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Approximation to the k-th derivatives by multiquadric quasi-interpolation method☆

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ABSTRACT

Quasi-interpolation is very useful in the study of approximation theory and its applications, since it can yield solutions directly without the need to solve any linear system of equations. Based on the good performance (simple to evaluate) of multiquadric functions, the advantages (approximation order, convexity and monotonlcity preserving) of multiquadric quasi-interpolation have been widely discussed. However, it is usually only for the approximation properties of the function itself and the good properties for derivatives, whereas the high order derivatives have been largely ignored. In this paper, we go further into the approximation properties to the k-th derivatives by using multiquadric quasi-interpolation. Furthermore, we develop two kinds of multiquadric quasi-interpolation schemes on bounded interval $[x_0, x_N]$, whose derivatives converge to the corresponding derivatives of the approximated functions. Finally, the numerical experiments are presented to confirm the accuracy of the presented scheme. Both theoretical results and numerical examples show this scheme provides good accuracy even if the data points in $[x_0, x_N]$ are irregularly distributed.

1. Introduction

Multiquadric functions were firstly proposed by in [1] in 1968 and they performed very well in many calculations including the numerical experiments that were reported in [2]. Micchelli [3] discussed the problem systematically in the theoretical category. For the meshless collocation (or interpolation) method for PDEs by using multiquadric functions, one is required to solve a large scaled system of linear equations; moreover, the coefficients matrix is usually very ill-conditioned and the results are sensitive to the shape parameter c, therefore multiquadric quasi-interpolation method has caught the attentions of many researchers. The most important advantage of quasi-interpolation is that one can evaluate the approximant directly without the need to solve any linear system of equations. Beatson and Powell proposed some quasi-interpolation schemes [4] by using multiquadric functions. Wu and Schaback [5] improved these schemes and discussed their approximation order and the shape preserving property. Beatson and Dyn [6] developed the theory in a wider category related to the topic. For applications in numerical solutions of PDEs using a collocation method with multiquadric functions, the readers are referred to [7]. Hon and Wu [8,9] used quasi-interpolation to simulate the solutions of shock wave equations. The multiquadric or radial basis functions method for solving PDEs has become one part of the new numerical method, which is named the meshless method. Beatson even used the multiquadric quasi-interpolation as a computer aided design tool in the film "The Lord of the Rings III".

The earliest case of quasi-interpolation is perhaps Bernstein's approximation [10], which uses the Bernstein polynomial to build a quasi-interpolation of an univariate function $f(x)$ on $[0, 1]$. It is a basic scheme in approximation theory and

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functional analysis, and, in addition, it is widely used in computer aided geometric design under the names of Beziér and de Casteljau. Another well-known quasi-interpolation scheme is formed by the reconstruction of bandlimited functions via the Whittaker–Shannon sampling series. There is also well-known B-spline series, which is included in any computer software for the representation of curves and surfaces [10]. For the advantages of multiquadric function, this paper surveys the properties of multiquadric quasi-interpolation. Defines the multiquadric function $\phi(x) = \sqrt{c^2 + x^2}$ and $\phi_j(x) = \phi(x - x_j)$, where c is shape parameter. Multiquadric quasi-interpolation of a function $f : [x_0, x_N] \mapsto R$ on the scattered knots

$$a = x_0 < x_1 < \cdots < x_N = b, \qquad h := \max_{1\leq j\leq N}(x_j - x_{j-1})$$

usually takes the form

$$(\mathcal{L}f)(x) = \sum f(x_j)\psi_j(x), \tag{1}$$

where $\psi_j(x)$ are the following linear combinations of the multiquadrics, that

$$\psi_j(x) = \frac{\phi_{j+1}(x) - \phi_j(x)}{2(x_{j+1} - x_j)} - \frac{\phi_j(x) - \phi_{j-1}(x)}{2(x_j - x_{j-1})}.$$

To improve the approximation behaviors near the boundary, Beatson and Powell [4] proposed three univariate multiquadric quasi-interpolation schemes, namely, $\mathcal{L}_\mathcal{A}$, $\mathcal{L}_\mathcal{B}$ and $\mathcal{L}_\mathcal{C}$, to approximate the function $\{f(x), x_0 \leq x \leq x_N\}$ from the space that is spanned by the multiquadrics and linear functions. Afterward, Wu and Schaback [5] presented the multiquadric quasi-interpolation $\mathcal{L}_\mathcal{D}$ on $[x_0, x_N]$ and proved that the scheme preserves convexity and monotonicity and is convergent. However, in these papers, the error estimates and the convexity, monotonicity preserving properties of the multiquadric quasi-interpolation, are only discussed for the function itself and its lower order derivatives but not for its high order derivatives. In order to use the quasi interpolation to solve PDEs numerically, one requires the approximation order for high order derivatives. This paper focuses on the approximate property of multiquadric quasi-interpolation for high order derivatives. Furthermore, we construct two kinds of multiquadric quasi-interpolation schemes (only a small correction of formula (1)), which possesses the optimal convergence property for the k-th derivative such that:

