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Detecting real singularities of a space curve from a real rational parametrization☆

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ABSTRACT

In this paper we give an algorithm that detects real singularities, including singularities at infinity, and counts local branches and multiplicities of real rational curves in the affine n-space without knowing an implicitization. The main idea behind this is a generalization of the D-resultant (see [van den Essen, A., Yu, J.-T., 1997. The D-resultant, singularities and the degree of unfaithfulness. Proc. Amer. Math. Soc. 25 (3), 689–695]) to n rational functions. This allows us to find all real parameters corresponding to the real singularities between the solutions of a system of polynomials in one variable.

0. Introduction

The interest in the algorithmic study of algebraic curves has increased in the last decades, due mainly to the creation of computer aided design tools (CAD), which are used almost in every branch of engineering and industrial design. CAD programs use, generally, rational parametric representation of curves; this is why the study and manipulation of curves from a parametrization has become a point of interest in Computer Algebra.

This paper is devoted to the problem of detecting all singularities and local branches of rational curves in the affine n-space, assuming a rational parametrization is known. The classical approach to compute singularities deals with the implicit equation of planar curves (see for instance Walker (1950)). Moreover, this problem has been approached algorithmically for the case of planar curves implicitly given (see for instance Sakkalis and Farouki (1990)).

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Many authors have addressed this problem when the planar curve is given parametrically, that is, assuming that its implicit equations are not available. In van den Essen and Yu (1997) the notion of Taylor resultant, introduced by Abhyankar for planar curves parametrized by polynomials over fields of characteristic zero, is generalized as the *D*-resultant of two polynomials over arbitrary fields; and it is used, among other purposes, to detect singularities of planar curves parametrized by polynomials. Afterwards, in Gutierrez et al. (2002b) the *D*-resultant of two rational functions is introduced and applied to compute singularities of rational planar curves over algebraically closed fields. In a recent paper (Pérez-Díaz, 2007), pointed out by the reviewer, singularities of rational planar curves are computed without introducing algebraic numbers.

For curves in the affine *n*-space, Park (see Park (2002)) gives a method which computes the singularities of polynomially parametrized curves over fields of characteristic zero by means of Gröbner basis. In Rubio et al. (2004) singularities of the image of a rational parametrization over algebraically closed fields are computed by means of generalized resultants. A more general approach is provided by Lazard (Lazard, 2006) for arbitrary varieties given by a rational parametrization; the author introduces the graph of a parametrization in order to obtain intersections and singularities, analyzing both real and complex cases; computations are made via Gröbner basis.

Other approaches to this problem are given for instance in Manocha and Canny (1992) and Li and Cripps (1997) where cusp and inflexion points of cubic curves in the space are computed. More recently, the notion of μ-basis, introduced by Sederberg and Chen, is used in Chen (2006) to detect singularities of rational parametrized planar curves .

In this work we generalize the main results in Gutierrez et al. (2002b), Park (2002) and Rubio et al. (2004) to any affine rational curve. More precisely, we give a method to compute all real singularities of a curve given by a proper rational real parametrization in the affine *n*-space, including singularities at infinity. Furthermore, we count the local branches at singular points, distinguish between ordinary and non-ordinary singularities and compute its multiplicity.

The *D*-resultant of two rational functions is defined in Gutierrez et al. (2002b) as the resultant of two symmetric bivariate polynomials associated to the rational functions. When the rational parametrization is proper, the set of zeroes of the *D*-resultant contains the parameters generating singular points. These bivariate polynomials are straightforwardly generalized for *n*-rational functions. Park shows that the projection of the set of zeroes of the bivariate polynomials associated to *n* polynomials contains the parameters of singular points and uses Gröbner basis to compute these parameters. Here we extend Park's result to rational parametrizations and we show that we can use generalized resultants in order to compute the projection of the set of zeroes of the bivariate polynomials. Afterwards, we adapt these results to compute all singular points of real rational curves.

In Section 1 we review some facts on proper parametrizations and generalized resultants. Section 2 contains the main results to detect singularities over the complex numbers and compute them using generalized resultants. The third section studies real singularities from a real parametrization, provides an algorithm to detect real singular points and count the local branches, including singularities at infinity, and shows some illustrative examples. The last section includes computation of multiplicity.

