On the $2k$-th Power Mean of Dirichlet L-Functions with the Weight of General Kloosterman Sums

Zhang Wenpeng, Yi Yuan, and He Xiali

Department of Mathematics, Northwest University, Xi’an, Shaanxi, People’s Republic of China

Communicated by D. Goss

Received April 1, 1999; revised November 25, 1999

The main purpose of this paper is using the classical estimation of the Kloosterman sum and the analytic method to study the $2k$-th power mean of Dirichlet L-functions with the weight of general Kloosterman sums and give an interesting $2k$-th mean value theorem.

Key Words: general Kloosterman sum; Dirichlet L-functions; distribution of the mean value.

1. INTRODUCTION

Let $q$ be an integer $\geq 2$ and $\chi$ a Dirichlet character modulo $q$. Then for any fixed integer $m$ and $n$, we define the general Kloosterman sum $S(m, n, \chi, q)$ as

$$S(m, n, \chi, q) = \sum_{a=1}^{\varphi(q)} \chi(a) e\left(\frac{ma+n\bar{a}}{q}\right),$$

where $\bar{a}$ denotes the solution $b$ of the congruence equation $ab \equiv 1 \pmod{q}$; that is, $b$ is the inverse of $a$ modulo $q$, and $e(y) = e^{2\pi i y}$.

The upper bound estimation of $S(m, n, \chi, q)$ has been studied by many people. For instance, S. Chowla [2] and A. V. Malyshev [3] have proved that

$$|S(m, n, \chi, p)| \ll (m, n, p)^{1/2} p^{1/2} + \varepsilon,$$

where $p$ is a prime, $\varepsilon$ is any fixed positive number, and $(m, n, p)$ denotes the greatest common divisor of $m$, $n$, and $p$.

For an arbitrary composite number $q$ we guess that

$$|S(m, n, \chi, q)| \ll q^{1/2} (m, n, q)^{1/2} d(q), \tag{1}$$

This work is supported by the N.S.F. and the P.S.F. of P. R. China.
where \( d(q) \) is the divisor function. But it has not been proved at present. In spite of that, we are quite sure about the correctness of (1). In fact, \( S(m, n, \chi, q) \) presents many good distribution properties in some number theory problems. The main purpose of this paper is using the mean value theorem of Dirichlet L-functions, the classical estimation of the Kloosterman sum, and the analytic method to study the asymptotic distribution of the 2k-th power mean

\[
\sum_{\chi \neq \chi_0} |L(1, \chi)|^{2k} |S(m, n, \chi, q)|^2,
\]

and to give a sharper mean value theorem. It appears that no one had studied this problem yet; at least, we have not seen such a mean value before. The problem is interesting because it can help us to find some relationships between Dirichlet L-functions and the general Kloosterman sum. In this paper, we shall prove the following:

**Theorem.** Let \( q \geq 2 \) be an integer. Then for any integer \( m \) and \( n \) with \( (m, q) = (n, q) = 1 \), we have the asymptotic formula

\[
\sum_{\chi \neq \chi_0} |S(m, n, \chi, q)|^2 |L(1, \chi)|^{2k} = \left( \frac{\pi^2}{6} \right)^{2k-1} \phi(q) \prod_{p|q} \left( 1 - \frac{1}{p^2} \right)^{2k-1} \prod_{p \nmid q} \left( 1 - \frac{1 - C_{p^2-2}}{p^2} \right) + O(q^{\frac{3}{2} + \varepsilon}),
\]

where \( \sum_{\chi \neq \chi_0} \) denotes the summation over all nonprincipal characters modulo \( q \), \( \phi(q) \) is the Euler function, \( \varepsilon \) denotes any fixed positive number, \( \prod_{p|q} \) denotes the product over all different prime divisors of \( q \), and \( C_n = m!/(n!(m-n)!) \).

In particular, from this theorem we immediately deduce:

**Corollary.** Let \( q \geq 3 \) be an integer; then for any integers \( m \) and \( n \) with \( (m, q) = (n, q) = 1 \), we have the asymptotic formula

\[
\sum_{\chi \neq \chi_0} |S(m, n, \chi, q)|^2 |L(1, \chi)|^4 = \frac{5}{72} \pi^2 \phi(q) \prod_{p|q} \frac{(p^2 - 1)^3}{p^4(p^2 + 1)} + O(q^{\frac{3}{2} + \varepsilon}).
\]

2. SOME LEMMAS

In order to complete the proof of the theorem, we need several lemmas. First, we have the following result of Estermann:
**Lemma 1.** Let \( m, n, \) and \( q \) be integers with \( q \geq 2 \). Then we have the estimates
\[
S(m, n, q) = \sum_{a=1}^{q} e\left(\frac{ma + na}{q}\right) \ll (m, n, q)^{1/2} q^{1/2} d(q),
\]
where \( \sum_{a} \) denotes the summation over all \( a \) such that \( (a, q) = 1 \) and \( e(y) = e^{2\pi iy} \).

