Multiple solutions for a two-point boundary value problem

F. Cammaroto *, A. Chinnì, B. Di Bella

Department of Mathematics, University of Messina, 98166 Sant’Agata-Messina, Italy
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Abstract

In this paper, using a recent result of Ricceri, we prove two multiplicity theorems for the problem
\[-u'' = \lambda f(u) + \mu g(x, u), u(0) = u(1) = 0,\]
extending a previous result that G. Bonanno obtained for \(\mu = 0\).

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1. Introduction

Since many years, the existence of multiple solutions for Dirichlet boundary value problem has been widely investigated. Several recent results, approached by variational methods, make use of Theorem 1 of [6] obtained by Ricceri (see, for example, [1,2]). Recently, in [4] a two local minima theorem was established; we recall it in a convenient form:

**Theorem 1.1.** [4, Theorem 4] Let \(X\) be a reflexive real Banach space, \(I \subseteq \mathbb{R}\) an interval, and \(\Psi : X \times I \to \mathbb{R}\) a function such that \(\Psi(x, \cdot)\) is concave in \(I\) for all \(x \in X\), while \(\Psi(\cdot, \lambda)\) is continuous, coercive and sequentially weakly lower semicontinuous in \(X\) for all \(\lambda \in I\). Further, assume that

\[
\sup_{\lambda \in I} \inf_{x \in X} \Psi(x, \lambda) < \inf_{x \in X} \sup_{\lambda \in I} \Psi(x, \lambda).
\]

* Corresponding author. Because of a surprising coincidence of names within the same Department, we have to point out that the author was born on August 4, 1968.

E-mail address: filippo@dipmat.unime.it (F. Cammaroto).
Then, for each $\rho > \sup_I \inf_X \Psi(x, \lambda)$ there exists a non-empty open set $A \subseteq I$ with the following property: for every $\lambda \in A$ and every sequentially weakly lower semicontinuous functional $\Phi : X \to \mathbb{R}$, there exists $\delta > 0$ such that, for each $\mu \in ]0, \delta[$, the functional $\Psi(\cdot, \lambda) + \mu \Phi(\cdot)$ has at least two local minima lying in the set $\{x \in X : \Psi(x, \lambda) < \rho\}$.

The aim of this paper is to obtain an application of Theorem 1.1 to the following Dirichlet problem:

$$\begin{cases}
-u'' = \lambda f(u) + \mu g(x, u) & \text{in } [0, 1], \\
u(0) = u(1) = 0,
\end{cases}$$

(D)

where $f : \mathbb{R} \to \mathbb{R}$ and $g : [0, 1] \times \mathbb{R} \to \mathbb{R}$ are two continuous functions and $\lambda$, $\mu$ two positive parameters.

For each $\xi \in \mathbb{R}$, put

$$F(\xi) = \int_0^\xi f(t) dt.$$

### 2. Main result

**Theorem 2.1.** Assume that there exist two positive numbers $c, d$, with $c < \sqrt{2}d$, such that

(i) $f(\xi) \geq 0$ for each $\xi \in [-c, \max\{c, d\}]$;

(ii) $\frac{F(c)}{c^2} < \frac{1}{4} \frac{F(d)}{d^2}$;

(iii) $\limsup_{|\xi| \to +\infty} \frac{F(\xi)}{\xi^2} \leq 0$.

Then, there exist a number $r$ and a non-degenerate compact interval $C \subseteq [0, +\infty[$ such that, for every $\lambda \in C$ and every continuous function $g : [0, 1] \times \mathbb{R} \to \mathbb{R}$, there exists $\delta > 0$ such that, for each $\mu \in ]0, \delta[$, the problem (D) has at least two classical solutions whose norms in $W^{1,2}_0([0, 1])$ are less than $r$.

**Proof.** Let $X = W^{1,2}_0([0, 1])$ endowed with the norm

$$\|u\| = \left( \int_0^1 |u'(t)|^2 \, dt \right)^{1/2} \quad \forall u \in X.$$

Let us define in $X$ two functionals $\Phi$ and $J$ by setting, for each $u \in X$,

$$\Phi(u) = \frac{1}{2} \|u\|^2 \quad \text{and} \quad J(u) = -\int_0^1 F(u(x)) \, dx.$$

Applying the well-known inequality

$$\max_{0 \leq x \leq 1} |u(x)| \leq \frac{1}{2} \|u\|,$$
satisfied for all \( u \in X \), we deduce that
\[
\sup_{u \in \Phi^{-1}([-\infty,r])} (-J(u)) \leq \max_{|\xi| \leq \sqrt{r/2}} F(\xi)
\]
for each \( r > 0 \).

