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Solving non-differentiable equations by a new one-point iterative method with memory

J.A. Ezquerro [a,*], M. Grau-Sánchez [b], M.A. Hernández [a]

[a] Department of Mathematics and Computation, University of La Rioja, 26004 Logroño, Spain
[b] Department of Applied Mathematics II, Technical University of Catalonia, 08034 Barcelona, Spain

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ABSTRACT

We construct a new iterative method for approximating the solutions of nonlinear operator equations, where the operator involved is not differentiable. The algorithm proposed does not need to evaluate derivatives and is more efficient than the secant method. For this, we extend a result of Traub for one-point iterative methods to one-point iterative methods with memory.

1. Introduction

We are interested in approximating a solution x^* of a nonlinear equation in Banach spaces,

$$F(x) = 0, \tag{1}$$

where $F : \Omega \subseteq X \to Y$ and Ω is a non-empty open convex domain in the Banach space X with values in the Banach space Y; we usually apply iterative methods of the form:

$$\begin{cases} x_0 \text{ given in } \Omega, \\ x_{n+1} = \Psi(x_n), \quad n \geq 0. \end{cases} \tag{2}$$

The choice of method (2) usually depends on the efficiency that interests us, taking into account that the efficiency of the method is determined by the R-order of convergence [5], the number of

* Corresponding author.
E-mail addresses: jezquer@unirioja.es (J.A. Ezquerro), miquel.grau@upc.edu (M. Grau-Sánchez), mahernan@unirioja.es (M.A. Hernández).

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evaluations of operators involved in the method and the operational cost required in the algorithm of the method. But, if the operator F is not differentiable, we have to choose (2) carefully. For this case, there are iterative methods, less studied, that do not use derivatives in their algorithms. This type of method usually uses divided differences [5] instead of derivatives. The best known iterative method of this type is the secant method [1,2], whose algorithm is:

$$\begin{cases} x_0, x_{-1} \text{ given in } \Omega, \\ x_{n+1} = \Phi(x_{n-1}, x_n) = x_n - [x_{n-1}, x_n; F]^{-1} F(x_n), \quad n \geq 0, \end{cases} \tag{3}$$

where $[u, v; F]$, $u, v \in \Omega$, is a first-order divided difference, which is a bounded linear operator such that [5]

$$[u, v; F] : \Omega \subset X \longrightarrow Y \quad \text{and} \quad [u, v; F](u - v) = F(u) - F(v).$$

The main aim of this paper consists of constructing a new iterative method for approximating the solutions of nonlinear operator equations, (1), where the operator involved F is not differentiable, such that its algorithm does not need to evaluate derivatives and it is more efficient than method (3).

According to [6], it is well known that we can obtain iterative methods with higher R-order of convergence than method (2) by using the following modification of (2):

$$\begin{cases} x_0 \text{ given in } \Omega, \\ y_n = \Psi(x_n), \\ x_{n+1} = y_n - [F'(x_n)]^{-1} F(y_n), \quad n \geq 0, \end{cases} \tag{4}$$

where Ψ is defined in (2). If we suppose that method (2) has R-order of convergence at least p, then we remember that method (4) has R-order of convergence at least $p + 1$.

On the other hand, to measure the efficiency of method (4), we consider that Eq. (1) represents a nonlinear system of m equations. In this particular case, if $\Psi(x_n)$ uses the matrix $F'(x_n)$ in the first step, then the operational cost of method (4) will not be increased highly, since the matrix $F'(x_n)$ of the second step has already been factorized. Moreover, if (2) is a one-point iterative method with R-order of convergence at least p, it is known that we have to evaluate $F(x_n), F'(x_n), \ldots, F^{(p-1)}(x_n)$ in each step. In consequence, when we apply method (4), we only need to evaluate one more function, $F(y_n)$. Therefore, taking into account the efficiency index and the computational efficiency to measure the efficiency of iterative methods, in the sense defined by Traub [6], the efficiency of method (4) will be higher than that of method (2) if $p = 1, 2$.

Following the previously mentioned Traub's idea for one-point iterative methods and taking into account the considerations done about the efficiency of this type of methods, we study a new situation in this paper, where one-point iterative methods with memory are considered instead of one-point iterative methods. For one-point iterative methods with memory, we emphasize that the evaluations of $F(x_n), F'(x_n), \ldots, F^{(p-1)}(x_n)$ in each step are not necessary to obtain R-order of convergence at least p and the R-order of convergence is not always a natural number.