**Proof.** (See Ref. [4]).

**Lemma 2.** Let \( f(x) \) be a polynomial of degree \( k \) with leading coefficient \( a_0 \), and define the difference operator \( \Delta \) by \( (\Delta f)(x) = f(x+1) - f(x) \). Then we have
\[
\Delta^l f(x) = k! a_0, \quad \Delta^l f(x) = 0 \quad (l \geq k+1)
\]
Especially for \( f(n) = (C_{k+n-1}^n)^2 \), we have
\[
\Delta^{2k-2} f(n) = C_{2k-2}^{k-1}.
\]

**Proof.** This result can be easily proved by the properties of the difference operator and mathematical induction.

**Lemma 3.** Let \( q \) be an integer with \( q \geq 3 \) and \( d_k(n) \) denote the \( k \)th divisor function (i.e., the number of solutions of the equation \( n_1 n_2 \cdots n_k = n \) in positive integers \( n_1, n_2, \ldots, n_k \)). Then for any complex variable \( s \) with \( \text{Re}(s) > 1 \), we have
\[
\sum_{n=1}^{\infty} \frac{d_k^2(n)}{n^s} = \zeta^{2k-1}(s) \prod_{p \mid q} \left(1 - \frac{1}{p^s}\right)^{2k-1} \prod_{p \not{\mid} q} \left(1 - \frac{1 - C_{2k-2}^{k-1}}{p^s}ight),
\]
where \( \zeta(s) \) denotes the Riemann zeta function.

**Proof.** It is clear that \( d_k^2(n) \) is a multiplicative function and the series \( \sum_{n=1}^{\infty} \frac{d_k^2(n)}{n^s} \) is absolutely convergent. Thus from Euler's infinite product representation we have
\[
\sum_{n=1}^{\infty} \frac{d_k^2(n)}{n^s} = \prod_{p \mid q} \left(1 + \frac{d_k^2(p)}{p^{2s}} + \frac{d_k^2(p^2)}{p^{4s}} + \cdots + \frac{d_k^2(p^r)}{p^{rs}} + \cdots \right).
\]

For \( n > 1 \), let \( n = p_1^{m_1} \cdots p_r^{m_r} \) denote the factorization of \( n \) into prime powers; then (see Ref. [6, 6.4.12])
\[
d_k(n) = \prod_{j=1}^{r} C_{k+m_j-1}^{m_j}.
\]
Hence

\[
\sum_{n=1}^{\infty} \frac{d_k^2(n)}{n^s} = \prod_{p \nmid q} \left( 1 + \frac{(C_1^1)^2}{p^s} + \frac{(C_1^2)^2}{p^{2s}} + \ldots + \frac{(C_1^{n-1})^2}{p^{ns}} + \ldots \right).
\]

Let

\[
S = 1 + \frac{(C_1^1)^2}{p^s} + \frac{(C_1^2)^2}{p^{2s}} + \ldots + \frac{(C_1^{n-1})^2}{p^{ns}} + \ldots;
\]

then from Lemma 2 we get

\[
S \left( 1 - \frac{1}{p^s} \right)^{2k-2} = 1 + \frac{C_{2k-2}^{k-1}}{p^s} + \frac{C_{2k-2}^{k-1}}{p^{2s}} + \ldots + \frac{C_{2k-2}^{k-1}}{p^{ns}} + \ldots
\]

\[
= 1 + \frac{C_{2k-2}^{k-1}}{p^s} \times \frac{1}{1 - 1/p^s},
\]

so

\[
S = \frac{1 - (1 - \frac{1}{C_{2k-2}^{k-1}})/p^s}{(1 - 1/p^s)^{2k-1}}.
\]

Note that \(\zeta(s) = \prod_p (1 - \frac{1}{p^s})^{-1}\); hence

\[
\sum_{n=1}^{\infty} \frac{d_k^2(n)}{n^s} = e^{2k-1}(s) \prod_{p \nmid q} \left( 1 - \frac{1}{p^s} \right)^{2k-1} \prod_{p \nmid q} \left( 1 - \frac{1 - C_{2k-2}^{k-1}}{p^s} \right).
\]

This proves Lemma 3.

