijn, et al. [2] showed that $\mathscr{P}(S)$ can be written as a disjoint union of symmetric chains [1, Theorem 3.1.1]. Any such union is called a *symmetric chain decomposition* of $\mathscr{P}(S)$. There are many different symmetric chain decompositions for $\mathscr{P}(S)$, but all of these consist of $\binom{n}{\lfloor n/2 \rfloor}$ chains. If as usual, the length of a chain is one less than the number of subsets in it, then a symmetric chain decomposition of $\mathscr{P}(S)$ will always have $1 = \binom{n}{0}$ chain of length n, $\binom{n}{1} - \binom{n}{0}$ chains of length $n-2$, $\binom{n}{2} - \binom{n}{1}$ chains of length $n-4$ and so on. The following easy lemma will be very useful.

Lemma 5. *Let $S = \{1, 2, \ldots, n\}$. Given an arbitrary symmetric chain $\mathscr{A}$ of subsets of S, we can find other symmetric chains so that these together with $\mathscr{A}$ form a symmetric chain decomposition of $\mathscr{P}(S)$.*

Proof. We know that $\mathscr{P}(S)$ has some symmetric chain decomposition, and because of the comment before the lemma, in this symmetric chain decomposition there will be at least one symmetric chain $\mathscr{B}$ with the same length as $\mathscr{A}$. It is not hard to see that there is a relabeling of elements of S that turns $\mathscr{B}$ into $\mathscr{A}$. Now if we apply this relabeling to the whole symmetric chain decomposition we get another symmetric chain decomposition and one of the new symmetric chains will be $\mathscr{A}$. $\square$

Let $\mathscr{A} : A_1 \subset A_2 \subset \cdots \subset A_r$ be a symmetric chain in $\mathscr{P}(S)$, and let $\mathscr{B} : B_1 \subset B_2 \subset \cdots \subset B_t$ be a symmetric chain in $\mathscr{P}(T)$. Then the *symmetric rectangle* $\mathscr{A} \times \mathscr{B}$ is defined to be

$$\begin{matrix} A_1 \cup B_1 & A_1 \cup B_2 & \cdots & A_1 \cup B_t \\ A_2 \cup B_1 & A_2 \cup B_2 & \cdots & A_2 \cup B_t \\ \vdots & \vdots & \ddots & \vdots \\ A_r \cup B_1 & A_r \cup B_2 & \cdots & A_r \cup B_t \end{matrix}$$

Katona [12] used symmetric chain decompositions in each color class to study three-part Sperner families, and Griggs [9] used symmetric rectangles to give a simple proof of the Katona–Kleitman theorem on the size of a maximum 2-part Sperner family [1, Theorem 11.2.1]. The importance of symmetric rectangles is that a 2-part Sperner family can have at most one member from each row and each column of such a rectangle. The following Lemma is a slight strengthening of the Katona–Kleitman theorem on the size of a maximum 2-part Sperner family.

Lemma 6. *Let $S = R \cup W$ be a finite set with n elements, and with $R \cap W = \emptyset$. Let $\mathscr{A}$ be a symmetric chain in $\mathscr{P}(R)$ of length t, and let $\mathscr{B}$ be a symmetric chain in $\mathscr{P}(W)$ of length s. Then*

1. *The number of subsets of size $\lfloor n/2 \rfloor$ in $\mathscr{A} \times \mathscr{B}$ is $min(t+1, s+1)$.*
2. *Let $\mathscr{F}$ be a maximum 2-part Sperner family in $\mathscr{P}(S)$; then $\mathscr{F}$ contains exactly $min(t+1, s+1)$ subsets from $\mathscr{A} \times \mathscr{B}$, and no two of these are in the same row or column. The total number of subsets in $\mathscr{F}$ will be $\binom{n}{\lfloor n/2 \rfloor}$.*

Proof. 1. Since both chains are symmetric, for each subset in the shorter chain there is exactly one subset in the longer one such that there union has size $\lfloor n/2 \rfloor$.