After that, if we consider iterative methods that do not use derivatives in their algorithms, such as the secant method, we can see from (3) that these types of methods are one-point iterative methods with memory [6]. Remember that the secant method can be constructed from Newton's method by using the approximation $F'(x_n) \simeq [x_{n-1}, x_n, F]$, for all $n \geq 0$, so that the application of the secant method, for solving $F(x) = 0$, only needs one more previous approximation, x_{n-1}. Then, in a way similar to Traub for method (4), we extend this idea to the one-point iterative method with memory

$$\begin{cases} x_{-1}, x_0 \text{ given in } \Omega, \\ x_{n+1} = \varphi(x_{n-1}, x_n), \quad n \geq 0, \end{cases} \tag{5}$$

so that we present the following new iterative method,

$$\begin{cases} x_{-1}, x_0 \text{ given in } \Omega, \\ y_n = \varphi(x_{n-1}, x_n), \quad n \geq 0, \\ x_{n+1} = y_n - [x_{n-1}, x_n; F]^{-1} F(y_n), \quad n \geq 0, \end{cases} \tag{6}$$

where φ is given in (5). Next, we prove the relationship between the R-orders of convergence of iterative methods (5) and (6), and use it for obtaining our main aim stated above.

This paper is organized as follows. In Section 2, we extend Traub's result for one-point iterative methods to one-point iterative methods with memory. In Section 3, from the result obtained in the previous section, we construct a new one-point iterative method with memory more efficient than the secant method. In Section 4, we give a semilocal convergence result for the new method when non-differentiable operators in Banach spaces are used. Finally, in Section 5, we present an application where we illustrate the results obtained when a solution of a nonlinear and non-differentiable integral equation of mixed Hammerstein type is approximated by the new method.

Throughout the paper we denote $\overline{B(x,\sigma)} = \{y \in X;\ \|y-x\| \le \sigma\}$ and $B(x,\sigma) = \{y \in X;\ \|y-x\| < \sigma\}$.

2. An extension of Traub's result

As we have indicated in the introduction, the one-point iterative methods with memory given by (6) can be seen as an extension of the well known result given by Traub for the one-point iterative method (4). In this case, it is important to know the R-order of convergence of method (6), which is established in the next theorem.

Firstly, we introduce some notations that are needed. Let $\tilde{e}_n = y_n - x^* = K e_{n-1}^a e_n^b$, where $e_n = x_n - x^*$, $a, b \in \mathbb{N}$ and $K \in \mathscr{L}(X \times \overset{a+b}{\cdots} \times X, Y)$, the set of bounded linear operators.

Theorem 2.1. *If iterative method (5) has R-order of convergence at least* $\rho = \frac{1}{2}(b + \sqrt{b^2+4a})$, *then iterative method (6) has R-order of convergence at least* $\frac{1}{2}(b+\sqrt{b^2+4a+4})$. *More precisely, if* $F'(x^*) = \Gamma$ *is non-singular, then*

$$e_{n+1} = A_2 e_{n-1} K e_{n-1}^a e_n^b + o(e_{n-1}^{a+1} e_n^b),$$

where $A_2 = \frac{1}{2}\Gamma^{-1}F''(x^*) \in \mathscr{L}(X \times X, Y)$.

Proof. The divided difference operator can be written as (see [4])

$$[x_{n-1}, x_n; F] = \Gamma(I + A_2(e_{n-1}+e_n) + o(e_{n-1}))$$

and the expression of $F(y_n)$ in formal powers of $\tilde{e}_n$ is $F(y_n) = \Gamma(\tilde{e}_n + o(\tilde{e}_n))$. From (6) and the preceding results we get

$$\begin{aligned} e_{n+1} = x_{n+1} - x^* &= y_n - x^* - (I - A_2 e_{n-1} + o(e_{n-1}))\Gamma^{-1}\Gamma(\tilde{e}_n + o(\tilde{e}_n)) \\ &= A_2 e_{n-1}\tilde{e}_n + o(e_{n-1}\tilde{e}_n) \\ &= A_2 e_{n-1} K e_{n-1}^a e_n^b + o(e_{n-1}^{a+1} e_n^b). \end{aligned}$$

Taking norms in the last expression, we have $\|e_{n+1}\| \le \|A_2\|\,\|K\|\,\|e_{n-1}\|^{a+1}\|e_n\|^b$, whose associated equation is $t^2 - bt - (a+1) = 0$, which has only one positive real root, $\frac{1}{2}(b+\sqrt{b^2+4a+4})$, that indicates the R-order of convergence of method (6). □

