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Large solutions of semilinear elliptic equations under the Keller–Osserman condition ☆

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Abstract

We consider the equation $\Delta u = p(x) f(u)$ where p is a nonnegative nontrivial continuous function and f is continuous and nondecreasing on $[0, \infty)$, satisfies $f(0) = 0$, $f(s) > 0$ for $s > 0$ and the Keller–Osserman condition $\int_1^\infty (F(s))^{-1/2}\, ds = \infty$ where $F(s) = \int_0^s f(t)\, dt$. We establish conditions on the function p that are necessary and sufficient for the existence of positive solutions, bounded and unbounded, of the given equation.
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1. Introduction

We consider the problem

$$\Delta u = p(x) f(u), \quad x \in \Omega, \tag{1}$$

$$u(x) \to \infty, \quad x \to \partial\Omega, \tag{2}$$

where Ω is an open, connected subset of $\mathbf{R}^N$ ($N \geqslant 3$) with smooth boundary, the nonnegative nontrivial function p is continuous on $\overline{\Omega}$ and the nondecreasing continuous function f satisfies

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$f(0)=0$, $f(s)>0$ for $s>0$. Any solution of (1) and (2) is called a *large* solution of (1). If $\Omega=\mathbf{R}^N$, then (2) means that $u(x)\to\infty$ as $|x|\to\infty$ and in such case u is referred to as an *entire large* solution of (1).

Our purpose in this article is to extend some existence and nonexistence results of Keller [2] and Osserman [7] where $p(x)\equiv 1$ under the requirement that f satisfies

$$\int_1^\infty \left[F(s)\right]^{-1/2}\,ds=\infty \qquad \left(F(s)\equiv\int_0^s f(t)\,dt\right). \tag{3}$$

In particular, they prove that a necessary and sufficient condition for (1) (with $p=1$) to have an entire solution is that f satisfies (3). Such a solution will necessarily satisfy (2) and hence be a large solution. We extend this to the case where the function p is spherically symmetric. More generally, however, we are interested in the influence of the function p on existence results. In [3], we extended many of their results in a similar direction. We proved, for example, that if $p(x)\leqslant K|x|^{-\alpha}$, $\alpha>2$, for $|x|$ large, and consequently p satisfies

$$\int_0^\infty r\Big[\max_{|x|=r} p(x)\Big]\,dr<\infty,$$

then a necessary and sufficient condition for (1) to have an entire large solution is that the integral in (3) be finite. Here, instead of fixing the condition on p, we fix the condition on f (i.e., it satisfies (3)) and determine necessary and sufficient conditions on p that ensure that (1) has an entire solution and whether such solutions are bounded or unbounded and, perhaps, large. Our results somewhat parallel those of [5] where Wood and the author considered the case $f(s)=s^\gamma$, $0<\gamma\leqslant 1$. We note that if $f(s)=s^\gamma$, $\gamma>0$, then (3) is equivalent to $\gamma\leqslant 1$. Thus the present results include those of [5] as a special case.

Finally, we note that the study of large solutions for (1) when the integral in (3) is finite has been the subject of many articles. See, for example, [1,3] and their references.

2. Main results

Since Theorem 1 of [3] establishes the nonexistence of a large solution on any bounded domain whenever f satisfies (3), we concentrate here on the case $\Omega=\mathbf{R}^N$. (We note that condition (A) in [3] is not needed in the proof of necessity there.)

Theorem 1. *Let $\Omega=\mathbf{R}^N$ in* (1). *Suppose f satisfies* (3) *and there exists a positive number ε such that p satisfies*

$$\int_0^\infty t^{1+\varepsilon}\phi(t)\,dt<\infty, \quad \textit{where } \phi(t)=\max_{|x|=t} p(x), \tag{4}$$

and $r^{2N-2}\phi(r)$ is nondecreasing for large r. Then Eq. (1) *has a nonnegative nontrivial entire bounded solution on $\mathbf{R}^N$. If, on the other hand, p satisfies*

$$\int_0^\infty t\psi(t)\,dt=\infty, \quad \textit{where } \psi(t)=\min_{|x|=t} p(x), \tag{5}$$

and $r^{2N-2}\psi(r)$ *is nondecreasing for large* r*, then Eq.* (1) *has no nonnegative nontrivial entire bounded solution on* $\mathbf{R}^N$.