Part of this work was presented in the International Conference on Algebraic Geometry and Geometric Modeling (AGGM'06) held in Barcelona September 4–7, 2006 (see Rubio et al. (2006)).

1. On proper parametrizations and generalized resultants

Let C be an algebraic curve in $\mathbb{C}^n$, that is a one-dimensional set of solutions of a system of polynomial equations in *n* variables with complex coefficients.

Definition 1. A *rational parametrization* of an algebraic curve C in $\mathbb{C}^n$ is a map

$$\begin{aligned} \psi : \quad \mathbb{C} &\dashrightarrow \mathbb{C}^n \\ t &\longmapsto (f_1(t), \ldots, f_n(t)) \end{aligned}$$

where $f_i \in \mathbb{C}(t)$, $\forall i = 1, \ldots, n$, such that $C = \overline{Im\ \psi}$ (Zariski closure in $\mathbb{C}^n$ of the image of ψ). In this case we say that C is a *rational curve*.

Without loss of generality, given $\psi = (f_1, \ldots, f_n)$ we can suppose that f_i is not constant for $i = 1, \ldots, n$. Should any f_i were constant the curve might be embedded in a lower-dimensional space.

From now on given a rational function $f \in \mathbb{C}(t)$ we denote by f_N its numerator and by f_D its denominator assuming always that $\gcd(f_N, f_D) = 1$; i.e. $f = \frac{f_N}{f_D}$, with $f_N, f_D \in \mathbb{C}[t]$ being coprimes.

Definition 2. The *degree of a rational function* f is $\deg(f) = \max\{\deg(f_N), \deg(f_D)\}$.

Definition 3. Let $f \in \mathbb{C}(t)$. We define the *bivariate polynomial associated to* f as:

$$g(s,t) = \frac{f_N(s)f_D(t) - f_N(t)f_D(s)}{s-t}.$$

We denote by $g_1, \ldots, g_n \in \mathbb{C}[s,t]$ the bivariate polynomials associated to the rational functions $f_1, \ldots, f_n$ of the parametrization ψ.

Definition 4. The *generalized resultants* of $g_1, \ldots, g_n \in \mathbb{C}[s,t]$ are the polynomials $h_\alpha(t)$ defined by

$$\mathrm{Res}_s(g_1, u_2g_2 + \cdots + u_ng_n) = \sum_\alpha h_\alpha(t)\underline{u}^\alpha,$$

where $\underline{u} = (u_2, \ldots, u_n)$ are new variables.

For more details on generalized resultants see Cox et al. (1997).

Definition 5. The map ψ is a *proper parametrization* of C if there exist two Zariski open sets $U \subset \mathbb{C}$, $V \subset C$ such that $\psi|_U : U \longrightarrow V$ is bijective.

Remark 6. Let k be a field and $\psi : k \dashrightarrow k^n$ a rational parametrization over k. By Lüroth's Theorem ψ is proper if and only if $k(f_1, \ldots, f_n) = k(t)$. There exist more constructive characterizations of properness; for instance, using the bivariate polynomials $g_1, \ldots, g_n$ associated to $f_1, \ldots, f_n$, properness is equivalent to $\gcd(g_1, \ldots, g_n) \in k$ (see Gutierrez et al. (2002a)).

Furthermore, if the parametrization is not proper we can find a Lüroth's generator , i.e. a rational function h such that $k(f_1, \ldots, f_n) = k(h)$, and making $f_i = \tilde{f}_i(h)$, the new parametrization $(\tilde{f}_1, \ldots, \tilde{f}_n)$ is proper (see Sederberg (1986)).

The following characterization of properness using generalized resultants can be found in Rubio et al. (2004).

Proposition 7. *A rational parametrization ψ is proper if and only if* $\mathrm{Res}_s(g_1, u_2g_2 + \cdots + u_ng_n) \not\equiv 0$.

By definition a proper parametrization ψ of a curve C is almost bijective, i.e. all points of C, except a finite number of them, are reached by exactly one parameter and its image recovers C except a finite number of points. Proposition 42 in Andradas and Recio (2007) provides a more accurate result about surjectivity.