$$|f^{(k)}(x) - (\mathcal{L}_\mathcal{E}f)^{(k)}(x)| \leq \mathcal{O}(h^{\frac{2}{k+1}}), \quad k = 0, 1, 2, \ldots,$$
$$|f^{(k)}(x) - (\mathcal{L}_\mathcal{F}f)^{(k)}(x)| \leq \mathcal{O}(h^{\frac{2}{k+1}}), \quad k = 0, 1, 2, \ldots,$$

if we choose the parameter c according to k.

Like [4,5], this paper defines the normalized second order divided difference $\psi_j(x)$ of multiquadric function for every integer j as in the formula (1). Theoretically, at first we define a quasi-interpolation scheme $\mathcal{L}$ for $\{x_j\}_{j=-\infty}^{\infty}$ and get the convergence results of k-th derivatives. To solve the problems defined on a bounded interval, we provide two kinds of improved schemes, the one takes the form

$$f(x) \sim (\mathcal{L}_\mathcal{E}f)(x) = \sum [f(x_j) - P(x_j)] \cdot \psi_j(x) + P(x), \quad x \in [x_0, x_N],$$

where $P(x)$ is a polynomial, the another one takes the form

$$f(x) \sim (\mathcal{L}_\mathcal{F}f)(x) = \sum f(x_j) \cdot \psi_j(x) + \text{boundary term}, \quad x \in [x_0, x_N].$$

See next sections for details. Both theoretical results and numerical examples show that these schemes provide accuracy even if the data points in $[x_0, x_N]$ are irregularly distributed.

This paper is organized as follows. Section 2 introduces a multiquadric quasi-interpolation scheme $(\mathcal{L}f)(x)$ and proves its approximation order to functions and their derivatives, provided that the approximated function is smooth enough. Moreover, to approximate functions and their derivatives in a interval $[x_0, x_N]$, we develop two kinds of multiquadric quasi-interpolation schemes for functions on $[x_0, x_N]$. In order to test the accuracy and efficiency of the schemes, we present some numerical examples in Section 3. Section 4 ends this paper with a brief conclusion.

2. Convergence of multiquadric quasi-interpolation to *k*-th derivatives

We start by considering the following multiquadric quasi-interpolation scheme:

$$(\mathcal{L}f)(x) = \sum_{j=-\infty}^{\infty} f(x_j)\psi_j(x), \tag{2}$$

where $\psi_j(x)$ are defined as in formula (1). Buhmann [11] studied the accuracy of the quasi-interpolation operator $\mathcal{L}$ applied to function $f : R \mapsto R$ on uniform data points ($x_j = jh$) and showed that this scheme reproduces linear polynomials. Powell [12] proved that the quasi-interpolation operator $\mathcal{L}$ reproduces the linear polynomial even for non-uniformly spaced data points. The convergence estimates, however, need a more detailed analysis than the familiar ones from spline theory for instance, because compact support of the basis functions makes the proof techniques much simpler. Powell noted the asymptotic decay at an algebraic rate of the basis function $\psi_j(x)$ (not the multiquadric functions $\phi_j(x)$ but $\psi_j(x)$ the linear combinations of translates of the multiquadric function; it is important to distinguish carefully between those). Using this decay in tandem with polynomial recovery of a nontrivial order, one can prove the following theorem. For more details, we refer the readers to [4,11,5].

Theorem 1. *Let $f(x)$ be twice differentiable, such that $\|f'(x)\|_\infty$ and $\|f''(x)\|_\infty$ are bounded. Then the inequalities*

$$\|(\mathcal{L}f)(x) - f(x)\|_\infty \le \mathcal{O}(h^2)$$

and

$$\|(\mathcal{L}f)'(x) - f'(x)\|_\infty \le \mathcal{O}(h)$$

hold, provided c is small enough.

In order to use this method to solve PDEs numerically, we concentrate our attention on the approximation order of multiquadric quasi-interpolation scheme $(\mathcal{L}f)^{(k)}(x)$ to $f^{(k)}(x)$, $k \ge 2$, since for $k < 2$ the problem is already solved by a lot of authors (Theorem 1).