**Lemma 4.** Let \(q\) be an integer with \(q \geq 3\) and \(\chi\) a Dirichlet character modulo \(q\). Write \(A(y, \chi) = A(y, \chi, k) = \sum_{0 < n \leq y} \chi(n) d_k(n)\). Then we have the estimate

\[
\sum_{\chi \neq \chi_0} |A(y, \chi)|^2 \ll y^{2 - (4/2k) + \varepsilon} \phi^2(q),
\]

where \(\varepsilon\) denotes any fixed positive number.

**Proof.** We use mathematical induction (for \(k\)) to prove the estimate (3).
(i) If \( k = 1 \), then we have
\[
\sum_{n \leq y} \frac{\chi(n)}{n} = \frac{\chi(n)}{n} \sum_{n \leq y} \frac{\chi(n)}{n} \sum_{N \leq n \leq N + q} \frac{\chi(n) \chi(m)}{N + q} \ll \phi^2(q).
\]
So (3) holds for \( k = 1 \).

(ii) If \( k = 2 \), then from the partition identities we have
\[
A(y, z) = \sum_{n \leq y} \chi(n) \sum_{m \leq y/m} \chi(m) + \sum_{n \leq y} \chi(n) \sum_{m \leq y/m} \chi(m)
- \sum_{n \leq y} \chi(n) \sum_{m \leq y/m} \chi(m) + \sum_{n \leq y} \chi(n) \sum_{m \leq y/m} \chi(m).
\]
Applying the Cauchy inequality and (4) we have
\[
\sum_{n \leq y} \frac{\chi(n)}{n} \sum_{n \leq y} \frac{\chi(n)}{n} \sum_{m \leq y/m} \chi(m) \ll y^1 + \phi^2(q).
\]
That is, (3) holds for \( k = 2 \).

(iii) Assume (3) holds for \( k = s \geq 2 \), i.e.,
\[
\sum_{n \leq y} \frac{\chi(n)}{n} \sum_{n \leq y} \frac{\chi(n)}{n} \sum_{m \leq y/m} \chi(m) \ll y^{2^s - 4/2^s} + \phi^2(q).
\]
Then for \( k = s + 1 \), from the partition identities we have
\[
A(y, z) = \sum_{n \leq y} \chi(n) \sum_{n \leq y} \chi(n) \sum_{n \leq y} \chi(n) \cdot \chi(m)
= \sum_{n \leq y} \chi(m) \chi(r_{s+1}) d_j(m)
= \sum_{n \leq y} \chi(m) \chi(r_{s+1}) d_j(m) - \sum_{n \leq y} \chi(m) \chi(r_{s+1}) d_j(m) - \sum_{n \leq y} \chi(m) \chi(r_{s+1}) d_j(m)
\]
\[
\begin{align*}
&= \sum_{m \leq y^{1/25}} \chi(m) d_{\lambda}(m) \sum_{r_{\lambda+1} \leq y/m} \chi(r_{\lambda+1}) \\
&+ \sum_{r_{\lambda+1} \leq y^{2/5}} \chi(r_{\lambda+1}) \sum_{m \leq y/r_{\lambda+1}} \chi(m) d_{j}(m) \\
&- \sum_{m \leq N^{1/25}} \chi(m) d_{j}(m) \sum_{r_{\lambda+1} \leq N/m} \chi(r_{\lambda+1}) \\
&- \sum_{r_{\lambda+1} \leq N^{2/5}} \chi(r_{\lambda+1}) \sum_{m \leq N/r_{\lambda+1}} \chi(m) d_{j}(m) \\
&+ \sum_{m \leq N^{1/25}} \chi(m) d_{j}(m) \sum_{r_{\lambda+1} \leq N^{2/5}} \chi(r_{\lambda+1}).
\end{align*}
\]