Thanks to (i) and (ii) and to [1, Proposition 1] there exist \( \sigma > 0 \) and \( \bar{u} \in X \) such that
\[
\sup_{u \in \Phi^{-1}([-\infty,\sigma])} (-J(u)) < 2\sigma \left( \frac{-J(\bar{u})}{\|ar{u}\|^2} \right) = \sigma \frac{-J(\bar{u})}{\Phi(\bar{u})}.
\]

Let \( \rho \) be such that
\[
\sup_{u \in \Phi^{-1}([-\infty,\sigma])} (-J(u)) < \rho < \sigma \left( \frac{-J(\bar{u})}{\Phi(\bar{u})} \right)
\]
and
\[
\Psi(u, \lambda) = \Phi(u) + \lambda J(u) + \lambda \rho
\]
for each \( u \in X \), \( \lambda \in [0, +\infty[. \) Proposition 3.1 of [5] assures that
\[
\sup_{\lambda \geq 0} \inf_{u \in X} \Psi(u, \lambda) < \inf_{u \in X} \sup_{\lambda \geq 0} \Psi(u, \lambda).
\]

We apply Theorem 1.1 to the continuous functional \( \Psi \) by choosing \( I = [0, +\infty[. \) Clearly, \( \Psi(u, \cdot) \) is concave in \( I \) for all \( u \in X \).

Fix \( \lambda \in I \), let \( \varepsilon < 2/\lambda \). Since (iii) holds, there exists \( b_\varepsilon \in \mathbb{R} \) such that
\[
F(\xi) \leq \varepsilon |\xi|^2 + b_\varepsilon
\]
for all \( \xi \in \mathbb{R} \).

Fix \( u \in X \). From the last inequality we deduce that
\[
\int_0^1 F(u(x)) \, dx \leq \frac{\varepsilon}{4} \|u\|^2 + b_\varepsilon.
\]

So,
\[
\Psi(u, \lambda) \geq \left( \frac{1}{2} - \frac{\lambda \varepsilon}{4} \right) \|u\|^2 - \lambda b_\varepsilon + \lambda \rho,
\]
i.e. \( \Psi(\cdot, \lambda) \) is coercive.

A standard argument ensures that \( \Psi(\cdot, \lambda) \) is continuous and sequentially weakly lower semicontinuous in \( X \).

Fix \( \sigma > \sup_{\lambda \geq 0} \inf_{u \in X} \Psi(u, \lambda) \), we can apply Theorem 1.1. Therefore there exist a non-empty open set \( A \subseteq I \) with the following property: for every \( \lambda \in A \) and every continuous function \( g : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R} \), there exists \( \delta > 0 \) such that, for each \( \mu \in ]0, \delta[ \), the functional \( E_{\lambda, \mu}(u) = \Psi(u, \lambda) + \mu H(u) \) has at least two local minima lying in the set \( \{ u \in X : \Psi(u, \lambda) < \sigma \} \), where \( H \) is the sequentially lower semicontinuous functional defined by
\[
H(u) = -\int_0^1 \left( \int_0^{u(x)} g(x, t) \, dt \right) \, dx.
\]
for each \( u \in X \). These two local minima are critical points of \( E_{\lambda, \mu} \) and so, by well-known arguments, classical solutions of the problem (D).

Finally, let \([a, b] \subset A\) be any non-degenerate compact interval. Observe that

\[
\bigcup_{\lambda \in [a, b]} \{ u \in X : \Psi(u, \lambda) \leq \sigma \} \subseteq \{ u \in X : \Psi(u, a) \leq \sigma \} \cup \{ u \in X : \Psi(u, b) \leq \sigma \}.
\]

This clearly implies that the set \( S := \bigcup_{\lambda \in [a, b]} \{ u \in X : \Psi(u, \lambda) \leq \sigma \} \) is bounded. At this point, the conclusion follows taking \( r = \sup_{x \in S} \| x \| \).