To discuss the property of multiquadric functions, we first consider the standard multiquadric function

$$\varphi(x) = \sqrt{1+x^2}.$$

Lemma 1. *The kth-order ($k \ge 2$) derivatives of $\varphi(x)$ can be bounded as*

$$|\varphi^{(k)}(x)| \le \frac{C_k}{(1+x^2)^{\frac{k+1}{2}}},$$

where C_k is a constant, which depends on k.

Proof. $\varphi''(x) = 1/(1+x^2)^{3/2}$. Denote polynomial of degree k by $P_k(x)$. Assuming

$$\varphi^{(k)}(x) = P_{k-2}(x)/(1+x^2)^{(2k-1)/2},$$

then

$$\begin{aligned}\varphi^{(k+1)}(x) &= \varphi^{(k)'}(x) \\ &= P_{k-3}(x)/(1+x^2)^{(2k-1)/2} + 2xP_{k-2}(x)/(1+x^2)^{(2k+1)/2} \\ &= P_{k-1}(x)/(1+x^2)^{(2k+1)/2}.\end{aligned}$$

Furthermore by mathematical induction, we have

$$|\varphi^{(k)}(x)| \le \left|\frac{P_{k-2}(x)}{(1+x^2)^{(2k-1)/2}}\right| \le \frac{C_k}{(1+x^2)^{\frac{k+1}{2}}}. \quad \square \tag{3}$$

The multiquadric function with shape parameter c is defined to be

$$\phi(x) = \sqrt{c^2+x^2} = c\cdot\varphi\left(\frac{x}{c}\right),$$

and its derivatives can be bounded as

$$|\phi^{(k)}(x)| \le \frac{C_k\cdot c^2}{(c^2+x^2)^{\frac{k+1}{2}}}.$$

It is straightforward to derive the following two inequalities

$$|\phi^{(k)}(x)| \le \frac{C_k\cdot c^2}{|x|^{k+1}},$$

$$|\phi^{(k)}(x)| \le \frac{C_k}{c^{k-1}}.$$

Some results from approximation theory are also needed. We start by preparing the follow lemma.

Lemma 2. *If f can be represented by an inverse Fourier transform $f(x) = \int e^{ixw}\hat{f}(w)dw$, $\int \hat{f}(w)w^k dw$ exists, and*

$$|\widehat{\Phi}(\omega) - 1| \le \mathcal{O}(\omega^k), \quad \omega \to 0, \tag{4}$$

in which $\widehat{\Phi}(\omega)$ is the Fourier transform of $\Phi(x)$, then there is a constant C such that

$$|(\Phi_\varepsilon * f)(x) - f(x)| < C\cdot\varepsilon^k,$$

where $\Phi_\varepsilon(x) = 1/\varepsilon\Phi(x/\varepsilon)$.

The proof of the Lemma 2 can be found in [13]. The paper [14] showed that the condition (4) can be easily satisfied by modifying a given Φ, if $\widehat{\Phi}(0) \ne 0$ and Φ decays sufficiently fast. In particular, by a finite linear combination of shifts (or the scales) of the function Φ one can satisfy the condition (4).

Define $\Phi(x) = \frac{1}{2(1+x^2)^{\frac{3}{2}}} = \frac{\varphi''(x)}{2}$, then

$$\Phi_c(x) := \frac{\varphi''(\frac{x}{c})}{2c} = \frac{\phi''(x)}{2}.$$

By noting the fact that the Fourier transform of $\Phi(x)$ satisfies

$$|\widehat{\Phi}(\omega) - 1| \leq \mathcal{O}(\omega^2), \quad \omega \to 0,$$

we have following corollary

Corollary 1. *If $f \in C^2(R)$, then the following inequality*

$$\left| \int_{-\infty}^{\infty} f(t) \cdot \frac{\phi''(x-t)}{2} \mathrm{d}t - f(x) \right| \leq \mathcal{O}(c^2)$$

holds.

Now we will present the main result of this paper.

Theorem 2. *If $f(x) \in C^{(k+2)}(R)$ and $f^{(j)}(x)$ is bounded by a polynomial of degree $k+2-j$, then*

$$|(\mathcal{L}f)^{(k)}(x) - f^{(k)}(x)| \leq \mathcal{O}(h^{\frac{2}{k+1}})$$

holds, provided that $c = \mathcal{O}(h^{\frac{1}{k+1}})$.