Applying the Cauchy inequality and the inductive hypothesis we have
\[
\sum_{\chi \neq \chi_0} |A(y, \chi, s + 1)|^2 \\
\ll \left( \sum_{m \leq y^{1/25}} d_{\lambda}^2(m) \right) \sum_{m \leq y^{1/25}} \left| \sum_{\chi \neq \chi_0} \sum_{r_{\lambda+1} \leq y/m} \chi(r_{\lambda+1}) \right|^2 \\
+ y^{2/\lambda} \sum_{r_{\lambda+1} \leq y^{2/5}} \sum_{\chi \neq \chi_0} \left( \sum_{m \leq y/r_{\lambda+1}} \chi(m) d_{j}(m) \right)^2 \\
+ \sum_{\chi \neq \chi_0} \left( \sum_{m \leq y^{1/25}} \chi(m) d_{j}(m) \right)^2 \sum_{l_{\lambda+1} \leq y^{2/5}} \chi(r_{\lambda+1}) \right|^2 \\
\ll y^{2-4(2^{1/5} + \phi^2(q))} + y^{2-4(2^{1/5} + \phi^2(q))} + y^{2-4(2^{1/5} + \phi^2(q))} \\
\ll y^{2-4(2^{1/5} + \phi^2(q))}.
\]

This proves that (3) holds for \( k = s + 1 \). On combining (i), (ii), and (iii) it follows that the estimate (3) holds for all positive integer \( k \).

This completes the proof of Lemma 4.

**Lemma 5.** Let \( q \) be a positive integer with \( q \geq 3 \) and \( \chi \) be the Dirichlet character modulo \( q \). Then we have
\[
\sum_{d | p} \sum_{l=1}^{(q-1)/d} d^{1/2} \left| \sum_{\chi \neq \chi_0} \chi(ld + 1) |L(1, \chi)|^{2k} \right| = O(q^{1+\varepsilon}),
\]
where \( \chi_0 \) denotes the principal character modulo \( q \), and \( \varepsilon \) denotes any fixed positive number.
Proof. For convenience, we put

$$A(x, y) = \sum_{q \mid (ld+1)} \chi(n) d_k(n), \quad B(x, y) = \sum_{q < n \leq y} \chi(n) d_k(n).$$

Then for \( s > 1 \), the series \( L(s, \chi) \) is absolutely convergent, so applying Abel’s identity we have

$$L^k(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n) d_k(n)}{n^s} = \sum_{1 \leq n \leq q/(ld+1)} \frac{\chi(n) d_k(n)}{n^s} + s \int_{q/(ld+1)}^{+\infty} \frac{A(x, y)}{y^s} \frac{dy}{\chi(y)}$$

$$= \sum_{n=1}^{\infty} \frac{\chi(n) d_k(n)}{n^s} + s \int_{q/(ld+1)}^{+\infty} \frac{B(x, y)}{y^s} \frac{dy}{\chi(y)}.$$ 

It is clear that the above formula also holds for \( s = 1 \) and \( \chi \neq \chi_0 \). Hence from the definition of the Dirichlet L-function we have

$$\sum_{x \neq \chi_0} \chi(ld+1) |L(1, \chi)|^{2k}$$

$$= \sum_{x \neq \chi_0} \chi(ld+1) \left[ \sum_{n=1}^{\infty} \frac{\chi(n) d_k(n)}{n} \right]^2$$

$$= \sum_{x \neq \chi_0} \chi(ld+1) \left( \sum_{1 \leq n \leq q/(ld+1)} \frac{\chi(n) d_k(n)}{n} + \int_{q/(ld+1)}^{+\infty} \frac{A(x, y)}{y^s} \frac{dy}{\chi(y)} \right)$$

$$\times \left( \sum_{m=1}^{\infty} \frac{\chi(m) d_k(m)}{m} + \int_{q/(ld+1)}^{+\infty} \frac{B(x, y)}{y^s} \frac{dy}{\chi(y)} \right)$$

$$= \sum_{x \neq \chi_0} \chi(ld+1) \left( \sum_{1 \leq n \leq q/(ld+1)} \frac{\chi(n) d_k(n)}{n} \right) \left( \sum_{m=1}^{\infty} \frac{\chi(m) d_k(m)}{m} \right)$$

$$+ \sum_{x \neq \chi_0} \chi(ld+1) \left( \sum_{q \mid (ld+1)} \chi(n) d_k(n) \right) \left( \int_{q/(ld+1)}^{+\infty} \frac{A(x, y)}{y^s} \frac{dy}{\chi(y)} \right)$$

$$+ \sum_{x \neq \chi_0} \chi(ld+1) \left( \int_{q/(ld+1)}^{+\infty} \frac{A(x, y)}{y^s} \frac{dy}{\chi(y)} \right) \left( \sum_{q \mid (ld+1)} \chi(n) d_k(n) \right)$$

$$M_1 + M_2 + M_3 + M_4,$$
so that

$$\sum_{d \mid q} \sum_{l=1}^{(q-1)/d} d^{1/2} \left| \sum_{x \neq x_0} \chi(ld + 1) |L(1, \chi)|^2 \right| \leq \sum_{d \mid q} \sum_{l=1}^{(q-1)/d} d^{1/2} (|M_1| + |M_2| + |M_3| + |M_4|).$$

Now we shall estimate each term in the expression (6).