3. Construction of a new one-point iterative method with memory

Since our main aim is to construct iterative methods more efficient than the secant method, in view of Theorem 2.1, we can then construct method (6) with φ defined by Φ in (3) to obtain the following algorithm:

$$\begin{cases} x_0, x_{-1} \text{ given in } \Omega, \\ y_n = x_n - [x_{n-1}, x_n; F]^{-1} F(x_n), \\ x_{n+1} = y_n - [x_{n-1}, x_n; F]^{-1} F(y_n), \quad n \ge 0, \end{cases} \tag{7}$$

whose R-order of convergence is at least two, since method (5) is the secant method and consequently $a = b = 1$ in Theorem 2.1.

To compare the efficiency of methods (3) and (7), we use the efficiency index (EI) and the computational efficiency (CE), in the sense defined by Traub in [6], which are the order of convergence to the inverse power of the number of evaluations of functions and the order of convergence to the inverse power of the number of the involved products and divisions, respectively. To do this, we suppose that Eq. (1) represents a nonlinear system of dimension m; namely, $F(x_1, x_2, \dots, x_m) = 0$, where $F : \Omega \subseteq \mathbb{R}^m \to \mathbb{R}^m$ is a nonlinear function and $F \equiv (F_1, F_2, \dots, F_m)$ with $F_i : \Omega \subseteq \mathbb{R}^m \to \mathbb{R}$, $i = 1, 2, \dots, m$. From the algorithms of the methods, we count the evaluations of operators involved and analyse the operational cost needed to apply both methods.

To make the analysis of the efficiency easier, we write (3) as

$$\begin{cases} x_0, x_{-1} \text{ given in } \Omega, \\ [x_{n-1}, x_n; F]\delta_n = -F(x_n), \quad n \geq 0, \\ x_{n+1} = x_n + \delta_n, \end{cases}$$

so that, for solving nonlinear systems of m equations, the method requires the m functions F_i, $i = 1, 2, \dots, m$, and the $m(m-1)$ evaluations of functions in the divided difference matrix

$$[u, v; F]_{ij} = \frac{1}{u_j - v_j}\left(F_i(u_1, \dots, u_{j-1}, u_j, v_{j+1}, \dots, v_m) - F_i(u_1, \dots, u_{j-1}, v_j, v_{j+1}, \dots, v_m)\right).$$

$1 \leq i, j \leq m$, to evaluate per iteration; namely, m^2 evaluations of functions are required in total. Moreover, method (3) requires m^2 divisions to compute $[x_{n-1}, x_n; F]$, $(m^3 - m)/3$ products and divisions in the decomposition LU and m^2 products and divisions for solving two triangular linear systems. Therefore, $\frac{1}{3}(m^3 + 6m^2 - m)$ products and divisions are required in total. In consequence, $EI = (\frac{1+\sqrt{5}}{2})^{1/m^2}$ and $CE = (\frac{1+\sqrt{5}}{2})^{3/(m^3+6m^2-m)}$, since the R-order of convergence of the secant method is $\frac{1+\sqrt{5}}{2}$, see [2,5].

Next, we write (7) as

$$\begin{cases} x_0, x_{-1} \text{ given in } \Omega, \\ [x_{n-1}, x_n; F]\delta_n = -F(x_n), \quad n \geq 0, \\ y_n = x_n + \delta_n, \\ [x_{n-1}, x_n; F]\zeta_n = -F(y_n), \\ x_{n+1} = x_n + \delta_n + \zeta_n, \end{cases}$$

and we can easily see that the number of evaluations of functions required to apply (7) is $m^2 + m$, since a new evaluation of the operator F is needed, $F(y_n)$. Taking into account that the two linear systems to solve have the same matrix of coefficients, the decomposition LU is made only once, but two more triangular linear systems have to be solved (m^2 products and divisions), so that $\frac{1}{3}(m^3 + 9m^2 - m)$ is the total number of products and divisions required to apply (7). Consequently, for method (7), we obtain $EI = 2^{1/(m^2+m)}$ and $CE = 2^{3/(m^3+9m^2-m)}$.

In addition, method (7) is more efficient than the secant method, since its EI is higher for $m \geq 3$ and its CE is higher for $m \geq 2$, as we can see in Figs. 1–4, where we have drawn $\log(EI)$ and $\log(CE)$ instead of EI and CE, respectively, which is more realistic (it is the inverse of the work).