Proposition 8. *Let ψ be a proper parametrization of an algebraic complex curve C. Then $C \setminus \mathrm{Im}(\psi)$ contains at most the point $P_\infty = \lim_{t\to\infty} \psi(t)$.*

The point P_∞ is called the *critical point* of ψ. Note that the critical point exists if and only if $\deg(f_{i_N}) \le \deg(f_{i_D})$ for all $i = 1, \ldots, n$. In this case,

$$P_\infty = \left(\frac{N^{(1)}}{D^{(1)}}, \ldots, \frac{N^{(n)}}{D^{(n)}}\right),$$

where $N^{(i)}$ and $D^{(i)}$ are the coefficients of degree d_i of f_{i_N} and f_{i_D} respectively and $d_i = \deg(f_i)$.

The zero set of the generalized resultants of the bivariate polynomials associated with ψ not only give a test of properness for ψ, but they also describe the parameters given singularities of C as we show in the next section.

2. Singularities of rational curves

In this section we compute the singularities of a rational curve from a proper parametrization over the complex numbers.

Let $\psi = (f_1, \ldots, f_n)$ be a proper parametrization of C. The algebraic variety $V = V(g_1, \ldots, g_n) \subset \mathbb{C}^2$ defined by the bivariate polynomials g_i associated to f_i is essential for the forthcoming results.

Lemma 9. *The variety V can be rewritten as $V = A_\psi \cup B_\psi \cup C_\psi$ where*
$A_\psi = \left\{(s,t) \in \mathbb{C}^2 / \ s \neq t,\ g_i(s,t) = 0,\ f_{i_D}(s) \neq 0\ \forall i = 1, \ldots, n\right\}$,
$B_\psi = \left\{(t,t) \in \mathbb{C}^2 / \ g_i(t,t) = 0,\ f_{i_D}(t) \neq 0\ \forall i = 1, \ldots, n\right\}$,
$C_\psi = T^2 \cap V$ *where* $T = \left\{t \in \mathbb{C} \ / \ f_{i_D}(t) = 0 \text{ for some } i = 1, \ldots, n\right\}$.

Proof. It is obvious that the sets A_ψ, B_ψ and C_ψ are included in V. For the other inclusion, let $(s_0, t_0) \in V$. Then $g_i(s_0, t_0) = 0$ for all $i = 1, \ldots, n$. That is, $f_{i_N}(t_0) f_{i_D}(s_0) = f_{i_N}(s_0) f_{i_D}(t_0)$.
If $f_{i_D}(t_0) = 0$ for some i, $f_{i_N}(t_0) \neq 0$ and $f_{i_D}(s_0) = 0$. Hence $(s_0, t_0) \in C_\psi$.
If $f_{i_D}(t_0) \neq 0$ for all i, $(s_0, t_0) \in A_\psi \cup B_\psi$. □

From now on let π_1 and π_2 be the projections of $\mathbb{C}^2$ over the first and second coordinate, respectively.

Proposition 10. *Let ψ be a proper parametrization. Then,*

- *$P \in \psi(\pi_2(A_\psi))$ if and only if there exist different parameters $t, s \in \mathbb{C}$, $t \neq s$ with $\psi(t) = \psi(s) = P$. In other words, $\psi(\pi_2(A_\psi))$ is the set of all singular points, except at most P_∞, that belong to at least two local branches.*
- *$Q \in \psi(\pi_2(B_\psi))$ if and only if there exists a parameter $t \in \mathbb{C}$ satisfying that $\psi(t) = Q$ and $\psi'(t) = 0$. In other words, $\psi(\pi_2(B_\psi))$ is the set of singular points, except at most P_∞, that belong to a cusp-type branch.*

Proof. If $P \in \psi(\pi_2(A_\psi))$, then $P = \psi(t)$ for some $t \in \pi_2(A_\psi)$. Thus, there is $s \in \mathbb{C}$, $s \neq t$ such that $g_i(s,t) = 0$, $f_{i_D} \neq 0$ for all $i = 1, \ldots, n$. In consequence, $P = \psi(t) = \psi(s)$. Conversely, if $P = \psi(t) = \psi(s)$ with $s \neq t$ we have $s, t \in \pi_2(A_\psi)$. Therefore, P is a singular point that belongs at least to two local branches if and only if $P \in \psi(\pi_2(A_\psi))$ or possibly $P = P_\infty$.