(i) From the orthogonality relation for character sums modulo $q$, we know that for $(q, mn) = 1$, we have the identity

$$\sum_{x \mod q} \chi(n) \overline{\chi}(m) = \begin{cases} \phi(q), & \text{if } n \equiv m \mod q; \\ 0, & \text{otherwise.} \end{cases}$$

From this we can easily get

$$M_1 = \sum_{x \neq x_0} \chi(ld + 1) \left( \sum_{n \leq q/(ld + 1)} \frac{\chi(n) d_k(n)}{n} \left( \sum_{m \leq q} \frac{\overline{\chi}(m) d_k(m)}{m} \right) \right)$$

$$= \sum_{x \mod q} \chi(ld + 1) \left( \sum_{n \leq q/(ld + 1)} \frac{\chi(n) d_k(n)}{n} \left( \sum_{m \leq q} \frac{\overline{\chi}(m) d_k(m)}{m} \right) \right)$$

$$- \sum_{n \leq q/(ld + 1)} \frac{d_k(n)}{n} \sum_{m \leq q} \frac{d_k(m)}{m}$$

$$= \phi(q) \sum_{n \leq q/(ld + 1)} \sum_{\substack{m \leq q \backslash \text{m does not divide } q/(ld + 1) \backslash \text{m divides } q/(ld + 1) \backslash n \text{ divides } q/(ld + 1) \backslash mm \text{ divides } q}} \frac{d_k(n) d_k(m)}{nn} + O(\eta^q)$$

$$= \phi(q) \sum_{n \leq q/(ld + 1)} \frac{d_k(n) d_k((ld + 1) n)}{(ld + 1) n^2} + O(\eta^q)$$

$$\leq \phi(q) \frac{d_k((ld + 1)/ld + 1)}{ld + 1} \sum_{n \leq q/(ld + 1)} \frac{|d_k(n)|^2}{n^2} + O(\eta^q)$$

$$\leq \phi(q) \frac{d_k((ld + 1)/ld + 1)}{ld + 1} \sum_{n=1}^{\omega(q)} \frac{|d_k(n)|^2}{n^2} + O(\eta^q).$$

(7)
So from (7) we get

\[ \sum_{d \mid q} \sum_{l=1}^{(q-1)/d} d^{1/2} |M_1| = O \left( \sum_{d \mid q} \sum_{l=1}^{(q-1)/d} \frac{d^{1/2} \varphi(q) \varphi(ld+1)}{ld+1} \right) \]

+ \( q^s \sum_{d \mid q} \sum_{l=1}^{(q-1)/d} d^{1/2} \right) = O(q^{1+s}), \quad (8) \]

where we have used the estimation \( d_k(q) \ll q^{1/2} \).

(ii)

\[ \sum_{d \mid q} \sum_{l=1}^{(q-1)/d} d^{1/2} |M_2| \]

\[ = \sum_{d \mid q} \sum_{l=1}^{(q-1)/d} d^{1/2} \]

\[ \times \left| \sum_{y \neq q_0} \chi(ld+1) \left( \sum_{n \leq q/(ld+1)} \frac{\varphi(n) \varphi(n)}{n} \left( \int_{q}^{+\infty} B(q, y) \frac{dy}{y^2} \right) \right) \right| \]

\[ \ll \sum_{d \mid q} \sum_{l=1}^{(q-1)/d} d^{1/2} \left| \sum_{y \neq q_0} \chi(ld+1) \right| \]

\[ \times \left( \sum_{n \leq q/(ld+1)} \frac{\varphi(n) \varphi(n)}{n} \left( \int_{q}^{+\infty} B(q, y) \frac{dy}{y^2} \right) \right) \]

\[ + \sum_{d \mid q} \sum_{l=1}^{(q-1)/d} d^{1/2} \left| \sum_{y \neq q_0} \chi(ld+1) \right| \]