4. A semilocal convergence result for non-differentiable operators

Now we analyse the semilocal convergence of method (7). To do this, we use a technique based on proving first a system of recurrence relations. Firstly, we suppose that there exists a first-order divided difference $[x, y; F] \in \mathcal{L}(X, Y)$, for all $x, y \in \Omega$, where $\mathcal{L}(X, Y)$ denotes the space of bounded linear operators from X to Y. Secondly, we suppose that

(A1) $x_{-1}, x_0 \in \Omega$ are such that $\|x_0 - x_{-1}\| \leq \alpha$,

(A2) the bounded linear operator $A_0 = [x_{-1}, x_0; F]$ is invertible and such that $\|A_0^{-1}\| \leq \beta$, $\|A_0^{-1}F(x_0)\| \leq \eta$,

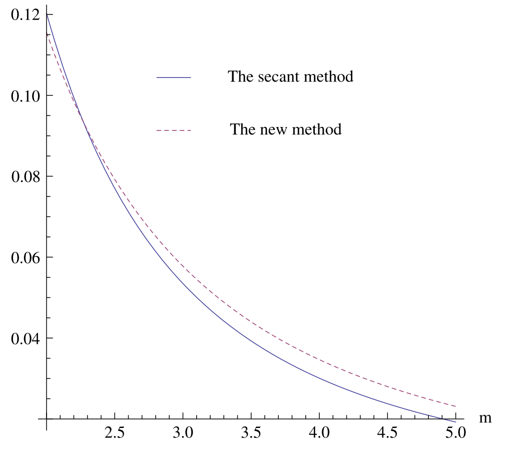

Fig. 1. *EI* if $m = 2, 3, 4, 5$.

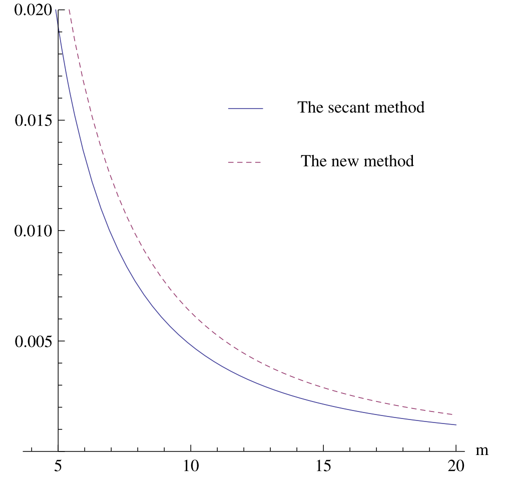

Fig. 2. *EI* if $m \geq 5$.

(A3) $\|[x, y; F] - [u, v; F]\| \leq \omega(\|x - u\|, \|y - v\|)$, $x, y, u, v \in \Omega$, where $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a continuous non-decreasing function in both arguments,

(A4) the equation

$$(1 + f(t))\eta = t(1 - f(t)(2 + f(t))), \tag{8}$$

where $f(t) = \frac{m}{1-\beta\omega(\alpha+t,t)}$, $m = \max\{m_1, m_2, m_3\}$, $m_1 = \beta\omega(\alpha, \eta)$, $m_2 = \beta\omega(\alpha, (1 + m_1)\eta)$ and $m_3 = \beta\omega((1 + m_1)\eta, \eta)$, has at least one positive root; the smallest positive root of (8) is denoted by R,

(A5) $f(R) \in (0, 0.3247\ldots)$,

(A6) $B(x_0, R) \subseteq \Omega$.

After that, we present three technical lemmas. In the first one, we give two expressions for the operator F that we need later.

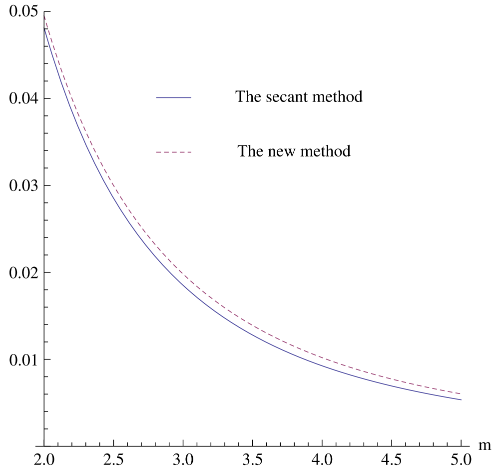

Fig. 3. *CE* if $m = 2, 3, 4, 5$.

