Available online at www.sciencedirect.com
ScienceDirect
Nuclear Physics B 808 (2009) 224–236
NUCLEAR PHYSICS B
www.elsevier.com/locate/nuclphysb

Cyclic universes from general collisionless braneworld models

E.N. Saridakis

Department of Physics, University of Athens, GR-15771 Athens, Greece

Received 27 October 2007; received in revised form 22 July 2008; accepted 15 September 2008

Available online 20 September 2008

Abstract

We investigate the full 5D dynamics of general braneworld models. Without making any further assumptions we show that cyclic behavior can arise naturally in a fraction of physically accepted solutions. The model does not require brane collisions, which in the stationary case remain fixed, and cyclicity takes place on the branes. We indicate that the cosmological constants play the central role for the realization of cyclic solutions and we show that its extremely small value on the observable universe makes the period of the cycles and the maximum scale factor astronomically large.

1. Introduction

The last decade proves to be really exciting for cosmology. Observational data indicated, among other very interesting results, that the expansion of the universe is accelerated [1]. At the same time the braneworld scenario appeared in the literature [2,3]. Though the exciting idea that we live in a fundamentally higher-dimensional spacetime which is greatly curved by vacuum energy was older [4], the new class of "warped" geometries offered a simple way of localizing the low energy gravitons on the brane.

In this novel background the old idea of a cyclic Universe was reheated. Started as ekpyrotic [5,6], enriched to ekpyrotic/cyclic [6–12] and recently to new ekpyrotic [13–16], the new paradigm tries to be established as an alternative to standard cosmology. According to its basic contents, our universe experiences an infinite or extremely large number of cycles, each one consisting of a hot bang phase, a phase of accelerated expansion, a phase of slow-ekpyrotic

E-mail address: msaridak@phys.uoa.gr.

0550-3213/$ – see front matter
doi:10.1016/j.nuclphysb.2008.09.022

contraction and a bounce-bang that triggers the next cycle. Starting with a simplified notional framework (infinite and not "created" time) cyclic cosmology have many advantages. It successfully faces the homogeneity, isotropy, topological and flatness problems, it handles the issue of initial conditions, it incorporates the dark energy and transforms it to an important factor, and it provides the mechanism of the generation of cosmic perturbations and of structure formation. However, there are some key issues that do not have a consistent and efficient approach so far, despite the great progress. These are the settlement of the singularity, although temperature and density remain finite, the entropy evolution, and the fate of the perturbations through the bounce. Through this research, cyclic scenarios have become more complicated, by the insertion of more complex potentials, of more branes [5], of the mechanism of ghost condensation [13,17], of more scalar fields [14] and of procedures which cancel the tachyonic instabilities [15].

Most of the works on cyclic cosmology involve, initially or at some stage, the transition to effective 4D equations. However, as it was mentioned in [18,19], such a procedure does not lead to reliable results since one cannot return to the 5D description self-consistently. Furthermore, the old 4D-singularity problem (of both Big Bang and traditional cyclic universes), has been replaced by a new one (equally annoying) concerning the singularity of extra dimension(s). This later case is accompanied by the brane collision phenomenon, which seems to be a basic constituent of the ekpyrotic scenario.

In this work we desire to investigate the full 5D dynamics of general braneworld models and examine if a cyclic behavior is possible. This is an essential procedure in order to consistently confront the arguments of the authors of [18], which claim that cyclic behavior cannot arise from a complete 5D description, and our study must not include any additional assumptions or fine tunings in order to remain general and therefore convincing. Secondly, we are interested to explore if a cyclic behavior of 5D dynamics is necessarily related to brane collisions. This work is organized as follows: In Section 2 we present the 5D braneworld model and we derive the equations of motion. In Section 3 we provide analytical solutions for two simplified stationary solution subclasses, while in Section 4 we investigate numerically the full stationary dynamics. Finally, in Section 5 we discuss the physical implications of our analysis and we summarize the obtained results.

2. The model

We consider quite general braneworld models, characterized by the action [20,21]:

$$\kappa_5^2 S = \frac{1}{2}\int d^4x\,dy\,\sqrt{-g}R + \int d^4x\,dy\,\sqrt{-g}\left[-\frac{1}{2}(\partial\phi)^2 - V(\phi)\right] - \sum_{i=1,2}\int_{b_i} d^4x\,\sqrt{-\gamma}\big\{[K] + U_i(\phi)\big\}, \tag{1}$$

where $\kappa_5^2 = \frac{1}{M_5^3}$ is a 5D gravitational constant, and all quantities are measured in units of M_5. The first term describes gravity in the five-dimensional bulk space. The second term corresponds to a minimally coupled bulk scalar field with the potential $V(\phi)$. The last term corresponds to two $(3+1)$-dimensional branes, which constitutes the boundary of the 5D space. We allow for a potential term $U(\phi)$ for the scalar field at each of the two branes, and we denote by γ the induced metric on them and by K their extrinsic curvature. Here and in the following the square brackets denote the jump of any quantity across a brane ($[Q] \equiv Q(y_+) - Q(y_-)$). The reason we use a second brane is to eliminate possible "naked" singularities, therefore by assuming $S^1/\mathbb{Z}_2$ symmetry across each brane we restrict our interest only in the interbrane space.