\[ \times \left( \sum_{n \leq q/(ld+1)} \frac{\varphi(n) \varphi(n)}{n} \left( \int_{q}^{+\infty} B(q, y) \frac{dy}{y^2} \right) \right) \]

\[ \ll \sum_{d \mid q} \sum_{l=1}^{(q-1)/d} d^{1/2} \left| \sum_{y \equiv q \mod q} \chi(ld+1) \right| \]

\[ \times \left( \sum_{n \leq q/(ld+1)} \frac{\varphi(n) \varphi(n)}{n} \left( \int_{q}^{+\infty} B(q, y) \frac{dy}{y^2} \right) \right) \]

\[ + \sum_{d \mid q} \sum_{l=1}^{(q-1)/d} d^{1/2} \left( \sum_{n \leq q/(ld+1)} \frac{\varphi(n) \varphi(n)}{n} \left( \int_{q}^{+\infty} B(q, y) \frac{dy}{y^2} \right) \right) \]

\[ + q^s \sum_{d \mid q} \sum_{l=1}^{(q-1)/d} d^{1/2} \int_{q}^{+\infty} \frac{1}{y^2} \sum_{y \neq q_0} |B(q, y)| \, dy. \quad (9) \]
Using the Cauchy inequality and Lemma 4 we may immediately get the estimate

\[
\sum_{x \not\in x_0} |B(x, y)| \leq \Phi^{1/2}(q) \left( \sum_{x \not\in x_0} |B(x, y)|^2 \right)^{1/2} \leq \Phi^{3/2}(q) y^{1-(2/\delta)+\epsilon}. \tag{10}
\]

So from (9) and (10) we have

\[
\sum_{d|q} \sum_{l=1}^{(q-1)/d} d^{1/2} |M_2| \leq \Phi(q) \int_{q}^{\infty} \frac{1}{y^2} \left[ \sum_{d|q} \sum_{l=1}^{(q-1)/d} \sum_{n \leq q/(ld+1)} \sum_{ q \leq m \leq y} d^{1/2} d_k(n) d_k(m) \frac{1}{n} \right] dy
\]

\[+ O\left( q^{3/2+\epsilon} \int_{q}^{\infty} \frac{1}{y^{2-1-(2/\delta)+\epsilon}} dy \right) \]

\[= O\left( \Phi(q) q^\delta \sum_{d|q} d^{1/2} \sum_{n \leq q/d} \frac{1}{n} \left( \int_{q}^{\infty} \frac{y^{1-(2/\delta)+\epsilon}}{y^2} dy \right) + O(q^{1+\epsilon}) \right) \]

\[= O(q^{1+\epsilon}). \tag{11}
\]

(iii) Similarly, we also have

\[
\sum_{d|q} \sum_{l=1}^{(q-1)/d} d^{1/2} |M_3| \leq \sum_{d|q} \sum_{l=1}^{(q-1)/d} d^{1/2} \left| \chi (ld + 1) \right|
\]

\[\times \left( \sum_{m \geq q} \frac{d_k(m)}{m} \right) \left( \int_{q/(ld+1)}^{+\infty} \frac{A(z, y)}{y^2} dy \right) \]

\[\leq \Phi(q) q^\delta \sum_{d|q} d^{1/2} \left| \chi (ld + 1) \right|
\]

\[\times \left( \sum_{m \geq q} \frac{d_k(m)}{m} \right) \left( \int_{q/(ld+1)}^{+\infty} \frac{A(z, y)}{y^2} dy \right) \]

\[+ \sum_{d|q} \sum_{l=1}^{(q-1)/d} d^{1/2} \left| \chi (ld + 1) \right|
\]

\[\times \left( \sum_{m \geq q} \frac{d_k(m)}{m} \right) \left( \int_{q^{1-(2/\delta)+\epsilon}}^{+\infty} \frac{A(z, y)}{y^2} dy \right) \]

\[+ \sum_{d|q} \sum_{l=1}^{(q-1)/d} d^{1/2} \left| \chi (ld + 1) \right|
\]