Let us present a variant of the main theorem in order to obtain three solutions of (D) instead of two and where the involved functional satisfies the Palais–Smale condition. We recall that a \( \hat{G} \)âteaux differentiable functional \( S \) on a real Banach space \( X \) is said to satisfy the Palais–Smale condition if each sequence \( \{ x_n \} \) in \( X \) such that \( \sup_{n} |S(x_n)| < +\infty \) and \( \lim_{n \to +\infty} \| S'(x_n) \| = 0 \) admits a strongly converging subsequence.

**Theorem 2.2.** Let the hypotheses of Theorem 2.1 hold. Then, there exists a non-empty open set \( A \subseteq [0, +\infty[ \) such that, for every \( \lambda \in A \) and every continuous function \( g : [0, 1] \times \mathbb{R} \to \mathbb{R} \) with

\[
(\text{iv}) \quad \limsup_{|\xi| \to +\infty} \frac{\sup_{x \in [0,1]} \int_{0}^{\xi} g(x, t) \, dt}{\xi^2} < +\infty,
\]

there exists \( \delta > 0 \) such that, for each \( \mu \in ]0, \delta[, \) the problem (D) has at least three classical solutions.

**Proof.** Let \( A \) and \( E_{\lambda, \mu} \) have the same meaning as in the proof of Theorem 2.1, \( g : [0, 1] \times \mathbb{R} \to \mathbb{R} \) being a continuous function satisfying (iv).

Let us check the Palais–Smale condition for the functional \( E_{\lambda, \mu} \).

From (iv) there exist two constants \( p, q \in \mathbb{R} \) with \( p \neq 0 \) and

\[
\int_{0}^{\xi} g(x, t) \, dt \leq p |\xi|^2 + q
\]

for all \( x \in [0, 1] \) and \( \xi \in \mathbb{R} \).

Fix \( u \in X \). From the last inequality we deduce that

\[
H(u) = -\int_{0}^{1} \int_{0}^{1} (g(x, t) \, dt) \, dx \geq -\frac{p}{4} \| u \|^2 - q.
\]

Let \( \delta < \min\{ \delta_1, \frac{1}{p} (2 - \lambda \varepsilon) \} \). So, for each \( \lambda \in A \) and \( \mu \in ]0, \delta[ \), one has

\[
E_{\lambda, \mu}(u) = \Psi(u, \lambda) + \mu H(u) \geq \left( \frac{1}{2} - \frac{\lambda \varepsilon}{4} - \frac{\mu p}{4} \right) \| u \|^2 - \lambda b \varepsilon + \lambda \rho - \mu q,
\]

for all \( u \in X \). This assures the coercivity of the functional \( E_{\lambda, \mu} \) for each \( \lambda \in A \) and \( \mu \in ]0, \delta[ \).

Now, since \( J' \) and \( H' \) are compact, the fact that \( E_{\lambda, \mu} \) satisfies the Palais–Smale condition follows from a classical result of [7, Example 38.25].
Since the functional $E_{\lambda,\mu}$ is $C^1$ in $X$, our conclusion follows by [3, Corollary 1] which assures that there exists a third critical point of the functional $E_{\lambda,\mu}$ which is a solution of problem (D).

Now, we exhibit an example in which the hypotheses of Theorem 2.2 (as well as of Theorem 2.1) are satisfied.

**Example 2.1.** Let $0 < \alpha < 1$, $\beta \leq 1$ and let $f : \mathbb{R} \to \mathbb{R}$ and $g : [0, 1] \times \mathbb{R} \to \mathbb{R}$ be defined by setting, for each $t \in \mathbb{R}$,

$$f(\xi) = \begin{cases} \xi^4 & \text{if } \xi \leq 1, \\ \xi^\alpha & \text{if } \xi > 1, \end{cases}$$

and

$$g(x, \xi) = x|\xi|^\beta.$$

In this case it is easy to observe that

$$F(\xi) = \begin{cases} \frac{\xi^5}{5} & \text{if } \xi \leq 1, \\ \frac{1}{\alpha+1}[\xi^{\alpha+1} + \frac{\alpha-4}{5}] & \text{if } \xi > 1, \end{cases}$$

and all the hypotheses of Theorem 2.2 are satisfied, if one chooses $c = \frac{1}{2}$ and $d = 1$.

**References**