As usual the two branes are taken parallel, y denotes the coordinate transverse to them and we assume isometry along three-dimensional $\mathbf{x}$ slices including the branes. For the metric we choose the conformal gauge [20,21]:

$$ds^2 = e^{2B(t,y)}\left(-dt^2 + dy^2\right) + e^{2A(t,y)}\, d\mathbf{x}^2. \tag{2}$$

This metric choice, along with the residual gauge freedom $(t, y) \to (t', y')$ which preserves the 2D conformal form, allows us to "fix" the positions of the branes. Without loss of generality we can locate them at $y = 0, 1$, having in mind that their physical distance is encoded in the metric component $B(t, y)$, and at a specific time it is given by [20,21]:

$$D(t) \equiv \int_0^1 dy\, \sqrt{g_{55}} = \int_0^1 dy\, e^{B(t,y)}, \tag{3}$$

quantity that is invariant under the residual gauge freedom in our coordinates. The reason we prefer the metric (2), instead of the usual form in the literature, is that in the later case the brane positions are in general time-dependent and the various boundary conditions are significantly more complicated. Thus, our coordinates are preferable for numerical calculations, despite the loss of simplicity in the definitions of some quantities. Eventually, the physical interpretation of the results is independent of the coordinate choice.

The non-trivial five-dimensional Einstein equations consist of three dynamical:

$$\begin{aligned}
&\ddot{A} - A'' + 3\dot{A}^2 - 3A'^2 = \frac{2}{3}e^{2B}V,\\
&\ddot{B} - B'' - 3\dot{A}^2 + 3A'^2 = -\frac{\dot{\phi}^2}{2} + \frac{\phi'^2}{2} - \frac{1}{3}e^{2B}V,\\
&\ddot{\phi} - \phi'' + 3\dot{A}\dot{\phi} - 3A'\phi' + e^{2B}V_{,\phi} = 0,
\end{aligned} \tag{4}$$

and two constraint equations:

$$\begin{aligned}
&-A'\dot{A} + B'\dot{A} + A'\dot{B} - \dot{A}' = \frac{1}{3}\dot{\phi}\phi',\\
&2A'^2 - A'B' + A'' - \dot{A}^2 - \dot{A}\dot{B} = -\frac{\dot{\phi}^2}{6} - \frac{\phi'^2}{6} - \frac{1}{3}e^{2B}V,
\end{aligned} \tag{5}$$

where primes and dots denote derivatives with respect to y and t respectively. It is easy to show that the constraints are preserved by the dynamical equations.

Additionally, from the boundary terms in the action for the branes we obtain the following junctions (Israel) conditions:

$$[A'] = \mp\frac{1}{3}Ue^B, \qquad [B'] = \mp\frac{1}{3}Ue^B, \qquad [\phi'] = \pm e^B U_{,\phi}, \tag{6}$$

where the upper and lower signs refer to the branes at $y = 0, 1$, respectively. Since we have imposed $\mathbb{Z}_2$ symmetry across the two branes, for any function Q we get:

$$[Q']_0 = 2Q'(0^+), \qquad [Q']_1 = -2Q'(1^-). \tag{7}$$

For the bulk potential $V(\phi)$ we assume a general form consistent with the stabilization mechanism [22]:

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \Lambda, \tag{8}$$

where Λ is the 5D bulk cosmological constant. For the brane potentials $U_i(\phi)$ we use the quite general quadratic form [20,21]:

$$U_i(\phi) = \frac{1}{2}M_i(\phi_i - \sigma_i)^2 + \lambda_i, \tag{9}$$

where λ_i stand for the brane tensions and ϕ_i for the value of ϕ on the ith brane. The "masses" m and M_i can be varying.

Finally, the induced 4D metrics of the two ("fixed"-position) branes in the conformal gauge are simply given by

$$ds^2 = -d\tau^2 + a^2(\tau)\,d\mathbf{x}^2, \tag{10}$$

with $d\tau_i = e^{B_i}\,dt$ and $a_i = e^{A_i}$ the proper times and scale factors of the two branes. Thus, for the Hubble parameter on the branes we acquire

$$H_i \equiv \frac{1}{a}\frac{da}{d\tau}\bigg|_i = e^{-B_i}\dot{A}_i, \tag{11}$$

which is invariant under residual gauge transformations. As usual, we identify the brane at $y = 0$ as the visible brane corresponding to our Universe. Note that relations (10) and (11) can be generalized to hold in every $3 + 1$ slice transverse to the fifth dimension.

Finally, let as comment on the behavior of gravity on the physical brane in our model. As was shown in [23], in such a two-brane model we re-obtain the correct Newton's law in the brane-universe. This becomes more transparent if we include the aforementioned brane stabilization mechanism [22], where the two-brane model leads naturally to the recovery of the Einstein gravity on the physical brane [20,24]. More generally, even in a generic two-brane model with arbitrary bulk and brane potentials, using the renormalization group flow of the 4D Newton's constant in IR and UV, it can be shown that Newton's law is recovered, plus one extremely small brane correction [25].

3. Stationary case. Analytical solutions

The model we have described is general enough and includes the full spacetime evolution of the 5D braneworld. Our aim is to look for physically accepted solutions that correspond to cyclic behavior. The main difficulty of solving the equation system (4)–(9) is that one has to satisfy the constraints (5) and boundary conditions (6) at $t = t_0$, ensuring that no divergencies are present in the interbrane space. Then, one has to assure that time evolution will not be unstable, or give rise to naked singularities in the bulk.

Let us first investigate a subclass of solutions, the so-called stationary ones. In this case we assume that

$$B(t, y) \to B(y), \qquad A(t, y) \to B(y) + Ht. \tag{12}$$

This case is characterized by a fixed bulk geometry and maximally symmetric (de Sitter, Minkowski or oscillatory) branes:

$$ds^2 = e^{2B(y)}\big(dy^2 - dt^2 + e^{2Ht}\,d\mathbf{x}^2\big). \tag{13}$$

Note that there is a name confusion in the literature, since spacetimes (12) are called stationary in [26] and when $H = 0$ they are called static (a convention that we follow in this paper), while in [20] they are called static even for $H \neq 0$.