Remark. If $f(s)=s^\gamma$, $\gamma>0$, then ε can be taken to be zero and the functions $r^{2N-2}\phi(r)$ and $r^{2N-2}\psi(r)$ need not be monotone for any r. (See [6].)

Open problem. It remains unknown as to whether (1) has an entire large solution if (5) is satisfied, even in the case $f(s)=s^\gamma$, $0<\gamma\leqslant 1$. The best related result seems to be in [4] where, in addition to requiring p to satisfy (5), it is also required that $\phi(s)-\psi(s)$ decay very rapidly to zero as $s\to\infty$.

Proof. Suppose (4) holds. We will show that (1) has a solution by finding an upper solution, v, and a lower solution, w, for which $w\leqslant v$. To do this, we first prove the existence of w to the equation

$$\Delta w=\phi(r)f(w). \tag{6}$$

We note that this equation becomes in this case

$$w''+\frac{N-1}{r}w'=\phi(r)f(w)$$

and that any solution w to the integral equation

$$w(r)=1+\int_0^r t^{1-N}\int_0^t s^{N-1}\phi(s)f\big(w(s)\big)\,ds\,dt,\quad r>0,$$

is a solution to (6). To establish a solution to this equation, we use successive approximation. Let $w_0=1$ and define the sequence $\{w_k\}$ by

$$w_k(r)=1+\int_0^r t^{1-N}\int_0^t s^{N-1}\phi(s)f\big(w_{k-1}(s)\big)\,ds\,dt\quad\text{for all } k \text{ and all } r\geqslant 0.$$

Clearly $w_0\leqslant w_1$ which, in turn, yields $w_1\leqslant w_2$ since f is nondecreasing. Hence the sequence $\{w_k\}$ is an increasing sequence of nonnegative, nondecreasing functions. We now show that the sequence $\{w_k\}$ is bounded above and hence converges. We note that w_k satisfies

$$\big(r^{N-1}w_k'\big)'=r^{N-1}\phi(r)f(w_{k-1}),\quad k\geqslant 1, \tag{7}$$

and the monotonicity of $\{w_k\}$ yields

$$\big(r^{N-1}w_k'\big)'\leqslant r^{N-1}\phi(r)f(w_k). \tag{8}$$

Choose $R>0$ so that $r^{2N-2}\phi(r)$ is nondecreasing for $r\geqslant R$. We first show that $w_k(R)$ and $w_k'(R)$, both of which are nonnegative, are bounded above independent of k. To do this, let $\Phi_R=\max\{\phi(r)\colon 0\leqslant r\leqslant R\}$. Using this and the fact that $w_k'\geqslant 0$, we note that (8) yields

$$w_k''\leqslant \Phi_R f(w_k),\quad 0\leqslant r\leqslant R.$$

Multiply this by w_k' and integrate to get

$$\big(w_k'(r)\big)^2\leqslant 2\Phi_R\int_1^{w_k(r)} f(s)\,ds,\quad 0\leqslant r\leqslant R, \tag{9}$$

which yields

$$\int_1^{w_k(R)} \left[\int_1^t f(s)\,ds \right]^{-1/2} dt \leqslant \sqrt{2\Phi_R} R.$$

From (3), we now conclude that $w_k(R)$ is bounded above independent of k and using this fact in (9) shows that the same is true of $w_k'(R)$. We now show that w_k is bounded for all $r \geqslant 0$ and all k. Multiplying (8) by $r^{N-1}w_k'$ and integrating gives

$$\big(r^{N-1}w_k'(r)\big)^2 \leqslant \big(R^{N-1}w_k'(R)\big)^2 + 2\int_R^r t^{2N-2}\phi(t)\frac{d}{dt}\int_1^{w_k(t)} f(s)\,ds\,dt \quad (r \geqslant R).$$