2. By Lemma 5, there is a symmetric chain decomposition of R that contains $\mathscr{A}$, and a symmetric chain decomposition of W that contains $\mathscr{B}$. Construct all possible symmetric rectangles. By the definition of a 2-part Sperner family no two elements of $\mathscr{F}$ can be in the same row or column of the same symmetric rectangle. Thus $\mathscr{F}$ contains at most m sets from each rectangle, where m is the minimum of the number of rows and the number of columns. By the first claim this minimum is equal to the number of subsets of size $\lfloor n/2 \rfloor$. Since every subset of S is in one of these rectangles, we have $|\mathscr{F}| \leqslant \binom{n}{\lfloor n/2 \rfloor}$. But we have equality since $\binom{S}{\lfloor n/2 \rfloor}$ is a 2-part Sperner family. This also means that $\mathscr{F}$ must include from each rectangle the maximum number possible, and hence the proof is complete. □

4. Proofs of the theorems

To prove each of the two theorems, we first need a lemma. The following notation will be fixed: $S = R \cup W$ is a finite set with n elements and $R \cap W = \emptyset$. We let $r = |R|$, and $w = |W|$.

Let $\mathscr{F}$ be a 2-part Sperner family, and let k_1, k_2, l_1, and l_2 be integers with $0 \leqslant k_1 < k_2 \leqslant r$, and $0 \leqslant l_1 < l_2 \leqslant w$. *We will say that* $\mathscr{F}$ satisfies condition (C1) for k_1, k_2, l_1, and l_2 if given any four sets $C_{11}, C_{12}, C_{21}, C_{22}$ such that

1. $C_{ij} \in \binom{R}{k_i}\binom{W}{l_j}$ for $i, j = 1, 2$.

$$2.\quad \begin{array}{ccc} C_{11} & \subset & C_{12} \\ \cap & & \cap \\ C_{21} & \subset & C_{22} \end{array}.$$

then $\mathscr{F}$ contains two of the four sets.

Of course in the above situation since $\mathscr{F}$ is a 2-part Sperner family, either C_{11} and C_{22} or C_{12} and C_{21} are in $\mathscr{F}$.

Lemma 7. *Let $\mathscr{F}$ be a 2-part Sperner family in $\mathscr{P}(S)$. Assume that $\mathscr{F}$ satisfies condition* (C1), *and furthermore at least one element of $\binom{R}{k_i}\binom{W}{l_j}$ is in $\mathscr{F}$, for some $i, j \in 1, 2$. Then every element of $\binom{R}{k_i}\binom{W}{l_j}$ will be in $\mathscr{F}$.*

Proof. We will prove one case. All other cases will be similar. Assume $\mathscr{F} \cap \binom{R}{k_1}\binom{W}{l_1}$ is not empty, and yet $\binom{R}{k_1}\binom{W}{l_1}$ is not completely contained in $\mathscr{F}$. Choose subsets $C_{11}, C'_{11} \in \binom{R}{k_1}\binom{W}{l_1}$ such that C_{11} is in $\mathscr{F}$, C'_{11} is not in $\mathscr{F}$, and $|C_{11} \cap C'_{11}|$ is as large as possible.

Let x be an element of C_{11} that is not in C'_{11}. Without loss of generality assume $x \in R$. Thus there exists another element of R, y, that is in C'_{11} but not in C_{11}. Define some subsets of S as follows:

$$C_{21} = C_{11} \cup \{y\} \cup \{k_2 - k_1 - 1 \text{ other arbitrary elements of } R\},$$

$$C_{12} = C_{11} \cup \{l_2 - l_1 \text{ other arbitrary elements of } W\},$$

$$C_{22} = C_{21} \cup C_{12}.$$

Now C_{11}, C_{12}, C_{21}, and C_{22} satisfy the conditions of condition (C1), and thus two of them must be in $\mathscr{F}$. C_{11} is already in $\mathscr{F}$, and so C_{22} must be in $\mathscr{F}$. Now define two other subsets as follows:

$$C''_{11} = (C_{11} - \{x\}) \cup \{y\},$$

$$C''_{12} = (C_{12} - \{x\}) \cup \{y\}.$$

Again $C''_{11}, C''_{12}, C_{21}$, and C_{22} satisfy the conditions of condition (C1), and thus two of them must be in $\mathscr{F}$. C_{22} is already in $\mathscr{F}$, which forces C''_{11} to be in $\mathscr{F}$. However, $|C''_{11} \cap C'_{11}| > |C_{11} \cap C'_{11}|$ which is a contradiction. $\square$

We are now ready to prove Theorem 1.