Under (12) the system of Eqs. (4) and (5) transits to the time-independent form:

$$B''(y) = -\frac{1}{6}e^{2B(y)} - \frac{\phi'^2(y)}{4},$$
$$\phi''(y) + 3B'(y)\phi'(y) - e^{2B(y)}\frac{\partial V(\phi)}{\partial \phi} = 0,$$
$$H^2 = B'^2(y) + \frac{1}{6}e^{2B(y)}V(\phi) - \frac{\phi'^2(y)}{12}, \tag{14}$$

where ϕ is also assumed to be time independent. Furthermore, boundary conditions (6) will be satisfied at all times, provided that they do so at $t = t_0$. Finally, according to definition (11) and under the ansatz (12), the Hubble parameter on the physical brane is simply $H_0^2 = e^{-2B(0)}H^2$.

Eqs. (14) and boundary conditions (6) cannot be solved analytically for the general bulk and brane potentials of (8) and (9). However, this is possible in two simplified cases: (A) Assuming $V(\phi) = 0$ with full $U_i(\phi)$, and (B) assuming $V(\phi) = \Lambda$, $U_0(\phi) = \lambda_0$ and $U_1(\phi) = \lambda_1$. In the following subsections we examine these two cases successively.

3.1. $V(\phi) = 0$ *and* $U_i(\phi) = \frac{1}{2}M_i(\phi_i - \sigma_i)^2 + \lambda_i$

If we set $V(\phi) = 0$ we can acquire analytical solutions depending on the sign of H^2. For $H^2 > 0$ and setting $H = |\sqrt{H^2}\,|$ we acquire:

$$B(y) = B(0) + \frac{1}{3}\log\left(\frac{B'(0)}{H}\sinh 3Hy + \cosh 3Hy\right) \tag{15}$$

and

$$\phi(y) = \phi(0) - \frac{2}{\sqrt{3}}\log\left[\frac{(e^{3Hy} - u)(1 + u)}{(e^{3Hy} + u)(1 - u)}\right], \tag{16}$$

where we set $u = \sqrt{\frac{B'(0)-H}{B'(0)+H}}$, and $|B'(0)| > H$ always in this case if we want $\phi'(0)$ to be real.

For $H^2 < 0$ and setting $\theta = |\sqrt{-H^2}|$ we get:

$$B(y) = B(0) + \frac{1}{3}\log\left(\frac{B'(0)}{\theta}\sin 3\theta y + \cos 3\theta y\right) \tag{17}$$

and

$$\phi(y) = \phi(0) - \frac{2}{\sqrt{3}}\log\left[\frac{\tan(\frac{3}{2}\theta y + \frac{1}{2}\arctan\frac{\theta}{B'(0)})}{\tan(\frac{1}{2}\arctan\frac{\theta}{B'(0)})}\right]. \tag{18}$$

Let us make some comments here. Firstly, in the expressions above we have eliminated the three integration constants in terms of $B(0)$, $B'(0)$ and $\phi(0)$, since it is more convenient, but one could equally use the values of three of B, B', ϕ and ϕ' at any y. Moreover, these solutions have to satisfy the boundary conditions (6). Eventually, all quantities, including H^2, are given in terms of the six parameters of the model ($M_0, \lambda_0, \sigma_0, M_1, \lambda_1, \sigma_1$, since we have set $m = 0$ and $\Lambda = 0$). We will return to this subject in the next section. Secondly, note that there is analytical continuation between the solutions for positive and negative H^2, i.e., the two solutions coincide for $H \to i\theta$. Thirdly, it is easy to see that in the limit $H^2 \to 0$, $B(y)$ and $\phi(y)$ become smoothly a negative logarithm of a linear in y function. Lastly, note that due to the non-linear form of the differential equations there exist more solution branches which can be obtained straightforwardly from (15)–(18) by sign changing. However, since all of them imply the same metric for the visible brane, which is given below and which is the central subject of this work, we do not write them explicitly.

In order for the solutions to be physically accepted, we have to assure that no naked singularities are present in the interbrane space, i.e., between $y=0$ and $y=1$. The investigation of the $H^2>0$ case was presented in [27] where we showed that H^2 values are restricted to very small values and only one 5D sub-surface of the 6D parameter space allows for arbitrary large H^2 values. Here we repeat the cogitations in the $H^2<0$ case. Solutions (17) and (18) have poles at $\frac{B'(0)}{\theta}\sin 3\theta y_p + \cos 3\theta y_p = 0$, i.e., at

$$y_{pn} = \frac{1}{3\theta}(y_0 + n\pi), \tag{19}$$

with $n \in \mathbb{Z}$, where $y_0 = \arctan(-\frac{\theta}{B'(0)})$ with $-\pi \leqslant y_0 < 0$. In this notation y_{p0} stands for the largest negative pole. Demanding none of these y_{pn} lying in $[0,1]$ we require that the smallest positive pole, i.e., y_{p1}, to be greater than 1. This leads to $y_0 > 3\theta - \pi$ and since $-\pi \leqslant y_0 < 0$ we acquire:

$$3\theta - \pi < \arctan\left(-\frac{\theta}{B'(0)}\right) < 0. \tag{20}$$

Thus, this relation provides a narrow and absolute window for θ, in a sense that there are no areas at all in the 6D parameter space that give θ larger than $\pi/3$.

Finally, we desire to explore the forms of the bulk and brane geometries in these cases. For $H^2 \geqslant 0$ we get de Sitter, anti-de Sitter or Minkowski branes ($3+1$ slices in general), that have been studied extensively in the literature. The bulk structure is fixed and this can be also confirmed by calculating the brane physical distance according to (3), which is obviously time independent (given analytically as an expression in terms of hypergeometric and hyperbolic functions of the parameters), i.e., the branes do not move.