The second statement follows from Lemma 1.7 in Gutierrez et al. (2002b) where it is shown that $g_i(s,s) = f_{i_D}(s)^2 f_i'(s)$. In consequence, Q ($\neq P_\infty$) is a singular point that belong to a cusp-type branch if and only if $Q \in \psi(\pi_2(B_\psi))$. □

From the proposition above, the next theorem follows.

Theorem 11. *If ψ is proper, then V is a finite set and*

$$\psi(\pi_2(V)) = \psi(\pi_2(A_\psi \cup B_\psi)) = \psi(\pi_1(A_\psi \cup B_\psi)) \subset C$$

describes all the singular points of C except at most P_∞.

The result below allows us to detect the parameters corresponding to singular points as solutions of a univariate polynomial system. Given a polynomial $G(s, \underline{x}) \in \mathbb{C}[s, \underline{x}]$, we denote by $\mathrm{lc}_s(G) \in \mathbb{C}[\underline{x}]$ the leading coefficient of G with respect to the variable s, that is, the coefficient of the monomial of greatest degree in the variable s.

Theorem 12. *Let $W = \{t_0 \in \mathbb{C} : \mathrm{lc}_s(g_1(s,t))(t_0) = 0, \mathrm{lc}_s(G(s,t,\underline{u}))(t_0, \underline{u}) = 0\}$, where $G(s,t,\underline{u}) = u_2 g_2(s,t) + \cdots + u_n g_n(s,t)$. If ψ is proper,*

$$V(h_\alpha) \setminus W \subset \pi_2(V) \cup \{t_0 \in \mathbb{C} \mid \psi(t_0) = P_\infty \in \mathbb{C}^n\} \subset V(h_\alpha)$$

Proof. Since ψ is proper, by Theorem 7, $\mathrm{Res}_s(g_1, G)(t, \underline{u}) \not\equiv 0$. Take $t_0 \in V(h_\alpha)\backslash W$. By the behaviour of the resultant under the evaluation homomorphism (see Winkler (1996)), $\mathrm{Res}_s(g_1(s, t_0), G(s, t_0, \underline{u})) = 0$. So there exists $s_0 \in \mathbb{C}$ such that $g_1(t_0, s_0) = 0$ and $G(t_0, s_0, \underline{u}) = 0$. Hence $g_i(s_0, t_0) = 0$ for all i and $t_0 \in \pi_2(V)$.

For the other inclusion, let $t_0 \in \pi_2(V)$. Then, there exists $s_0 \in \mathbb{C}$ such that $g_i(s_0, t_0) = 0$ for all i. On the other hand, there exist $p, q \in \mathbb{C}[s, t, u_2, \dots, u_n]$ such that $D(t, \underline{u}) = \mathrm{Res}_s(g_1, G) = pg_1 + qG$. If we evaluate this equality in $(s_0, t_0, u_2, \dots, u_n)$ we have $D(t_0, \underline{u}) = 0$ and $h_\alpha(t_0) = 0$ for all α. If $\psi(t_0) = P_\infty \in \mathbb{C}^n$, then the leading coefficients of the g_i's with respect to s vanish at t_0. □

Theorem 13. *There is a proper parametrization of C, $\widetilde{\psi}$, such that $\widetilde{\psi}(\pi_2(\widetilde{V}))$ describes all the singularities of C, where $\widetilde{V}$ is the zero set of the bivariate polynomials associated to $\widetilde{\psi}$.*

Proof. Consider a proper parametrization $\psi(t) = (f_1(t), \dots, f_n(t))$. If the critical point of ψ does not exist, we are done. Otherwise, take $t_0 \notin V(h_\alpha)$. Theorem 12 implies that $\psi(t_0)$ is not a singular point of C. Then, it suffices to take an invertible change of parameter $\sigma(t) = \dfrac{t_0 t + b}{t - t_0}$ which makes $\psi(t_0)$ to be the critical point. □

3. Detecting real singularities

Theorems 12 and 13 provide a method to detect singularities of rational curves over the complex numbers by computing the generalized resultants of the bivariate polynomials associated to the parametrization. In this section we adapt these ideas to the real case.