Proof. By rearranging the scheme

$$\begin{aligned}
\mathcal{L}f(x) &= \sum_{j=-\infty}^{\infty} f(x_j)\psi_j(x) \\
&= \sum_{j=-\infty}^{\infty} f(x_j) \cdot \left[\frac{\phi_{j+1}(x) - \phi_j(x)}{2(x_{j+1} - x_j)} - \frac{\phi_j(x) - \phi_{j-1}(x)}{2(x_j - x_{j-1})} \right] \\
&= \sum_{j=-\infty}^{\infty} \left[\frac{f(x_{j+1}) - f(x_j)}{2(x_{j+1} - x_j)} - \frac{f(x_j) - f(x_{j-1})}{2(x_j - x_{j-1})} \right] \cdot \phi_j(x),
\end{aligned}$$

we deduce that

$$(\mathcal{L}f)^{(k)}(x) = \sum_{j=-\infty}^{\infty} \left[\frac{f(x_{j+1}) - f(x_j)}{2(x_{j+1} - x_j)} - \frac{f(x_j) - f(x_{j-1})}{2(x_j - x_{j-1})} \right] \cdot \phi_j^{(k)}(x).$$

Due to the fact that

$$\frac{f(x_{j+1}) - f(x_j)}{x_{j+1} - x_j} = f'(x_j) + \frac{1}{2} f''(x_j) \cdot (x_{j+1} - x_j) + \frac{1}{6} f'''(\xi_j) \cdot (x_{j+1} - x_j)^2$$

and

$$\frac{f(x_j) - f(x_{j-1})}{x_j - x_{j-1}} = f'(x_j) - \frac{1}{2} f''(x_j) \cdot (x_j - x_{j-1}) + \frac{1}{6} f'''(\eta_j) \cdot (x_j - x_{j-1})^2,$$

where $\xi_j \in (x_j, x_{j+1})$, $\eta_j \in (x_{j-1}, x_j)$, we have

$$(\mathcal{L}f)^{(k)}(x) = \sum_{j=-\infty}^{\infty} \left[f''(x_j) \cdot \frac{(x_{j+1} - x_{j-1})}{2} \right] \cdot \phi_j^{(k)}(x) + \sum_{j=-\infty}^{\infty} \left[\frac{1}{6} f'''(\xi_j)(x_{j+1} - x_j)^2 + \frac{1}{6} f'''(\eta_j)(x_j - x_{j-1})^2 \right] \cdot \phi_j^{(k)}(x).$$

The intermediate value theorem of integration is applied on each interval $[\frac{x_{j-1}+x_j}{2}, \frac{x_j+x_{j+1}}{2}]$ to get

$$\begin{aligned}
I &:= \left| \int_{-\infty}^{\infty} f''(t) \cdot \frac{\phi^{(k)}(x-t)}{2} \mathrm{d}t - \sum_{j=-\infty}^{\infty} \left[f''(x_j) \cdot \frac{(x_{j+1} - x_{j-1})}{2} \right] \cdot \phi_j^{(k)}(x) \right| \\
&= \left| \sum_{j=-\infty}^{\infty} \int_{\frac{x_{j-1}+x_j}{2}}^{\frac{x_j+x_{j+1}}{2}} f''(t) \cdot \frac{\phi^{(k)}(x-t)}{2} \mathrm{d}t - \sum_{j=-\infty}^{\infty} \left[f''(x_j) \cdot \frac{(x_{j+1} - x_{j-1})}{2} \right] \cdot \phi_j^{(k)}(x) \right|
\end{aligned}$$

$$
\begin{aligned}
&= \left| \sum_{j=-\infty}^{\infty} [f''(\zeta_j)\phi^{(k)}(x-\zeta_j) - f''(x_j)\phi^{(k)}(x-x_j)] \frac{(x_{j+1}-x_{j-1})}{2} \right| \\
&= \left| \sum_{j=-\infty}^{\infty} \left[f'''(\delta_j)\phi^{(k)}(x-\delta_j) + f''(\delta_j)\phi^{(k+1)}(x-\delta_j)\right] \cdot (\zeta_j - x_j) \frac{(x_{j+1}-x_{j-1})}{2} \right| \\
&\le h \sum_{j=-\infty}^{\infty} |f'''(\delta_j)\phi^{(k)}(x-\delta_j) + f''(\delta_j)\phi^{(k+1)}(x-\delta_j)| \frac{(x_{j+1}-x_{j-1})}{2},
\end{aligned}
$$

where $\zeta_j \in [\frac{x_{j-1}+x_j}{2}, \frac{x_j+x_{j+1}}{2}]$ and $\delta_j \in [\zeta_j, x_j]$.