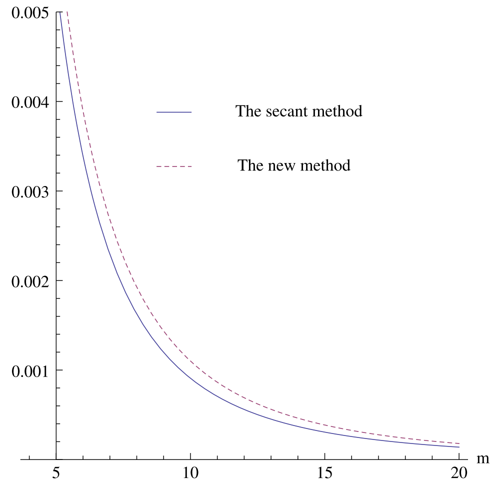

Fig. 4. *CE* if $m \geq 5$.

Lemma 4.1. *If $A_n = [x_{n-1}, x_n; F]$ and $B_n = [x_n, y_n; F]$, then, for all $n \in \mathbb{N}$, we have:*

$$F(y_n) = (B_n - A_n)(y_n - x_n),$$

$$F(x_{n+1}) = (A_{n+1} - A_n)(x_{n+1} - x_n) + (A_n - B_n)(y_n - x_n).$$

Lemma 4.2. *Suppose* (A1)–(A6). *Then*

$$(1 + f(R)) \sum_{i=0}^{n} f(R)^i (2 + f(R))^i \eta < \frac{(1 + f(R))\eta}{1 - f(R)(2 + f(R))} = R.$$

Lemma 4.3. *Suppose* (A1)–(A6). *Then, for $n \geq 1$, we have the following recurrence relations:*

[I] *There exists $A_n^{-1} = [x_{n-1}, x_n; F]^{-1}$ and $\|A_n^{-1}\| \leq \frac{\beta}{1-\beta\omega(\alpha+R,R)}$,*
[II] $\|y_n - x_n\| \leq f(R)(2+f(R))\|y_{n-1} - x_{n-1}\| \leq f(R)^n(2+f(R))^n\|y_0 - x_0\|$,
[III] $\|y_n - x_0\| \leq (1+f(R))\sum_{i=0}^{n} f(R)^i(2+f(R))^i\|y_0 - x_0\| < R$,
[IV] $\|x_{n+1} - x_n\| \leq (1+f(R))\|y_n - x_n\| \leq (1+f(R))f(R)^n(2+f(R))^n\|y_0 - x_0\|$,
[V] $\|x_{n+1} - x_0\| \leq (1+f(R))\sum_{i=0}^{n} f(R)^i(2+f(R))^i\|y_0 - x_0\| < R$.

Proof. Observe that $\|y_0 - x_0\| \leq \eta$, $\|z_0 - x_0\| \leq \eta$ and

$$\|x_1 - x_0\| \leq \|A_0^{-1}\|\,\|A_0 - B_0\|\,\|y_0 - x_0\| \leq m_1\|y_0 - x_0\| \leq (1+f(R))\eta < R.$$

Now, we can prove that **[I]**–**[VI]** is true for $n = 1$ and assume that **[I]**–**[VI]** are true for $k = 1, 2, \ldots, n-1$. Then,

[I] Since $\|I - A_0^{-1}A_n\| \leq \beta\omega(\|x_{n-1} - x_{-1}\|, \|x_n - x_0\|) \leq \beta\omega(\alpha + R, R) < 1$, then, by Banach's lemma, it follows

$$\|A_n^{-1}\| \leq \frac{1}{1 - \beta\omega(\|x_{n-1} - x_{-1}\|, \|x_n - x_0\|)} \leq \frac{\beta}{1 - \beta\omega(\alpha + R, R)}.$$

[II]

$$\begin{aligned}\|y_n - x_n\| &\leq \|A_n^{-1}\|\,\|F(x_n)\| \\ &\leq \|A_n^{-1}\|\omega(\|x_{n-1} - x_{n-2}\|, \|x_n - x_{n-1}\|)\|x_n - x_{n-1}\| \\ &\quad + \omega(\|x_{n-1} - x_{n-2}\|, \|y_{n-1} - x_{n-1}\|)\|y_{n-1} - x_{n-1}\| \\ &\leq f(R)(2+f(R))\|y_{n-1} - x_{n-1}\| \leq f(R)^n(2+f(R))^n\|y_0 - x_0\|,\end{aligned}$$