Take a proper parametrization $\psi(t) = (f_1(t), \dots, f_n(t))$ with $f_i \in \mathbb{R}(t)$ for $i = 1, \dots, n$, then the Zariski closure of $\psi(\mathbb{R})$ in $\mathbb{R}^n$ is a real curve $C_\mathbb{R}$. From Proposition 42 in Andradas and Recio (2007) we have the following:

Proposition 14. *$\psi(\mathbb{R})$ contains every point of $C_\mathbb{R}$ except at most the isolated ones and possibly one extra point $P_\infty = \lim_{t\to\infty} \psi(t) \in C_\mathbb{R}$.*

Applying the results of the previous section to $\psi : \mathbb{C} \dashrightarrow \mathbb{C}^n$ and taking

$$C = \overline{\{\psi(t) : t \in \mathbb{C}\}} \subset \mathbb{C}^n$$

we can recover this “missing points" as follows:

- To recover the critical point P_∞ it suffices to change the parameter as in Theorem 13, taking for instance $t = \dfrac{t_0 t + 1}{t - t_0}$ with $t_0 \notin V(h_\alpha)$, $t_0 \in \mathbb{R}$.
- The isolated points correspond to pairwise complex conjugated branches. Thus they can be detected in A_ψ as real points corresponding to pairwise complex conjugated parameters.

Remark that Theorem 12 allows us to compute the parameters associated to each affine singular point, therefore we have the parameter corresponding to each local branch through these points. We distinguish between *smooth branch* and *cusp* depending on whether the corresponding parameter is not in the projection of B_ψ or it is, (see Proposition 10).

A singular point is called *ordinary* if all its branches are smooth with different tangents. Otherwise, we say that it is *non-ordinary*.

The following algorithm computes the singularities and branches of a real rational curve using a real parametrization.

Algorithm 1 (*Detecting Singular Points*).

Input: $\psi = (f_1, \dots, f_n) \in \mathbb{R}(t)^n$ a proper parametrization with f_i nonconstant for every $i = 1, \dots, n$.

Output: The real singular points of the curve defined by $(f_1, \dots, f_n)$, the number of real smooth branches and cusp through each one and the character ordinary or non-ordinary of each one.

Step 1: *Compute* $g_1, \dots, g_n$ *the bivariate polynomials associated to* $f_1, \dots, f_n$.
Step 2: *Compute* $S = V(h_\alpha) \subset \mathbb{C}$ *where* $\mathrm{Res}_s(g_1, u_2g_2 + \cdots + u_ng_n) = \sum_\alpha h_\alpha(t)\underline{u}^\alpha$.
Step 3: *If* $\deg f_{i_N} > \deg f_{i_D}$ *for some i then go to Step 5.*
Step 4: *Let* $t_0 \in \mathbb{R}\setminus S$ *and* $\sigma = \dfrac{t_0t+1}{t-t_0}$. *Let* $\psi = \psi(\sigma), g_1, \dots, g_n$ *the associated bivariate polynomials and* $S = \sigma(S) \cup \{t_0\}$.
Step 5: *Let* $\{P_1, \dots, P_l\} = \psi(S) \cap \mathbb{R}^n$.
Step 6: *For each i take* $t_{i1}, \dots, t_{im_i} \in S \cap \mathbb{R}$ *such that* $\psi(t_{ij}) = P_i$ *for all j*
— *If there is no* t_{ij} *for* P_i, *return* P_i *is an isolated point.*
Step 7: *For each* $j = 1, \dots, m_i$, *compute* $sl_{ij} = (g_1(t_{ij}, t_{ij}), \dots, g_n(t_{ij}, t_{ij}))$.
Let $c_i = \#\left\{j \in \{1, \dots, m_i\} \mid sl_{ij} = (0, \dots, 0)\right\}$.
— *If* $m_i = 1$ *and* $c_i = 0$ *take another i.*
— *If* $c_i > 0$, *return* P_i *is a non-ordinary singularity with* $m_i - c_i$ *smooth branches and* c_i *cusps.*
— *If* $m_i > 1$ *and* $c_i = 0$, *compute for each* $j = 1, \dots, m_i$:
$T_j = ((g_1/f_{1_D}^2)(t_{ij}, t_{ij}), \dots, (g_n/f_{n_D}^2)(t_{ij}, t_{ij}))$ *and* $\tau_i = \#\left\{T_j/\|T_j\| \ : \ j \in \{1, \dots, m_i\}\right\}$.
- *If* $\tau_i = m_i$, *return* P_i *is an ordinary singularity with* m_i *smooth branches.*
- *Otherwise, return* P_i *is an non-ordinary singularity with* m_i *smooth branches.*