For $H^2<0$, which is the case of interest in this work, we acquire also a fixed bulk with constant brane distance. For the branes we get an interesting oscillatory behavior. In particular, the 5D metric: $ds^2 = e^{2B(y)}(-dt^2 + dy^2 + e^{2i\theta t}\,d\mathbf{x}^2)$, implies a 4D metric of the form (10) on the visible brane, with the scale factor given as:

$$a^2(\tau) = c\sin^2\left(e^{-B(0)}\theta\tau\right), \tag{21}$$

where τ is the visible brane proper time defined above as $d\tau = e^{B(0)}\,dt$ (for clarity in the following we omit the index i which distinguishes the proper times of the two branes, using τ for the visible brane and τ_1 for the hidden one). The constant c depends on the exact relation between τ and bulk time t (it is equal to $e^{2B(0)}$ for $\tau = e^{B(0)}t$). Moreover, one should add a constant phase which would be determined by the $a^2(\tau)$ value at a specific τ. Without loss of generality we have performed a shift in τ such that $a^2(0)=0$ (note that $\tau=0$ is not an initial time but just a random one). Definitely, one could achieve a relation similar to (21) in terms of bulk time t, but we desire to present the results from the visible brane observer's point of view.

3.2. $V(\phi) = \Lambda$, $U_0(\phi) = \lambda_0$ and $U_1(\phi) = \lambda_1$

In the case where the branes have just tensions, the bulk potential is comprised of a non-zero cosmological constant and there is no scalar field, analytical solutions, depending on the sign of H^2, can be derived.

For $H^2 > 0$, with $H = |\sqrt{H^2}|$ we acquire:

$$B(y) = -\log\left(\frac{c_1 e^{Hy}}{2H^2} + \frac{\Lambda e^{-Hy}}{12c_1}\right), \tag{22}$$

where $c_1 = H^2[e^{-B(0)} - \sqrt{e^{-2B(0)} - \Lambda/(6H^2)}\,]$, similar to the results of [28]. Expression (22), apart from $y \to \pm\infty$, has a naked singularity at $y_p = \frac{1}{2H}\log(-\frac{H^2\Lambda}{6c_1^2})$ which must lie outside [0, 1]. It is obvious that when $\Lambda < 0$ it is singularity-free and is the well studied AdS bulk case of the literature. Furthermore, solution (22), in the specific case where $\Lambda < 0$, $\lambda_0 = \sqrt{-6\Lambda}$ and $\lambda_1 = -\sqrt{-6\Lambda}$, corresponds to the two-brane Randall–Sundrum model [2], where H^2 acquires a zero value and $B(y)$ becomes a negative logarithm of a term linear in y.

For $H^2 < 0$, and setting $\theta = |\sqrt{-H^2}|$ we get:

$$B(y) = \log\left[\frac{\theta}{\sqrt{-\frac{\Lambda}{6}}}\frac{1}{\sin(\theta y + \arcsin\frac{\theta e^{-B(0)}}{\sqrt{-\frac{\Lambda}{6}}})}\right]. \tag{23}$$

Note that in this case Λ must be negative, as it is easily implied from the last of Eqs. (14) if we demand $B(y)$ to be real, i.e., the bulk is always AdS. Expression (23) possesses singularities at

$$y_{pn} = \frac{1}{\theta}(y_0 + n\pi), \tag{24}$$

with $n \in \mathbb{Z}$, where $y_0 = -\arcsin(\frac{\theta e^{-B(0)}}{\sqrt{-\frac{\Lambda}{6}}})$ with $-\pi \leqslant y_0 < 0$. As previously, y_{p0} stands for the largest negative pole. Forcing the smallest positive pole, i.e., y_{p1}, to be greater than 1 we acquire $y_0 > \theta - \pi$, and since $-\pi \leqslant y_0 < 0$ we finally obtain:

$$\theta - \pi < -\arcsin\left(\frac{\theta e^{-B(0)}}{\sqrt{-\frac{\Lambda}{6}}}\right) < 0. \tag{25}$$

Thus, in this case there are no areas at all in the 3D parameter space that lead to solutions with θ larger than π.

Let as describe the spacetime geometry that corresponds to these solutions (we repeat that there are more solution branches arising form (22) and (23) by sign changing, but they correspond to the same brane metric). In the $H^2 \geqslant 0$ case we acquire two stabilized branes, with their time-independent physical distance given analytically through (3) in terms of inverse hyperbolic trigonometric functions of the parameters. The 4D induced geometry on the two branes can be de Sitter, anti-de Sitter or Minkowski and has been investigated widely in the literature. For $H^2 < 0$ we also obtain a fixed bulk with the time-independent physical distance of the two branes given easily analytically. However, the geometry on the branes acquires the interesting oscillatory behavior described in the previous subsection, with a scale factor of the form:

$$a^2(\tau) = c\sin^2\left(e^{-B(0)}\theta\tau\right), \tag{26}$$

identically to relation (21).

Relations (21) and (26) correspond to cyclic Universes. Indeed, they imply that the scale factor pulsates between the following extremum values:

$$a_{\min} = 0, \qquad a_{\max} = \sqrt{c}, \tag{27}$$

where as we mentioned c depends on the specific relation between t and τ ($c = e^{2B(0)}$ for $\tau = e^{B(0)}t$). The constant period of the cycles is given as:

$$T = \frac{\pi e^{B(0)}}{\theta}. \tag{28}$$

In (27), (28) all quantities are measured in units of M_5. The aforementioned behavior corresponds to a sequence of expanding and contracting phases on the branes, while the physical brane positions stay constant. The fifth dimension remains unaffected, that is there are no brane collisions. The bulk dynamics determines also the quantitative characteristics of the cycles. Indeed, the value of $B(y)$ on the physical brane specifies the maximum value of the scale factor and can be arbitrary. Moreover, it designates the cycles period. Fortunately, the fact that θ is bounded from above (according to (20) and (25)) restricts completely its effect on decreasing the period. We will return to these subjects in the discussion section below, where we show that only astronomically large maximum scale factors and periods are possible. Finally, note that in the aforementioned analysis cyclicity arises naturally from the 5D dynamics, and the oscillatory behavior of the scale factor is explicit and not a result of Hubble constant sign change in special cases [9,11,13,29]. Furthermore, it emerges from general braneworld models, without Gauss–Bonnet terms [30] or the assumption of charged AdS bulk black holes which charge (along with fine tuning) is responsible for restricted cyclic behavior [10].