[III] $\|y_n - x_0\| \leq \|y_n - x_n\| + \|x_n - x_0\| \leq (1+f(R))\sum_{i=0}^{n} f(R)^i(2+f(R))^i\|y_0 - x_0\| < R$,
[IV]

$$\begin{aligned}\|x_{n+1} - x_n\| &\leq (1 + \|A_n^{-1}\|\,\|A_n - B_n\|)\|y_n - x_n\| \\ &\leq \left(1 + \frac{m}{1 - \beta\omega(\alpha + R, R)}\right)\|y_n - x_n\| \leq (1+f(R))\|y_n - x_n\| \\ &\leq (1+f(R))f(R)^n(2+f(R))^n\|y_0 - x_0\|.\end{aligned}$$

[V]

$$\begin{aligned}\|x_{n+1} - x_0\| &\leq \|x_{n+1} - x_n\| + \|x_n - x_0\| \\ &\leq (1+f(R))\sum_{i=0}^{n} f(R)^i(2+f(R))^i\|y_0 - x_0\| < R,\end{aligned}$$

so that **[I]**–**[VI]** are true for all positive integers k by mathematical induction. $\square$

Hence we are ready to prove the semilocal convergence of method (7) when it is applied to non-differentiable operators that satisfy conditions (A1)–(A6).

Theorem 4.4. *Let X and Y be two Banach spaces and let $F : \Omega \subseteq X \to Y$ be a nonlinear operator defined on a non-empty open convex domain Ω. We suppose that there exists $[x, y; F] \in \mathcal{L}(X, Y)$, for all $x, y \in \Omega$, and conditions* (A1)–(A6) *are satisfied. Then, sequence* (7)*, starting from x_{-1} and x_0, converges to a unique solution x^* of $F(x) = 0$. Moreover, the solution x^* and the iterates x_n belong to $\overline{B(x_0, R)}$ and x^* is unique in $B(x_0, R)$.*

Proof. From Lemma 4.3, it follows that (2) is a Cauchy sequence, since

$$\begin{aligned}\|x_{n+k}-x_n\| &\le \|x_{n+k}-x_{n+k-1}\|+\|x_{n+k-1}-x_{n+k-2}\|+\cdots+\|x_{n+1}-x_n\|\\ &\le (1+f(R))(\|y_{n+k-1}-x_{n+k-1}\|+\|y_{n+k-2}-x_{n+k-2}\|+\cdots+\|y_n-x_n\|)\\ &\le (1+f(R))\sum_{i=n}^{n+k-1} f(R)^i(2+f(R))^i\|y_0-x_0\|\\ &< (1+f(R))(2+f(R))^n\frac{1-f(R)^k(2+f(R))^k}{1-f(R)(2+f(R))}\eta.\end{aligned}$$

Consequently, $\{x_n\}$ is convergent and then $\lim_n x_n = x^* \in \overline{B(x_0,R)}$.

Observe now that

$$\|F(x_n)\| \le \omega(\eta,\eta)(2+f(R))\|y_{n-1}-x_{n-1}\|$$

and $\|y_{n-1}-x_{n-1}\| \to 0$ as $n\to\infty$, so that $F(x^*)=0$.

To prove the uniqueness of the solution x^*, we first assume that y^* is another solution of $F(x)=0$ in $B(x_0,R)$. Next, we consider the operator $P=[x^*,y^*;F]$, so that if P is invertible, we have $x^*=y^*$, since $P(y^*-x^*)=F(y^*)-F(x^*)$. Indeed,

$$\begin{aligned}\|A_0^{-1}P-I\| &\le \|A_0^{-1}\|\,\|P-A_0\| \le \|A_0^{-1}\|\,\|[y^*,x^*;F]-[x_{-1},x_0;F]\|\\ &\le \beta\omega(\|y^*-x_{-1}\|,\|x^*-x_0\|) \le \beta\omega(\alpha+R,R) < 1,\end{aligned}$$

and the operator P^{-1} exists. □

5. Application

In this section, we present an application of the previous analysis to the following nonlinear and non-differentiable integral equation of mixed Hammerstein type

$$x(s)=1+\frac{1}{2}\int_0^1 G(s,t)(|x(t)|+x(t)^2)\mathrm{d}t,\quad s\in[0,1], \tag{9}$$

where $x\in C[0,1]$, $t\in[0,1]$, and the kernel G is $G(s,t)=\begin{cases}(1-s)t, & t\le s,\\ s(1-t), & s\le t.\end{cases}$ We determine where a solution of (9) is located and is unique.