Step 8: *If no* P_i *is returned, then the curve is nonsingular.*

Proof. In Step 4 we reparametrize the curve, if it is necessary, in order to have a new proper parametrization such that the point corresponding to $t = \infty$ is not singular. In Step 6 we collect all the parameters that give the same point and in Step 7 we count the cusps and discard the nonsingular points (i.e. points that are neither nodes, $m_i = 1$, nor cusps, $\psi' \neq 0$). Moreover, also in Step 7, for each singular point P_i we compute the tangent vector T_j, using $f_i'(s) = \frac{g_i(s,s)}{f_{i_D}(s)^2}$ (see Lemma 1.7 in Gutierrez et al. (2002b)), when it exists. So we can distinguish between ordinary and non-ordinary singular points. □

The previous algorithm has been implemented in Maple 10 as "singRES". Let us show some examples.

Example 1. Let C be the rational curve parametrized by

$$f_1 := \frac{t^3+1}{t^3+t}, \qquad f_2 := \frac{3t^4-4t^3+t^2+3t-3}{t^5+3t^4-2t^3+3t^2-3t}, \qquad f_3 := \frac{t^2}{t^2+3t-3}.$$

If we perform "singRES" for this parametrization we have the following output

$$[[[0, -1/4, 1/4], \ \textit{isolated}], [[1, 0, 1], \ 2, \ 0, \ \textit{ordinary singularity}]].$$

Then we can conclude that $C_\mathbb{R}$ has two singular points,

- an isolated point in $(0, -1/4, 1/4)$, and
- an ordinary node in $(1, 0, 1)$ (2, 0 indicate that there are two smooth branches and zero cusps).

Let us analyze some steps of the algorithm. The set $V(h_\alpha)$ contains three complex parameters $\left\{1/2 + \sqrt{3}/2\mathrm{i}, \ 1/2 - \sqrt{3}/2\mathrm{i}, \ 1\right\}$. Since $P_\infty = (1, 0, 1)$, we make the change of parameter $t = 1/t$, that is, taking $t_0 = 0 \notin V(h_\alpha)$ in Step 4. The new set of parameters is $\left\{\frac{2}{1+\sqrt{3}\mathrm{i}}, \ \frac{2}{1-\sqrt{3}\mathrm{i}}, \ 1, \ 0\right\}$. If we compute the corresponding points in the new parametrization we obtain

- $(0, -1/4, 1/4)$ corresponds to $t = \frac{2}{1+\sqrt{3}\mathrm{i}}$ and $t = \frac{2}{1-\sqrt{3}\mathrm{i}}$, hence it is an isolated point.
- $(1, 0, 1)$ corresponds to $t = 1$ and $t = 0$. In Step 7 we compute the tangent vectors at each of these branches, verifying that they are different and non-zero. Therefore, $(1, 0, 1)$ is an ordinary node with two branches. Note that for the former parametrization there was a missing branch at $(1, 0, 1)$.

The possible singularities of the curve at infinity can be also computed. It suffices to embed the curve into the projective space of coordinates $(u : x_1 : \cdots : x_n)$ and apply Algorithm 1 to the converted parametrization in different charts as follows:

Algorithm 2 (*Singular Points at Infinity*).

Input: $(f_1, \ldots, f_n) \in \mathbb{R}(t)^n$ a proper parametrization with f_i nonconstant for every $i = 1, \ldots, n$.

Output: The real singular points $(0 : a_1 : \cdots : a_n)$ at infinity of the curve defined by $(f_1, \ldots, f_n)$, the number of real smooth branches and cusp through each one and the character ordinary or non-ordinary of each one.