We close this section by referring to the stability of the aforementioned solutions in both metric and scalar field components. As it was shown in [26,31], a two-brane model with the bulk and brane potentials considered above, has stable solutions, free of tachyonic modes. In particular, the presence of the stabilization mechanism [22], which forbids branes to move, makes all physically accepted solutions (i.e., those which are singularity-free and satisfy boundary conditions) to be stable under perturbations [24]. This feature has been also confirmed by numerical investigation in [20] and has been verified by us, too. Thus, the stability of the solutions provides the necessary physical hypostasis to our model.

4. Stationary case. Numerical results

In the previous section we derived analytical solutions for the full 5D equations, in the stationary ansatz with two simplified potential cases. We expressed the solutions in terms of the values of $B(0)$, $B'(0)$ and $\phi(0)$ and we stated that these must satisfy the boundary conditions (6). The usual, in the literature, method to achieve this is to randomly choose H^2 and the solution values at $y = 0$ and then fine-tune the model parameters. However, this procedure definitely does not reveal the properties and the rich structure of the solutions. On the contrary, as we showed in [27], it restricts the investigation in a small subclass of solutions. The natural way to encounter the problem is to choose randomly and uniformly the potential parameters at first, and then seek for physically accepted solutions, i.e., divergencies-free expressions which satisfy Eqs. (14) and boundary conditions (6). This is a hard task in general. The method we use is the following: We first choose randomly the values of the model parameters, uniformly distributed in a hyper-cube (6D in the case of Section 3.1 and 3D in that of Section 3.2). The obtained results do not depend on the hyper-cube's size, but on its effectual covering (number of parameter multiplets used in the calculation). We use the globally convergent Schmelcher–Diakonos algorithm [32] to solve the transcendental equation system with accuracy 10^{-13}. We find hexads (or triads) of parameters corresponding to acceptable solutions, and calculate H_0^2 through $H_0^2 = e^{-2B(0)} H^2$.

Numerical investigation reveals that only a small fraction of parameter choices (we use 10^6 parameter multiplets) allows for solutions to exist ($\approx 15\%$ in case Section 3.1 and $\approx 5\%$ in case Section 3.2). Note that, as we mentioned in [27], the parameter sub-space that leads to solutions is neither compact nor uniform. The question if it consists from continuous areas or from independent points does not have a clear answer [33]. Now, inside these percentages only a small fraction corresponds to solutions with $H^2 < 0$, i.e., to oscillatory ones ($\approx 2\%$ in case Section 3.1 and $\approx 6\%$ in case Section 3.2). The reason of the significantly smaller appearance of $H^2 < 0$ solutions is that they posses many singularities, in comparison with the $H^2 > 0$ case, and therefore it is harder to find a solution with no singularity at [0, 1]. In conclusion, in total only about $\approx 10^{-1}\%$ of the random and uniform parameter choices correspond to oscillatory behavior of the metric of the visible brane. In these cases, (21) and (26) are numerically verified and the analysis of the cyclic solution is valid.

Since we have confirmed our analytical calculations numerically, we can proceed to the numerical investigation of the general stationary case, i.e., with the full (8) and (9) potentials, where analytical solutions cannot be obtained. Again, we randomly choose the values of the 8 model parameters (10^6 octads of $M_0, \lambda_0, \sigma_0, M_1, \lambda_1, \sigma_1, m$ and Λ), from a uniform distribution, and we solve the transcendental equation system of (14) and (6). Note that the parameter space hypercube is taken large and symmetric (its edge extends from -10^3 to 10^3) and we do not impose any constraints on the parameter values (assuming for example positive “masses” or opposite tension branes) in order to remain as general as possible. In this case only $\approx 0.6\%$ of the parameter multiplets corresponds to physically accepted solutions, and within them only $\approx 4\%$ corresponds to $H^2 < 0$, i.e., in total only $\approx 10^{-2}\%$ of the parameter choices lead to cyclic branes. We mention that these percentages increase radically if we restrict the investigation in specific parameter signs. In Fig. 1 we present the evolution of the scale factor of the visible brane for one such solution. We conclude that even in the absence of analytical calculations in the general-potential stationary case, we do obtain fixed and cyclic $3+1$ branes and relations (21) (or (26)) and (27), (28) are satisfied. In other words, if the requirements for a complete 5D solution to exist are fulfilled, then (21) (or (26)) and (27), (28) are valid analytically, with the numerics only determining $B(0)$ and θ. The only numerical restriction arises from the presence of $B(0)$ in an exponential, which prevents us from handling arbitrary large values as in the analytical calculations. However, this can be solved by the additional scaling transformation proposed in [20].

Let us make a comment here concerning the “singular” points where the scale factor vanishes. These points correspond to the so-called “bounce”, which is always present in all cyclic-cosmology models. Although there have been many attempts in the literature to avoid the involved singularity, none of them is completely satisfactory up to now. These approaches, such is the insertion of quantum fluctuations [13,15,34] or the use of loop quantum gravity modifications [35], could be included in our analysis, leading to a smoothing out of the behavior of Fig. 1 and of relation (21) (or (26)), i.e., making the scale factor non-zero at the bounce. However, in this work we desire to present the basic characteristics of cyclic behavior in general collisionless braneworld models, and thus we will not examine in detail the (in any case non-complete) handling of the singularity. Our model shares this disadvantage of all cyclic models. We will return to this subject in the discussion section.