\[\times \left( \sum_{m \geq q} \frac{d_k(m)}{m} \right) \left( \int_{q^{1-(2/\delta)+\epsilon}}^{+\infty} \frac{A(z, y)}{y^2} dy \right) \]
\[
\sum_{d \mid q} \sum_{l=1}^{(q-1)/d} d^{1/2} \sum_{x \mod q} \chi(ld+1) \\
\times \left( \sum_{m \leq q} \frac{\chi(m) \cdot d_k(m)}{m} \right) \left( \frac{q^{1/2} \cdot 2^{l-2}}{q^{(ld+1) \cdot n \leq y}} \sum_{n \leq y} Z(n) \frac{d_k(n)}{y^2} \right) \right| \\
+ \sum_{d \mid q} \sum_{l=1}^{(q-1)/d} d^{1/2} \left( \sum_{m \leq q} \frac{d_k(m)}{m} \right) \left( \int_{q^{(ld+1)}}^{+\infty} \frac{1}{y^2} \sum_{x \neq z} |A(\chi, y)| \ dy \right) \\
\leq \phi(q) \sum_{d \mid q} \sum_{l=1}^{(q-1)/d} d^{1/2} \frac{1}{y^2} \\
\times \left[ \sum_{m \leq q, q^{(ld+1) \cdot n \leq m(q)}} d_k(m) \frac{d_k(n)}{m} \right] dy + O(q^{1+\epsilon}) \\
= \phi(q) \sum_{d \mid q} \sum_{l=1}^{(q-1)/d} d^{1/2} \frac{1}{y^2} \\
\times \left[ \int_{d}^{+\infty} \frac{1}{y^2} \sum_{n \leq y} d_k(m) \frac{d_k(n)}{m} \right] dy + O(q^{1+\epsilon}).
\] (12)

Now for any fixed positive integers \(m\) and \(n\), there exists at most one \(l\) with
\(1 \leq l \leq \frac{q-1}{d}\) such that the congruence equation:
\((ld+1) \cdot n \equiv m(q)\).

If \(m, n,\) and \(l\) satisfy this congruence equation, then \(m \equiv n(d)\) and \(m \neq n\) for
\(1 \leq n \leq d\). Thus from (12) we have
\[
\sum_{d|q} \sum_{l=1}^{(q-1)/d} d^{1/2} |M_s| \\
\leq \phi(q) \sum_{d|q} \left( \frac{q^{1/2} - 1}{y^2} \right) \left[ \sum_{m \leq q} \sum_{1 \leq n \leq y} \frac{d_{1}(m) d_{1}(n)}{m} \right] dy \\
+ \phi(q) \sum_{d|q} \left( \frac{q^{1/2} - 1}{y^2} \right) \left[ \sum_{m \leq q} \sum_{1 \leq n \leq y} \frac{d_{1}(m) d_{1}(n)}{m} \right] dy + O(q^{1+\epsilon}) \\
= O\left( \phi(q) \sum_{d|q} d^{1/2} \sum_{m \leq q} \frac{1}{m} \left( \frac{q^{1/2} - 1}{y^2} \right) dy \right) \\
+ O\left( \phi(q) \sum_{d|q} d^{1/2} \sum_{s=1}^{d} \frac{1}{s} \left( \frac{q^{1/2} - 1}{y^2} \right) dy \right) + O(q^{1+\epsilon}) \\
= O(q^{1+\epsilon}),
\tag{13}
\]
where we have used the estimation \( d(n) \ll n^\epsilon \).

(iv) Using the same method we can also get the estimates

\[
\sum_{d|q} \sum_{l=1}^{(q-1)/d} d^{1/2} \sum_{x \neq x_0} \varphi(ld + 1) \left( \int_{q/(ld + 1)}^{rf} A(\varphi y) \varphi y^2 dy \right) \left( \int_{q}^{rf} B(\varphi y) \varphi y^2 dy \right) \\
= O(q^{1+\epsilon}).
\tag{14}
\]

Combining (6), (8), (11), (13), and (14) we immediately obtain

\[
\sum_{d|q} \sum_{l=1}^{(q-1)/d} d^{1/2} \sum_{x \neq x_0} \varphi(ld + 1) |L(1, \varphi)|^{2k} = O(q^{1+\epsilon}).
\]

This completes the proof of Lemma 5.