Step 1: *Let $i = 1$ and L an empty list.*
Step 2: *While $i \geq n$*
- *Compute $\psi_{x_i} = (1/f_i, f_1/f_i, \ldots, f_{i-1}/f_i, f_{i+1}/f_i, \ldots, f_n/f_i)$.*
- *Apply Algorithm 1 to $\psi = \psi_{x_i}$.*
- *For any singular point of type $(0, a_1, \ldots, a_{i-1}, a_{i+1}, \ldots a_n)$ returned by Algorithm 1, append $(0 : a_1 : \cdots : a_{i-1} : 1 : a_{i+1} : \cdots : a_n)$ to L if it is not already in L.*
- *Set $i := i + 1$.*

Step 3: *If L is empty return the curve has nonsingular point at infinity, else return L.*

Proof. We embed the curve into the projective space of coordinates $(u : x_1 : \cdots : x_n)$ as $(1, f_1, \ldots, f_n)$. In the affine chart putting $x_i = 1$, we take coordinates $(u/x_i, x_1/x_i, \ldots, x_{i-1}/x_i, x_{i+1}/x_i, \ldots, x_n/x_i)$. Thus, we have the curve parametrized by ψ_{x_i}. Then singular points at infinity must be singular points of ψ_{x_i} having the first coordinate equal to 0. □

Example 2. Take now the rational curve $C_{\mathbb{R}}$ in $\mathbb{R}^3$ given by the parametrization

$$f_1 = \frac{t(t^3+1)^2}{t^3+t+1}, \qquad f_2 = \frac{(t^3+1)t^2}{t^3+t+1}, \qquad f_3 = \frac{t(t^3+1)^3}{(t^3+t+1)^2}.$$

Note that P_∞ does not exist.
The zero set of the generalized resultant $V(h_\alpha)$ contains 7 points, distributed as follows:

- Three of them are the roots of $t^3 + t + 1$, that is, elements of the projection of C_ψ that do not generate affine points.
- Two of them are complex conjugated and correspond to the point (0, 0, 0).
- The last two are $t_1 = 0$ and $t_2 = -1$ and correspond also to the point (0, 0, 0).

The output of "singRES" for this curve is

[[[0, 0, 0], 2, 0, *ordinary singularity*]]

which means that (0, 0, 0) is the only singular point of $C_{\mathbb{R}}$, being an ordinary singularity with two smooth branches, i.e. it is an ordinary node.

Note that if we consider the complex curve, (0, 0, 0) is also the only singular point, but it has four local branches.

Furthermore, we can look for singularities at infinity applying Algorithm 2. Let us analyze Step 2.

- For $i = 1$ we obtain, after applying Algorithm 1,

  [[[0, 0, 1], 0, 1, *non-ordinary singularity*]].

  Then, $L = \{[[0 : 1 : 0 : 1], 0, 1$ *non-ordinary singularity*$]\}$.
- For $i = 2$ we obtain, after applying Algorithm 1,

  [].

  Then, $L = \{[[0 : 1 : 0 : 1], 0, 1$ *non-ordinary singularity*$]\}$.

– For $i = 3$ we obtain, after applying Algorithm 1,

$$[[[0, 1, 0],\ 0,\ 1,\ \textit{non–ordinary singularity}]].$$

Then, $L = \{[[0 : 1 : 0 : 1],\ 0,\ 1\ \textit{non-ordinary singularity}]\}$.

In consequence, this rational curve has an ordinary node at (0, 0, 0) and a cusp at infinity.

Remark 15. Algorithm 1 computes real affine singular points of rational curves over $\mathbb{R}$. Moreover, it counts local real branches and distinguishes between ordinary and non-ordinary singularities. It could be given a similar algorithm for complex rational curves because it is based on Theorems 12 and 13 that hold over $\mathbb{C}$.

4. Computing multiplicities

The procedure exhibited in the previous section provides some information about multiplicity of singular points. For each point, we get the number of local branches with their character of smooth or cusp, and the number of different tangents in order to distinguish between ordinary and non-ordinary singular points. For ordinary singular points and non-ordinary ones with only smooth branches, the multiplicity can be computed counting the number of branches. However, if the points belong to some cusp we do not have complete information about its multiplicity. In this section we show how to compute the multiplicity of a rational curve given by a proper rational parametrization.

The *multiplicity of C at P* can be obtained as the sum of the multiplicities of the local branches of C at P (see Walker (1950) for more details). Therefore, our objective is to determine the multiplicity of each local branch.

A local branch γ of a curve C in $\mathbb{K}^n$ ($\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$) at the origin is the image of a punctured disk in $\mathbb{K}^n$ by a parametrization of type $(q_1(s), \dots, q_n(s))$, where $q_i(t) \in \mathbb{K}[[s]]$ is a convergent power series. We say that $(q_1(s), \dots, q_n(s))$ is primitive if the greater common divisor of all exponents of the q_i is equal to 1.