Using the monotonicity of $t^{2N-2}\phi(t)$ for $t \geqslant R$, we get ($C \equiv (R^{N-1}w_k'(R))^2$)

$$\big(r^{N-1}w_k'(r)\big)^2 \leqslant C + 2r^{2N-2}\phi(r)F\big(w_k(r)\big),$$

which yields

$$w_k'(r) \leqslant \sqrt{C}r^{1-N} + \sqrt{2\phi(r)}\big[F\big(w_k(r)\big)\big]^{1/2} \tag{10}$$

and hence

$$\frac{d}{dr}\int_1^{w_k(r)} \big[F(t)\big]^{-1/2}\,dt \leqslant \sqrt{C}r^{1-N}\big[F\big(w_k(r)\big)\big]^{-1/2} + \sqrt{2\phi(r)}.$$

Integrating this and using the fact that

$$\sqrt{2\phi(r)} = \sqrt{2r^{(1+\varepsilon)/2}\phi(r)r^{(-1-\varepsilon)/2}} \leqslant r^{1+\varepsilon}\phi(r) + r^{-1-\varepsilon}$$

for every $\varepsilon > 0$, we have

$$\begin{aligned}\int_{w_k(R)}^{w_k(r)} \big[F(t)\big]^{-1/2}\,dt &\leqslant \sqrt{C}\int_R^r t^{1-N}\big[F\big(w_k(t)\big)\big]^{-1/2}\,dt \\ &\quad + \int_R^r t^{1+\varepsilon}\phi(t)\,dt + \int_R^r t^{-1-\varepsilon}\,dt \\ &\leqslant \sqrt{C}\big[F\big(w_k(R)\big)\big]^{-1/2}\int_R^r t^{1-N}\,dt \\ &\quad + \int_R^r t^{1+\varepsilon}\phi(t)\,dt + \frac{1}{\varepsilon R^{\varepsilon}}.\end{aligned} \tag{11}$$

Since for each $\varepsilon > 0$ the right side of this inequality is bounded independent of k (note that $w_k(R) \geqslant 1$), so is the left side and hence, in light of (3), the sequence $\{w_k\}$ is a bounded sequence. Thus $w_k \uparrow w$ as $k \to \infty$ and hence w is a solution to (6). Furthermore, $w' \geqslant 0$ and since the sequence $\{w_k\}$ is bounded above, so is w. We let M be the least upper bound of w and note that $M = \lim_{r\to\infty} w(r)$. Now let v be the positive increasing bounded solution of

$$v(r) = M + \int_0^r t^{1-N} \int_0^t s^{N-1}\psi(s) f\big(v(s)\big)\,ds\,dt, \quad r > 0,$$

which, of course, satisfies (6) with w replaced with v. It is also clear that $v \geqslant M$. (The proof of the existence of v and that it has the properties mentioned is virtually identical to the proof for w above and is therefore omitted.) Thus we have that w and v satisfy, respectively,

$$\Delta w \geqslant p(x) f(w), \qquad \Delta v \leqslant p(x) f(v)$$

on $\mathbf{R}^N$ and $w \leqslant v$. Hence the standard upper–lower solution principle (see [8]) implies that (1) has a solution u such that $w \leqslant u \leqslant v$ on $\mathbf{R}^N$, which is the desired solution.

Now assume that (5) holds and that (1) has a nontrivial nonnegative entire bounded solution, u. Let $\bar{u}$ be the spherical mean of u, defined as in [6] and given by

$$\bar{u}(r) = \frac{1}{v_0(S^{N-1}r)} \int_{|x|=r} u(x)\,d\sigma_r \equiv \int_{|x|=r} u(x)\,d\sigma,$$

where $v_0(S^{N-1}r)$ is the volume of the $(N-1)$-dimensional sphere of radius r and σ_r is the measure on the sphere. We have

$$\begin{aligned}\Delta\bar{u} = \bar{u}'' + \frac{N-1}{r}\bar{u}' = \int_{|x|=r} \Delta u\,d\sigma = \int_{|x|=r} p(x) f(u)\,d\sigma \\ \geqslant \psi(r) \int_{|x|=r} f(u)\,d\sigma \equiv \psi(r)\overline{f(u)},\end{aligned}$$

which yields

$$\bar{u}(r) \geqslant \bar{u}(0) + \int_0^r t^{1-N} \int_0^t s^{N-1}\psi(s)\overline{f(u)}(s)\,ds\,dt.$$