**Lemma 6.** Let \( q \) be an integer with \( q \geq 3 \). Then we have the asymptotic formula

\[
\sum_{x \neq x_0} |L(1, \varphi)|^{2k} \leq \phi(q) \zeta^{2k-1/2} \prod_{p|q} \left( 1 - \frac{1}{p} \right)^{2k-1} \prod_{p \nmid q} \left( 1 - \frac{1 - C_{p-2}^{k-1}}{p^{2k-2}} \right) + O(q^\epsilon).
\]
Proof. Let \( A(y, \chi) = \sum_{n=1}^{\infty} \chi(n) d_0(n) \); then from the definition of Dirichlet L-functions and the method of proving Lemma 5 we have

\[
\sum_{x \neq x_0} |L(1, \chi)|^{2k} = \sum_{x \neq x_0} \left( \sum_{n=1}^{\infty} \frac{\chi(n) d_k(n)}{n} \right)^2
\]

\[
= \sum_{x \neq x_0} \sum_{n=1}^{\infty} \frac{\chi(n) d_k(n)}{n} \left( \sum_{1 \leq m \leq q^2} \frac{\overline{\chi}(m) d_k(m)}{m} \right)^2
\]

\[
+ \sum_{x \neq x_0} \left( \sum_{1 \leq n \leq q^2} \frac{\chi(n) d_k(n)}{n} \right) \times \left( \int_{q^{2k}}^{+\infty} \frac{A(y, \chi)}{y^2} dy \right)
\]

\[
+ \sum_{x \neq x_0} \left( \sum_{1 \leq m \leq q^2} \frac{\overline{\chi}(m) d_k(m)}{m} \right) \times \left( \int_{q^{2k}}^{+\infty} \frac{A(z, \overline{\chi})}{z^2} dz \right)
\]

\[
= \phi(q) \sum_{n=1}^{\infty} \frac{|d_0(n)|^2}{n^2} + O(q^\epsilon). \quad (15)
\]

Applying Lemma 3 and (15) we obtain

\[
\sum_{x \neq x_0} |L(1, \chi)|^{2k} = \phi(q) \epsilon^{2k-1}(2) \prod_{p \mid q} \left( 1 - \frac{1}{p^2} \right)^{2k-1}
\]

\[
	imes \prod_{p \mid q} \left( 1 - \frac{1 - C_{k-1}^{2k-2}}{p^2} \right) + O(q^\epsilon).
\]

This proves Lemma 6.

3. PROOF OF THE THEOREM

In this section, we shall complete the proof of the theorem. First we have

\[
\sum_{x \neq x_0} |L(1, \chi)|^{2k} |S(m, n, \chi, q)|^2
\]

\[
= \sum_{r=1}^{q} \sum_{s=1}^{q} e \left( \frac{(r-s)m + (\bar{r}-\bar{s})n}{q} \right) \sum_{x \neq x_0} \chi(rs) |L(1, \chi)|^{2k}
\]
\[ \begin{align*}
\phi(q) \sum_{x \neq x_0} |L(1, x)|^{2k} &+ \sum_{s=1}^{q} e \left( \frac{r(1 - \tilde{s}) m + \tilde{r}(1 - s) n}{q} \right) \\
&\sum_{x \neq x_0} \chi(s) |L(1, x)|^{2k} \\
&\times \sum_{x \neq x_0} \chi(s) |L(1, x)|^{2k}.
\end{align*} \]

From (16) and Lemmas 1 and 6 we get

\[ \begin{align*}
\sum_{x \neq x_0} |L(1, x)|^{2k} |S(m, n, x, q)|^2
&= \left( \frac{\pi^2}{6} \right)^{2k-1} \phi(q) \prod_{p \mid q} \left( 1 - \frac{1}{p^2} \right)^{2k-1} \prod_{p \nmid q} \left( 1 - \frac{1 - C^{k-1}_{2k-2}}{p^2} \right) + O(q^6) \\
&+ O \left( q^{1/2} d(q) \sum_{s=2}^{q} (s - 1, m, (\tilde{s} - 1, n, q)^{1/2} \left| \sum_{x \neq x_0} \chi(s) |L(1, x)|^{2k} \right) \right).
\end{align*} \]

From (17) and Lemma 5 we obtain

\[ \begin{align*}
\sum_{x \neq x_0} |L(1, x)|^{2k} |S(m, n, x, q)|^2
&= \left( \frac{\pi^2}{6} \right)^{2k-1} \phi(q) \prod_{p \mid q} \left( 1 - \frac{1}{p^2} \right)^{2k-1} \prod_{p \nmid q} \left( 1 - \frac{1 - C^{k-1}_{2k-2}}{p^2} \right) + O(q^6) \\
&+ O \left( q^{1/2} d(q) \sum_{s=2}^{q} (s - 1, m, (\tilde{s} - 1, n, q)^{1/2} \left| \sum_{x \neq x_0} \chi(s) |L(1, x)|^{2k} \right) \right)
\end{align*} \]
ACKNOWLEDGMENT

The authors express their gratitude to the referee for very helpful and detailed comments.

REFERENCES