Proof of Theorem 1. To prove one direction assume that $\mathscr{F} = \bigcup_{i=0}^{\lfloor w/2 \rfloor} Z_i$ as in the statement of the theorem. $\mathscr{F}$ is clearly a 2-part Sperner family and we can calculate its size easily. Since n is even, r and w are both even or both odd. If r is even, we have $|X_0| = |Y_0| = \binom{r}{\lceil r/2 \rceil}\binom{w}{\lfloor w/2 \rfloor}$. For r even and $i > 0$, or for r odd and $i \geqslant 0$, we have:

$$|X_i| = |Y_i| = \binom{r}{\lceil r/2 \rceil + i}\binom{w}{\lfloor w/2 \rfloor - i} + \binom{r}{\lfloor r/2 \rfloor - i}\binom{w}{\lceil w/2 \rceil + i}.$$

Now $|\mathscr{F}| = \sum_{i=0}^{\lfloor w/2 \rfloor} |Y_i|$, which is exactly the number of ways to pick $\lceil r/2 \rceil + \lfloor w/2 \rfloor = \lfloor r/2 \rfloor + \lceil w/2 \rceil = \lfloor n/2 \rfloor$ elements from S. Thus $|\mathscr{F}| = \binom{n}{\lfloor n/2 \rfloor}$, and $\mathscr{F}$ is a maximum 2-part Sperner family by Lemma 6.

For the other direction, we will show that $Z_i \subset \mathscr{F}$ for $i = 0, \dots, \lfloor w/2 \rfloor$. We use induction on i. For $i = 0$ there are two cases:

Case 1: $r = 2l$, and $w = 2m$ are both even. Then $X_0 = \binom{R}{l}\binom{W}{m} = Y_0$. Let $A \in \binom{R}{l}$ and $B \in \binom{W}{m}$. Both singletons $\{A\}$ and $\{B\}$ are symmetric chains of respectively R and W, and the 1×1 symmetric rectangle made from the product of these two chains will have only one element: $A \cup B$. Thus Lemma 6 applies and $A \cup B$ must be in $\mathscr{F}$. Thus $Z_0 \subset \mathscr{F}$.

Case 2: $r = 2l + 1$ and $w = 2m + 1$, are both odd. We claim that $\mathscr{F}$ must satisfy condition (C1) for $l, l+1, m, m+1$. Assume that we are given any four sets $C_{ij} \in \binom{R}{l+i}\binom{W}{m+j}$ for $i, j = 0, 1$, such that

$$\begin{array}{ccc} C_{00} & \subset & C_{01} \\ \cap & & \cap \\ C_{10} & \subset & C_{11}. \end{array}$$

The above is a symmetric rectangle that can be obtained from two symmetric chains of length one. Thus by Lemma 6 two of these sets must be in $\mathscr{F}$ and the condition (C1) is satisfied. Now Lemma 7 applies and we get that X_0 or Y_0 are completely contained in $\mathscr{F}$. Thus $Z_0 \subset \mathscr{F}$.

Now consider the case $i > 0$. We assume by induction that $Z_i \subset \mathscr{F}$ for $i = 0, \dots, m-1$, and will show that $Z_m \subset \mathscr{F}$.

Let $k_1 = \lfloor r/2 \rfloor - m, k_2 = \lceil r/2 \rceil + m, l_1 = \lfloor w/2 \rfloor - m$, and $l_2 = \lceil w/2 \rceil + m$. We claim that $\mathscr{F}$ satisfies condition (C1) for k_1, k_2, l_1, and l_2 and hence by Lemma 7 all of X_m or all of Y_m is contained in $\mathscr{F}$, which would finish the proof.

Let C_{ij} be given as in the definition of condition (C1). For $i = 1, 2$ we can find $A_i \in \binom{R}{k_i}$ and $B_j \in \binom{W}{l_j}$ such that $A_1 \subset A_2$, $B_1 \subset B_2$, and $C_{ij} = A_i \cup B_j$.