The multiplicity of γ is the minimum of $\{ord_s(q_1(s)), \dots, ord_s(q_n(s))\}$, where $ord_s(q(s))$ denotes the minimal i such that $a_i \neq 0$ being $q(s) = \sum a_i s^i$.

Suppose now that C is a rational curve given by a rational parametrization $\psi = (f_1(t), \dots, f_n(t))$ as above.

Proposition 16. *Let γ_a be a local branch of C at $P = (p_1, \dots, p_n) = \psi(a)$ with $a \in \mathbb{K}$. Then the multiplicity of γ_a is the minimum order in s of the power series expansions of $f_1(s+a) - p_1, \dots, f_n(s+a) - p_n$.*

Proof. Up to a change of coordinates $(x_n - p_1, \dots, x_n - p_n)$ we can assume that P is the origin.

By Andradas and Recio (2007), γ_a corresponds to a discrete (real if $\mathbb{K} = \mathbb{R}$) valuation ring W of the rational function field $\mathbb{K}(C)$ of C and W is isomorphic to $\mathbb{K}[t]_{(t-a)}$ via the rational function field isomorphism induced by ψ. The *multiplicity of* γ_a is the minimal valuation of elements in the maximal ideal $(x_1, \dots, x_n)$ modulo the ideal of C. Then, taking $t = s + a$ we are done. □

Let $P = (p_1, \dots, p_n)$ be a singular point of C. We can assume that P is not a critical point for ψ, because otherwise we reparametrize (see Theorem 13). From Algorithm 1 we have the set of all parameters $t_1, \dots, t_m$ given P. Each parameter t_i represents a local branch of C through P and we also know if γ_{t_i} is smooth or cusp. Suppose that we have $t_1, \dots, t_c$ gives c cusp and $t_{c+1}, \dots, t_m$ gives $m-c$ smooth branches.

- For $i = c+1, \dots, m$, γ_{t_i} has multiplicity is 1.
- For $i = 1, \dots, c$, we compute the multiplicity l_i of γ_{t_i} using Proposition 16.

Then, the multiplicity of C at P is $l_1 + \dots + l_c + (m - c)$.

Example 3. Take now the rational curve $C_{\mathbb{R}}$ in $\mathbb{R}^3$ given by the parametrization

$$f_1 = \frac{(t-1)^4(1+4t+7t^2)}{1-4t+17t^2-5t^6-13t^4+20t^5+48t^3},$$
$$f_2 = \frac{(1-4t+22t^2-4t^3+t^4)(1+t)^2}{1-4t+17t^2-5t^6-13t^4+20t^5+48t^3},$$
$$f_3 = \frac{(1-4t+22t^2-4t^3+t^4)(1+t)^2}{1-4t+17t^2-5t^6-13t^4+20t^5+48t^3}.$$

If we perform "singRES" for this parametrization we have the following output

$$[[[-1,0,0],\ 0,\ 1,\ \textit{non-ordinary singularity}]\ ,[[0,1,1],\ 0,\ 1,\ \textit{non-ordinary singularity}]].$$

The singular points of this curve consist of two cusps.

In order to compute multiplicities we need the corresponding parameters.

- $(-1, 0, 0)$ corresponds to $t = -1$. Then, we compute the expansion around 0 of $f_1(s-1)+1$, $f_2(s-1)$, $f_3(s-1)$ and we obtain
$$f_1(s-1)+1 = -\frac{3}{4}s^2 - \frac{15}{8}s^3 - \frac{61}{16}s^4 - \frac{241}{32}s^5 + O(s^6)$$
$$f_2(s-1) = f_3(s-1) = -\frac{1}{2}s^2 - \frac{5}{4}s^3 - \frac{5}{2}s^4 - \frac{39}{8}s^5 + O(s^6)$$
Therefore, the multiplicity of $(-1, 0, 0)$ is 2.
- $(0, 1, 1)$ corresponds to $t = 1$. Analogously, if we compute the expansion around 0 of $f_1(s+1)$, $f_2(s+1)-1$, $f_3(s+1)-1$ we obtain that the multiplicity of $(0, 1, 1)$ is 4.

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