In the above investigation we have been restricted to stationary solutions of the form (12), case in which stable time evolution is implied easily. The question is what can be said about the

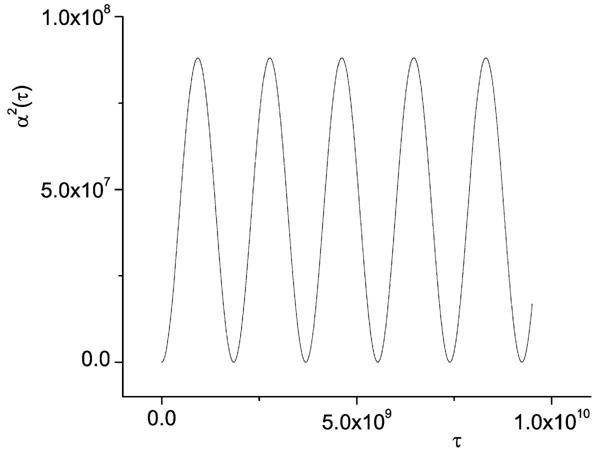

Fig. 1. Visible brane scale factor evolution for one random solution of the stationary case with the general bulk and brane potentials of (8) and (9). The edge of the parameter space hyper-cube extends from -10^3 to 10^3. The solution arises from $m = 81.03$, $\Lambda = -505.8$, $\lambda_0 = -212.5$, $\lambda_1 = -464.2$, $M_0 = 77.94$, $M_1 = 180.7$, $\sigma_0 = 1.308$, $\sigma_1 = -3.556$ (for simplicity we provide the first four relevant digits only). The obtained $B(0)$ and θ values are correspondingly 9.148 and 0.1522, and τ is calculated as $e^{B(0)}t$. All quantities are expressed in M_5 units.

full dynamics of Eqs. (4) and (5), where H_0^2 and physical brane distance $D(t)$ can be varying. Fortunately, solutions of the full dynamics seem to consist of stationary ones and the transitions between them [20,33]. Therefore our stationary investigation is sufficient. We will return to this subject in the next section.

5. Discussion–conclusions

In the aforementioned analysis we considered general braneworld models characterized by the action (1), the conformal metric (2), and the general potentials (8) and (9). Performing both analytical and numerical calculations we showed that the full 5D dynamics allows for stationary solutions corresponding to oscillatory scale factor of the physical brane and therefore to cyclic universes. In statistical terms cyclicity corresponds to $\approx 4\%$ of the physical solutions. Our investigation is completely 5D, cyclic behavior arises naturally and is induced on the brane by the full dynamics, and it is not a result of a modified 4D dynamics, with fine-tuned parameters or specific assumptions in the Friedmann equation. Furthermore, we do not use an explicit brane state equation, considering just the bulk scalar field (the decays and interactions of which will eventually fill the physical brane with the conventional content [36]). As we mentioned in the introduction this full 5D approach is necessary in order to confront the arguments of the authors of [18]. Indeed, their allegations that one cannot transit to an effective 4D theory (integrating the action over y), solve the equations there and then return naively to the 5D description (adding time-dependence by hand), are correct. Doing so, the results are not self-consistent (especially the boundary conditions are not satisfied) and the authors of [18] use this fact as a central argument against the cyclic scenario. However, our consistent 5D analysis reveals that cyclic behavior is possible.

Another important feature of the present study is that cyclic universes do not require brane collisions. Thus, we avoid the known problems concerning such a description, which force ekpyrotic model to successively more complicated versions. On the contrary, the branes do not move at all and the system is stable (stationary solutions are a stable fixed point [20,33]). Furthermore, in our model, expansion and contraction take place in the $3+1$ branes, and in all $3+1$ slices in general, while the fifth dimension remains unaffected. The 4 spacial dimensions shrink periodically to an 1D string and re-expand. This is in a radical contrast with the cyclic models with extra dimensions, where the extra dimension is the one that gets contracted (the fifth in [7] or the eleventh in [6]). Cyclicity seems to re-obtain its "physical" meaning.

Our 5D investigation is general and does not involve extra assumptions, fine-tunings or specific potential forms. We result to periodic, cyclic, homogeneous and isotropic universes, where the scale factor changes smoothly from expanding to contracting. An observer on the physical brane feels successively accelerated expansion, decelerated expansion, turnabout, accelerated contraction, decelerated contraction, bounce, etc., and a promising signature of the cyclic behavior would be the measure of the varying rate of the Hubble constant. The cycles period, given in (28), can be arbitrary, depending on $B(0)$, i.e., on the value of the warp factor on the physical brane (θ is bounded from above and therefore cannot act as a period-decreasing factor). A very interesting conclusion comes from the insertion of observational results in our model, which was not made above in order to remain as general as possible. Explicitly, if we use the fact that the cosmological constant of our Universe is extremely small ($\approx \mathcal{O}(10^{-47})$ GeV^4), and assuming a reasonable M_5 value of $\mathcal{O}(10^{19})$ GeV, the first two boundary conditions in relation (6) provide in general a huge value for $e^{B(0)}$ ($\approx \mathcal{O}(10^{45})$). This is in consistency with the scaling transformation of [20], which allows us, in a solution, to scale the parameters by e^{-S} and add to the warp factor the constant S, and acquire another solution. Therefore, the extremely small cosmological constant of the observable universe leads the cyclicity period to be around $T \approx \mathcal{O}(10^{13})$ years and the maximum scale factor value, given by (27), to be $a_{\max} \approx \mathcal{O}(10^{28})$ m (where the decimal exponents in these rough estimations can vary by 1 or 2, depending on $B'(0)$ and θ values). Luckily enough, the smallness of the cosmological constant excludes oscillatory models with small periods in astronomical terms. In more foundational words, the reason that made the cosmological constant that small, is the same that makes the cycle period and the size of the Universe that large.