Now $|A_1| + |A_2| = \lfloor r/2 \rfloor + \lceil r/2 \rceil = r$, and similarly $|B_1| + |B_2| = w$. Hence, the chains $A_1 \subset A_2$ and $B_1 \subset B_2$ can be refined into symmetric chains $\mathscr{A}$ and $\mathscr{B}$ of $\mathscr{P}(R)$ and $\mathscr{P}(W)$ respectively. Both $\mathscr{A}$ and $\mathscr{B}$ are of length $2m + \lceil r/2 \rceil - \lfloor r/2 \rfloor$. By Lemma 6, $\mathscr{A} \times \mathscr{B}$ must contain exactly $2m + 1 + \lceil r/2 \rceil - \lfloor r/2 \rfloor$ elements of $\mathscr{F}$. $\mathscr{A} \times \mathscr{B}$ is a symmetric square and looks as follows:

$$\begin{array}{ccc} A_1 \cup B_1 & \cdots & A_1 \cup B_2 \\ \vdots & \boxed{} & \vdots \\ A_2 \cup B_1 & \cdots & A_2 \cup B_2 \end{array}$$

By the inductive hypothesis $2m - 1 + \lceil r/2 \rceil - \lfloor r/2 \rfloor$ elements from the inside square are already in $\mathscr{F}$, and hence two of the corner elements must be in $\mathscr{F}$. Thus condition (C1) is satisfied, and the theorem is proved using Lemma 7. □

The proof of Theorem 3 parallels the above. We need a condition similar to condition (C1), and a lemma in place of Lemma 7. Because of the similarities we only sketch the proofs here.

Let $\mathscr{F}$ be a 2-part Sperner family, and let k_1, k_2, k_3, l_1, and l_2 be integers with $0 \leqslant k_1 < k_2 < k_3 \leqslant r$, and $0 \leqslant l_1 < l_2 \leqslant w$. We will say that $\mathscr{F}$ satisfies condition (C2) for k_1, k_2, k_3, l_1, and l_2 if given any six sets $C_{11}, C_{12}, C_{21}, C_{22}, C_{31}, C_{32}$ such that

1. $C_{ij} \in \binom{R}{k_i}\binom{W}{l_j}$ for $i = 1, 2, 3$, and $j = 1, 2$,

2. $$\begin{array}{ccc} C_{11} & \subset & C_{12} \\ \cap & & \cap \\ C_{21} & \subset & C_{22} \\ \cap & & \cap \\ C_{31} & \subset & C_{32}. \end{array}$$

then $\mathscr{F}$ contains one of $\{C_{21}, C_{22}\}$, and one of the other four sets.

Lemma 8. *Let $\mathscr{F}$ be a 2-part Sperner family in $\mathscr{P}(S)$. Assume that $\mathscr{F}$ satisfies condition* (C2), *and furthermore at least one element of $\binom{R}{k_2}\binom{W}{l_j}$ is in $\mathscr{F}$, for some $j \in 1, 2$. Then every element of $\binom{R}{k_2}\binom{W}{l_j}$ will be in $\mathscr{F}$.*

Proof. We will consider one case here. The other is very similar. Assume C_{21} and C'_{21} are both elements of $\binom{R}{k_2}\binom{W}{l_1}$, such that C_{21} is in $\mathscr{F}$ and C'_{21} is not and $|C_{21} \cap C'_{21}|$ is as large as possible. Let x be an element of C_{21} but not of C'_{21}. We do the case when $x \in R$ here. So there is another element $y \in R$ such that y is in C'_{21} but not C_{21}. Just as in the proof of Lemma 7 by using two instances of condition (C2) we can show that the set we get by substituting y for x in C_{21} must be in $\mathscr{F}$. This new set, however, has a larger intersection with C'_{21}, and this contradiction proves the lemma. □

There is a corresponding condition (C3) and lemma if the six sets were from $\binom{R}{k_i}\binom{W}{l_j}$ for $i = 1, 2$, and $j = 1, 2, 3$.

Proof of Theorem 3 (*Sketch*). The proof proceeds similarly to that of Theorem 1. It is easy to see that the sizes of the $\mathscr{F}$ defined is the same as the size of a maximum 2-part Sperner family. For the other direction, since one of r and w is odd and the other even, we do not have symmetric squares, and we have to consider symmetric rectangles where the number of rows is one more or one less than the number of columns. As before we proceed by induction and at each stage, the first rectangle considered will provide part of the hypothesis of condition (C2) for the larger rectangle. At each stage depending on our previous choices we will be able to recognize condition (C2) or condition (C3) and apply Lemma 8. □

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