Assuming $M = \sup_{x\in\mathbf{R}^N} u(x)$ and knowing that $\bar{u}' \geqslant 0$, we get $\lim_{r\to\infty}\bar{u}(r) = M$. Thus there exists $R > 0$ such that $\bar{u}(r) \geqslant 3M/4$ for $r \geqslant R$. Now define the function $g : [0, M] \to [0, M]$ by $g(s) = 0$ for $0 \leqslant s \leqslant M/2$ and

$$g(s) = \frac{2f(M/2)}{M}s - f(M/2) \quad \text{for } M/2 \leqslant s \leqslant M.$$

Then g is nonnegative nondecreasing convex and $g \leqslant f$ on $[0, M]$. Hence, for $r \geqslant R$,

$$\overline{f(u)}(r) \geqslant \int_{|x|=r} g(u)\,d\sigma \geqslant g\bigg(\int_{|x|=r} u\,d\sigma\bigg) \geqslant g(3M/4) \equiv c_0 > 0.$$

Thus

$$\bar{u}(r) \geqslant \bar{u}(0) + c_0 \int_0^r t^{1-N} \int_0^t s^{N-1}\psi(s)\,ds\,dt \to \infty \quad \text{as } r \to \infty,$$

a contradiction to the boundedness of u. This completes the proof. □

3. The spherically symmetric case

We now consider the radial case (i.e., $p(x) = p(|x|)$), and show that (1) has an entire solution. We would like to show that a necessary and sufficient condition for such a solution to be large is that p satisfies

$$\int_0^\infty rp(r)\,dr = \infty. \tag{12}$$

This is true, for example, if $f(s) = s^\gamma$, $0 < \gamma \leqslant 1$. (See [6].) However, we have been unable to prove this under the Keller–Osserman condition (3). The best we have been able to establish is given here in Theorem 2 below. Thus it remains an open problem.

Theorem 2. *Suppose that p is spherically symmetric (i.e., $p(x) = p(|x|)$) and $\Omega = \mathbf{R}^N$. If f satisfies* (3)*, then Eq.* (1) *has a nonnegative nontrivial entire solution. Suppose furthermore that $r^{2N-2}p(r)$ is nondecreasing for large r. If p satisfies* (12)*, then any nonnegative nontrivial entire solution u of* (1) *is large. Conversely, if* (1) *has a nonnegative entire large solution, then p satisfies*

$$\int_0^\infty r^{1+\varepsilon}p(r)\,dr = \infty \tag{13}$$

for every $\varepsilon > 0$.

Proof. For any $a > 0$ a solution of

$$v(r) = a + \int_0^r t^{1-N}\int_0^t s^{N-1}p(s)f\big(v(s)\big)\,ds\,dt \tag{14}$$

exists for, at least, small r. Since $v' \geqslant 0$, the only way that the solution can become singular at R is for $v(r) \to \infty$ as $r \uparrow R$. Thus, if we can show that, for each $R > 0$, there exists $C_R > 0$ so that $v(R) \leqslant C_R$, we will have existence. To this end, let $M_R = \max\{p(r)\colon\ 0 \leqslant r \leqslant R\}$ and consider the equation

$$w(r) = b + M_R\int_0^r t^{1-N}\int_0^t s^{N-1}f\big(w(s)\big)\,ds\,dt,$$

where $b > a$. The solution to this equation exists for all $r \geqslant 0$ (see [7, Lemma 3]) and of course, it is a solution to $\Delta w = M_R f(w)$ on $\mathbf{R}^N$. We now show that $v(r) \leqslant w(r)$ for all $0 \leqslant r \leqslant R$ and hence complete the proof of existence. Clearly $v(0) < w(0)$ so that $v(r) < w(r)$ for at least all r near zero. Let $r_0 = \sup\{r\colon\ v(s) < w(s) \text{ for all } s \in [0, r]\}$. If $r_0 = R$, then we are done. Thus assume that $r_0 < R$. Then we have

$$v(r_0) = a + \int_0^{r_0} t^{1-N}\int_0^t s^{N-1}p(s)f\big(v(s)\big)\,ds\,dt$$

$$< b + M_R \int_0^{r_0} t^{1-N} \int_0^t s^{N-1} f\big(w(s)\big)\, ds\, dt = w(r_0).$$

Thus there exists $\varepsilon > 0$ so that $v(r) < w(r)$ for all $[0, r_0 + \varepsilon)$, contradicting the definition of r_0. Thus we conclude that $v \leqslant w$ on $[0, R]$ for all $R > 0$ and hence v is a nontrivial entire solution of (1).