In this work we have been restricted to stationary solutions, where the subclass of them that possesses $H^2 < 0$ corresponds to eternal cyclic behavior with constant period. Numerical investigation of the full dynamics seem to consist of such stationary solutions and the transitions between them [20,33]. In such transitions H_0^2 on the physical brane can chance sign, leading to a form of "chaotic cyclicity", where large intervals of (non-periodic in general) oscillatory behavior could be followed by large intervals of conventional evolution and vice versa. In this case, an initial Big Bang and/or a final Big Rip or Big Crunch (in conventional terms) could be possible. Another interesting possibility would be the exploration of our model with cosmological constants being piecewise constant functions of time, reflecting cosmological phase transitions, which could also lead to chaotic cyclicity. Note however that numerical confirmation of such behaviors is very hard due to the small probability of cyclic stationary solutions ($\approx 10^{-2}\%$ as we have already mentioned). These subjects are under investigation.

In order for a model to serve as a description of nature, it has to explain the basic physical key issues. Especially for cyclic cosmology, amongst others these are the entropy evolution and, probably the most pressing issue, that of a fuller understanding of the bounce and the handling of the singularity. Our model provides a consistent background for cyclicity and it reveals how such a behavior arises from the full 5D dynamics. However, since braneworlds and brane cosmology in general arise as limits of a multi-dimensional theory unknown up to now, the 5D results have a phenomenological character and must be considered from this point of view. Definitely, a complete explanation and apprehension, and a successful confrontation of the aforementioned subjects, can only come through a higher-dimensional, fundamental theory of nature. For the moment we have to rely on the relevant research on cyclic cosmology, linearized gravity, M-theory and strings, which has improved our knowledge on these issues. These results can be embodied in our analysis. The most hopeful effort is the use of quantum fluctuations in order to tame the singularity, which effectively is translated into a modification of gravity by the scalar field [13,15,34]. Alternatively, using loop quantum gravity we could modify non-perturbatively the dynamical equations leading to a singularity resolution as in [35]. Concerning the entropy, we could include the relevant discussion in our investigation. The argument of the authors of [11,12] about maximum amount of entropy possible in de Sitter spacetime, may lead our model to have a maximum cycle number between 10^{20} and 10^{30}. However, the idea of the causal patch [12] is probably the best way of handling the entropy problem so forth, and there are some interesting recent works on the subject which give a boost on cyclic cosmology [37].

Let us close this discussion section with some comments on the role of the brane tensions and of the bulk cosmological constant in our model. As can be numerically confirmed, setting them to zero makes it almost impossible to satisfy the boundary conditions obtaining $H^2 < 0$ and singularity absence in [0, 1] (this can be achieved only through a careful fine-tuning since our random choice procedure gives an one-digit number of such solutions in 10^6 parameter multiplets). On the other hand, as we showed in Section 3.2, in the case where Λ, λ_0 and λ_1 are the only non-zero parameters, an $\approx 10^{-1}\%$ of the random parameter choices, or $\approx 6\%$ of the solutions, correspond to $H^2 < 0$. In mathematical terms, Λ, λ_0 and λ_1 are requisite in order to acquire a solution with $H^2 < 0$ in the full dynamics, in a natural and not in a fine-tuning way. In terms of physics, it is the dark energy that lies in the background of the oscillatory mechanism and allows for cyclicity to realize. Adding the fact that it determines the cycles period and the maximum scale factor value, we conclude that dark energy is crucial in the described model. This brings it closer to the ekpyrotic paradigm of the literature.

In this work we examine general braneworld models and we show that cyclic behavior can naturally arise from the full 5D dynamics. One important feature is that brane collisions are not required, on the contrary the branes remain stable, and the cyclicity takes place on the 4D geometry not on the extra dimension. Another significant result is that the smallness of the cosmological constant of the observable universe pushes the cyclic period and the scale factor to astronomical large values, an essential requirement for the establishment of cyclic cosmology as a realistic alternative paradigm. Furthermore, we indicate the possibility of a "chaotic cyclicity", that is extremely large, non-periodic, cyclic intervals followed by extremely large intervals of conventional evolution and vice versa. After these, the model shares both the advantages and disadvantages of cyclic cosmology.

Acknowledgements

The author acknowledges partial financial support through the research program "Pythagoras" of the EPEAEK II (European Union and the Greek Ministry of Education).

References

[1] A. Balbi, et al., Astrophys. J. 545 (2000) L1–L4 (MAXIMA-1);
N.A. Bahcall, J.P. Ostriker, S. Perlmutter, P.J. Steinhardt, Science 284 (1999) 1481.
[2] L. Randall, R. Sundrum, Phys. Rev. Lett. 83 (1999) 3370, hep-ph/9905221;
L. Randall, R. Sundrum, Phys. Rev. Lett. 83 (1999) 4690, hep-th/9906064.