Then, from the study about the efficiency presented in Section 3, we choose method (7) for approximating the solution of (9).

To approximate numerically a solution of (9), we approximate the integral by a Gauss–Legendre quadrature with eight nodes,

$$\int_0^1 g(t)\mathrm{d}t \simeq \sum_{j=1}^{8} w_j g(t_j).$$

If we denote the approximations of $x(t_i)$ by x_i, $i=1,2,\ldots,8$, we obtain the following nonlinear system:

$$x_i = 1+\frac{1}{2}\sum_{j=1}^{8} b_{ij}(|x_j|+x_j^2),\quad i=1,2,\ldots,8, \tag{10}$$

where

$$b_{ij}=\begin{cases} w_j t_j(1-t_i) & \text{if } j\le i,\\ w_j t_i(1-t_j) & \text{if } j>i.\end{cases}$$

System (10) can now be written as $\mathbf{x}=\mathbf{1}+\frac{1}{2}B(\hat{\mathbf{x}}+\bar{\mathbf{x}})$, or

$$F:\mathbb{R}^8 \longrightarrow \mathbb{R}^8,\qquad F(\mathbf{x})\equiv \mathbf{x}-\mathbf{1}-\frac{1}{2}B(\hat{\mathbf{x}}+\bar{\mathbf{x}})=0, \tag{11}$$

Table 1
Numerical solution $\mathbf{x}^*$ of (10).

i	x_i^*	i	x_i^*	i	x_i^*	i	x_i^*
1	1.011501...	3	1.109892...	5	1.148143...	7	1.054678...
2	1.054678...	4	1.148143...	6	1.109892...	8	1.011501...

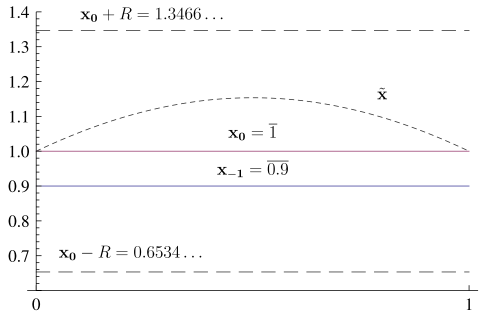

Fig. 5. Approximated solution $\tilde{\mathbf{x}}$ of Eq. (9).

where $\mathbf{x} = (x_1, x_2, \ldots, x_8)^T$, $\mathbf{1} = (1, 1, \ldots, 1)^T$, $B = (b_{ij})_{i,j=1}^8$, $\hat{\mathbf{x}} = (|x_1|, |x_2|, \ldots, |x_8|)^T$ and $\bar{\mathbf{x}} = (x_1^2, x_2^2, \ldots, x_8^2)^T$.

Moreover, for $\mathbf{u}, \mathbf{v} \in \mathbb{R}^8$, $[\mathbf{u}, \mathbf{v}; F] = ([\mathbf{u}, \mathbf{v}; F]_{ij})_{i,j=1}^8 \in \mathcal{L}(\mathbb{R}^8, \mathbb{R}^8)$, where

$$[\mathbf{u}, \mathbf{v}; F]_{ij} = \frac{1}{u_j - v_j}(F_i(u_1, \ldots, u_j, v_{j+1}, \ldots, v_8) - F_i(u_1, \ldots, u_{j-1}, v_j, \ldots, v_8)).$$

$\mathbf{u} = (u_1, u_2, \ldots, u_8)^T$ and $\mathbf{v} = (v_1, v_2, \ldots, v_8)^T$, so that, for the previous F, we have $[\mathbf{u}, \mathbf{v}; F] = I - \frac{1}{2}(C + D)$, where $C = (c_{ij})_{i,j=1}^8$ with $c_{ij} = b_{ij}(u_j + v_j)$ and $D = (d_{ij})_{i,j=1}^8$ with $d_{ij} = b_{ij}\frac{|u_j|-|v_j|}{u_j - v_j}$.