Now let u be any nonnegative nontrivial entire solution of (1) and suppose p satisfies (12). Then the proof that $u(r) \to \infty$ as $r \to \infty$ is very similar to part of the proof of Theorem 1 of [6] so we provide details here only of the difference between the two proofs. Indeed since u is nontrivial and nonnegative, there exists $R > 0$ so that $u(R) > 0$. Since $u' \geqslant 0$, we get $u(r) \geqslant u(R) > 0$ for $r \geqslant R$, and thus from (14) (since u will satisfy that equation for all $r \geqslant 0$) we get

$$\begin{aligned} u(r) &= u(0) + \int_0^r t^{1-N} \int_0^t s^{N-1} p(s) f\big(u(s)\big)\, ds\, dt \\ &\geqslant u(R) + f\big(u(R)\big) \int_R^r t^{1-N} \int_R^t s^{N-1} p(s)\, ds\, dt \to \infty \quad \text{as } r \to \infty, \end{aligned}$$

where we have applied Eq. (8) in [5] to establish the limit.

To prove the converse we use an argument similar to part of the proof of Theorem 1 above. Indeed, if f satisfies (3) and w is a nonnegative entire large solution of (1), then w satisfies

$$\big(r^{N-1} w'\big)' = r^{N-1} p(r) f(w).$$

Now, as in the proof of Theorem 1, we multiply this by $r^{N-1} w'(r)$, integrate, and use the monotonicity of $r^{2N-2} p(r)$ for $r \geqslant R$ to get (see inequality (10) above and its derivation)

$$w'(r) \leqslant \sqrt{C} r^{1-N} + \sqrt{2p(r)} \big[F\big(w(r)\big)\big]^{1/2}$$

and hence, as with (11), we get

$$\begin{aligned} \int_{w(R)}^{w(r)} \big[F(t)\big]^{-1/2}\, dt &\leqslant \sqrt{C}\, \big[F\big(w(R)\big)\big]^{-1/2} \int_R^r t^{1-N}\, dt + \int_R^r t^{1+\varepsilon} p(t)\, dt + \frac{1}{\varepsilon R^{\varepsilon}} \\ &\leqslant C_R + \int_R^r t^{1+\varepsilon} p(t)\, dt, \end{aligned}$$

where $C_R = \sqrt{C}[F(w(R))]^{-1/2} R^{N-2}/(N-2) + 1/(\varepsilon R^{\varepsilon})$. Letting $r \to \infty$, we find that p satisfies (13) since w is large and f satisfies (3). This completes the proof. □

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References

[1] F.S. Cirstea, V.D. Radulescu, Blow-up boundary solutions of semilinear elliptic problems, Nonlinear Anal. 48 (2002) 521–534.
[2] J.B. Keller, On solutions of $\Delta u = f(u)$, Comm. Pure Appl. Math. 10 (1957) 503–510.
[3] A.V. Lair, A necessary and sufficient condition for existence of large solutions to semilinear elliptic equations, J. Math. Anal. Appl. 240 (1999) 205–218.
[4] A.V. Lair, Nonradial large solutions of sublinear elliptic equations, Appl. Anal. 82 (2003) 431–437.
[5] A.V. Lair, A.W. Wood, Large solutions of semilinear elliptic problems, Nonlinear Anal. 37 (1999) 805–812.
[6] A.V. Lair, A.W. Wood, Large solutions of sublinear elliptic equations, Nonlinear Anal. 39 (2000) 745–753.
[7] R. Osserman, On the inequality $\Delta u \geqslant f(u)$, Pacific J. Math. 7 (1957) 1641–1647.
[8] D. Sattinger, Topics in Stability and Bifurcation Theory, Springer-Verlag, Berlin, 1973.