[3] P. Horawa, E. Witten, Nucl. Phys. B 460 (1996) 506;
P. Horawa, E. Witten, Nucl. Phys. B 475 (1996) 94;
A. Lukas, B.A. Ovrut, K.S. Stelle, D. Waldram, Phys. Rev. D 59 (1999) 086001, hep-th/9803235;
P. Binetruy, C. Deffayet, D. Langlois, Nucl. Phys. B 565 (2000) 269, hep-th/9905012.
[4] K. Akama, Lect. Notes Phys. 176 (1982) 267, hep-th/0001113;
V.A. Rubakov, M.E. Shaposhnikov, Phys. Lett. B 125 (1983) 136;
I. Antoniadis, C. Bachas, D. Lewellen, T. Tomaras, Phys. Lett. B 207 (1988) 441;
N. A-Hamed, S. Dimopoulos, G. Dvali, Phys. Lett. B 429 (1998) 263, hep-ph/9803315.
[5] J. Khoury, B.A. Ovrut, P.J. Steinhardt, N. Turok, Phys. Rev. D 64 (2001) 123522, hep-th/0103239.
[6] J. Khoury, B.A. Ovrut, N. Seiberg, P.J. Steinhardt, N. Turok, Phys. Rev. D 65 (2002) 086007, hep-th/0108187.
[7] P.J. Steinhardt, N. Turok, Science 296 (2002) 1436;
P.J. Steinhardt, N. Turok, Phys. Rev. D 65 (2002) 126003, hep-th/0111098.
[8] S. Tsujikawa, R. Brandenberger, F. Finelli, Phys. Rev. D 66 (2002) 083513, hep-th/0207228.
[9] T. Clifton, J.D. Barrow, Phys. Rev. D 75 (2007) 043515, gr-qc/0701070.
[10] S. Mukherji, M. Peloso, Phys. Lett. B 547 (2002) 297, hep-th/0205180;
Y.S. Myung, Class. Quantum Grav. 20 (2003) 935, hep-th/0208086.
[11] P.J. Steinhardt, N. Turok, New Astron. Rev. 49 (2005) 43, astro-ph/0404480.
[12] N. Turok, P.J. Seinhardt, Phys. Scr. T 117 (2005) 76, hep-th/0403020.
[13] E.I. Buchbinder, J. Khoury, B.A. Ovrut, Phys. Rev. D 76 (2007) 123503, hep-th/0702154.
[14] K. Koyama, S. Mizuno, D. Wands, Class. Quantum Grav. 24 (2007) 3919, hep-th/0704.1152.
[15] E.I. Buchbinder, J. Khoury, B.A. Ovrut, JHEP 0711 (2007) 076, hep-th/0706.3903.
[16] J.-L. Lehners, N. Turok, Phys. Rev. D 77 (2008) 023516, hep-th/0708.0743;
J.K. Erickson, S. Gratton, P.J. Steinhardt, N. Turok, Phys. Rev. D 75 (2007) 123507, hep-th/0607164;
J.-L. Lehners, P. McFadden, N. Turok, P.J. Steinhardt, Phys. Rev. D 76 (2007) 103501, hep-th/0702153.
[17] N. A-Hamed, H.-C. Cheng, M.A. Luty, S. Mukohyama, JHEP 0405 (2004) 074, hep-th/0312099.
[18] R. Kallosh, L. Kofman, A.D. Linde, A.A. Tseytlin, Phys. Rev. D 64 (2001) 123524, hep-th/0106241.
[19] P.L. McFadden, N. Turok, P.J. Steinhardt, hep-th/0512123.
[20] J. Martin, G.N. Felder, A.V. Frolov, M. Peloso, L. Kofman, Phys. Rev. D 69 (2004) 084017, hep-th/0309001.
[21] E.N. Saridakis, JCAP 0804 (2008) 020, astro-ph/0712.2672.
[22] W.D. Goldberger, M.B. Wise, Phys. Rev. Lett. 83 (1999) 4922, hep-ph/9907447;
C. Csaki, M. Graesser, L. Randall, J. Terning, Phys. Rev. D 62 (2000) 045015, hep-ph/9911406.
[23] E. Flanagan, N.T. Jones, H. Stoica, S.-H.H. Tye, I. Wasserman, Phys. Rev. D 64 (2001) 045007, hep-ph/0012129.
[24] T. Tanaka, X. Montes, Nucl. Phys. B 582 (2000) 259, hep-th/0001092.
[25] S. Nojiri, O. Obregon, S.D. Odintsov, S. Ogushi, Phys. Rev. D 62 (2000) 064017, hep-th/0003148.
[26] E. Flanagan, S.-H.H. Tye, I. Wasserman, Phys. Lett. B 522 (2001) 155, hep-th/0110070.
[27] F.K. Diakonos, E.N. Saridakis, hep-th/0708.3143.
[28] A. Lukas, D. Skinner, JHEP 0109 (2001) 020, hep-th/0106190.
[29] J.-F. Zhang, X. Zhang, H.-Y. Liu, Eur. Phys. J. C 52 (2007) 693, hep-th/0708.3121.
[30] P.S. Apostolopoulos, N. Brouzakis, N. Tetradis, E. Tzavara, Phys. Rev. D 76 (2007) 084029, hep-th/0708.0469;
H. Maeda, V. Sahni, Y. Shtanov, Phys. Rev. D 76 (2007) 104028, gr-qc/0708.3237;
E.N. Saridakis, Phys. Lett. B 661 (2008) 335, gr-qc/0712.3806.
[31] E. Flanagan, R.J. Hill, N.T. Jones, S.-H.H. Tye, I. Wasserman, Phys. Lett. B 515 (2001) 161, hep-th/0105293;
D. Iakubovskyi, Y. Shtanov, Class. Quantum Grav. 22 (2005) 2415, gr-qc/0408093;
D. Maity, S. SenGupta, S. Sur, Phys. Lett. B 643 (2006) 348, hep-th/0604195.
[32] F.K. Diakonos, P. Schmelcher, O. Biham, Phys. Rev. Lett. 81 (1998) 4349, hep-th/0001145.
[33] N. Tetradis, Phys. Lett. B 509 (2001) 307, hep-th/0012106.
[34] M. Bojowald, Phys. Rev. D 64 (2001) 084018, gr-qc/0105067.
[35] M. Bojowald, R. Maartens, P. Singh, Phys. Rev. D 70 (2004) 083517, hep-th/0407115.
[36] F.K. Diakonos, E.N. Saridakis, N. Tetradis, Phys. Lett. B 605 (2005) 1, hep-th/0409025;
P.S. Apostolopoulos, N. Brouzakis, E.N. Saridakis, N. Tetradis, Phys. Rev. D 72 (2005) 044013, hep-th/0502115.
[37] L. Baum, P.H. Frampton, Phys. Rev. Lett. 98 (2007) 071301, hep-th/0610213;
V.N. Yershov, physics/0702113;
L. Baum, P.H. Frampton, Mod. Phys. Lett. A 23 (2008) 33, physics/0703162;
P.H. Frampton, Mod. Phys. Lett. A 22 (2007) 2587, physics/0705.2730;
X. Zhang, arXiv: 0708.1408 [gr-qc].