We now choose as starting points $\mathbf{x_{-1}} = (9/10, 9/10, \ldots, 9/10)^T$ and $\mathbf{x_0} = (1, 1, \ldots, 1)^T$. We work with the max-norm and obtain $\alpha = 1/10$, $\beta = 1.2116\ldots$, $\eta = 0.1459\ldots$ and $\omega(s, t) = \frac{1}{2}(0.1235\ldots)(s + t + 2)$. We also obtain that $m = 0.1734\ldots$, the solution of (8) is $R = 0.3466\ldots$ and $f(R) = 0.2192\ldots \in (0, 0.3247\ldots)$. In consequence, the hypotheses of Theorem 4.4 are satisfied and method (7) converges, after five iterations, to the solution $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_8^*)^T$ given in Table 1. Note that 32 significative figures are used in the computations.

Furthermore, the existence of the solution is guaranteed in the ball $\overline{B(\mathbf{x_0}, 0.3466\ldots)}$ and the uniqueness in $B(\mathbf{x_0}, 0.3466\ldots)$.

Finally, we interpolate the points of Table 1 and taking into account that the solution of (9) satisfies $x(0) = x(1) = 1$, an approximation $\tilde{\mathbf{x}}$ of the numerical solution $\mathbf{x}^*$ is obtained (see Fig. 5). Observe that the interpolated approximation $\tilde{\mathbf{x}}$ lies within the existence domain of the solution obtained above.

Finally, we consider the computational efficiency index (CEI) of an iterative method with local order of convergence at least ρ, in the sense defined in [3], as

$$CEI(m) = \rho^{1/\mathcal{C}(m)},$$

where the computational cost is $\mathcal{C}(m) = \widetilde{P}(m) + P(m)$ with $\widetilde{P}(m)$ the number of product and divisions in the evaluation of F and $P(m)$ the number of product and divisions to do an iteration. If we analyse F in (11), we have to do $m + 2$ products and divisions each evaluation of the m scalar functions F_i of F ($i = 1, 2, \ldots, m$). Then, for iterative method (3), we have $\widetilde{P}_{(3)}(m) = (m + 2)m^2$, $P_{(3)}(m) = \frac{1}{3}(m^3 + 6m^2 - m)$ and

$$CEI_{(3)}(m) = \left(\frac{1 + \sqrt{5}}{2}\right)^{3/(4m^3+12m^2-m)}$$

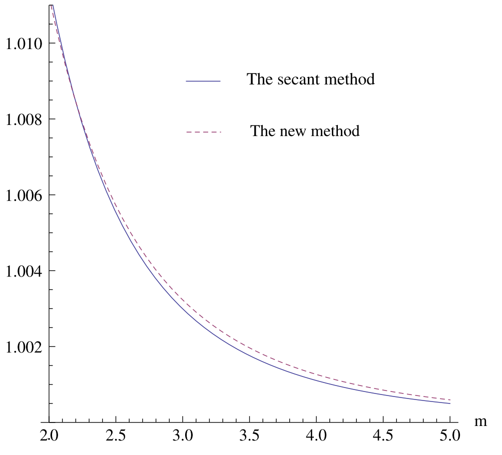

Fig. 6. *CEI* if $m = 2, 3, 4, 5$.

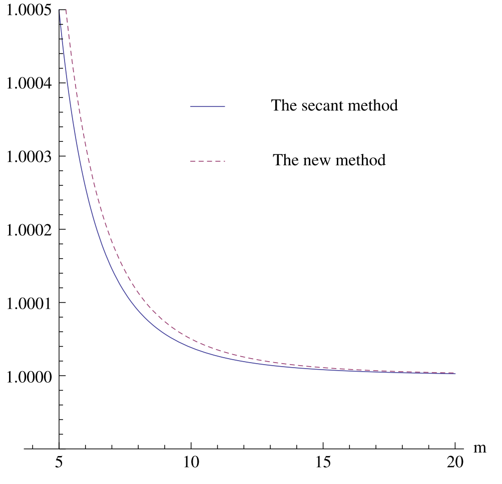

Fig. 7. *CEI* if $m \geq 5$.

and, for (7), we obtain $\widetilde{P}_{(7)}(m) = (m+2)(m^2+m)$, $P_{(7)}(m) = \frac{1}{3}(m^3+9m^2-m)$ and

$$CEI_{(7)}(m) = 2^{3/(4m^3+18m^2-m)}.$$

Consequently, as we can see in Figs. 6 and 7, $CEI_{(7)}(m) \geq CEI_{(3)}(m)$, for $m \geq 3$.

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