Note the estimate (3) and the conditions for the f and its derivatives. Splitting the sum in two parts with $|x - x_j| \le c$ and the rest, the two estimates

$$
|\phi^{(k)}(x)| \le \frac{C_k}{c^{k-1}}
$$

$$
|\phi^{(k)}(x)| \le \frac{C_k \cdot c^2}{|x|^{k+1}}
$$

in Lemma 2 are applied to get

$$
I \le \mathcal{O}(h/c^{k-1}),
$$

where the coefficient of the (h/c^{k-1}) depends on x and is bounded by a polynomial of x of degree k. Similarly, by using the bound of $\phi_j^{(k)}(x)$ and $f'''(x)$ (i.e. estimate (3) and the conditions of the approximand) we get

$$
\sum_{j=-\infty}^{\infty} \left[\frac{1}{6} f'''(\xi_j)(x_{j+1}-x_j)^2 + \frac{1}{6} f'''(\eta_j)(x_j - x_{j-1})^2\right] \cdot \phi_j^{(k)}(x) \le \mathcal{O}(h/c^{k-1}).
$$

Now we have

$$
\begin{aligned}
(\mathcal{L}f)^{(k)}(x) &= \int_{-\infty}^{\infty} f''(t) \cdot \frac{\phi^{(k)}(x-t)}{2} \mathrm{d}t + \mathcal{O}(h/c^{k-1}) \\
&= \int_{-\infty}^{\infty} f^{(k)}(t) \cdot \frac{\phi''(x-t)}{2} \mathrm{d}t + \mathcal{O}(h/c^{k-1}).
\end{aligned} \tag{5}
$$

If $f(x)$ satisfies the conditions of Lemma 2, then by using Corollary 1, one can deduce the following inequality easily

$$
\left| \int_{-\infty}^{\infty} f^{(k)}(t) \cdot \frac{\phi''(x-t)}{2} \mathrm{d}t - f^{(k)}(x) \right| \le \mathcal{O}(c^2).
$$

That is to say

$$
|(\mathcal{L}f)^{(k)}(x) - f^{(k)}(x)| \le \mathcal{O}(c^2) + \mathcal{O}(h/c^{k-1}).
$$

By setting $c = \mathcal{O}(h^{\frac{1}{k+1}})$, we get the following optimal error estimates:

$$
|f^{(k)}(x) - (\mathcal{L}f)^{(k)}(x)| \le \mathcal{O}(h^{\frac{2}{k+1}}). \quad \square
$$

Corollary 2. *If the shape parameter c is selected as above, then for any $l \le k$,*

$$
|f^{(l)}(x) - (\mathcal{L}f)^{(l)}(x)| \le \mathcal{O}(h^{\frac{2}{k+1}});
$$

for $l = k + 1$,

$$
|f^{(l)}(x) - (\mathcal{L}f)^{(l)}(x)| \le \mathcal{O}(h^{\frac{1}{k+1}}). \quad \square
$$

So far, the discussion focused on the multiquadric quasi-interpolation scheme $(\mathcal{L}f)(x)$ for data points $\{x_j\}_{j=-\infty}^{\infty}$ and $\lim_{j\to\pm\infty} x_j = \pm\infty$. In applications, however, the problem usually appears only for the bounded interval: given the data points $\{x_0, \ldots, x_N\}$, the data $\{f(x_0), \ldots, f(x_N)\}$ and extreme derivatives $\{f'(x_0), \ldots, f^{(k+2)}(x_0), f'(x_N), \ldots, f^{(k+2)}(x_N)\}$, one wants to get an approximation of $f^{(k)}(x)$ on $[x_0, x_N]$. There are several ways to handle this problem. We will provide two solutions to settle it.

The first one is by Hermitian interpolation. We can get $P(x)$ which is a polynomial of degree $2k+5$ such that $P^{(l)}(x_0) = f^{(l)}(x_0)$ and $P^{(l)}(x_N) = f^{(l)}(x_N)$ for $0 \le l \le k+2$. Therefore

$$\bar{f}(x) = \begin{cases} f(x) - P(x), & x \in [x_0, x_N] \\ 0, & \text{otherwise} \end{cases}$$

is a compactly supported C^{k+2} function satisfing the condition of the Lemma 2. Then we apply the operator $\mathcal{L}$ to $\bar{f}(x)$ and get

$$(\mathcal{L}_{\mathcal{E}} f)^{(k)}(x) = \mathcal{L}\bar{f}^{(k)}(x) + P^{(k)}(x) = \sum_{j=0}^{N} [f(x_j) - P(x_j)]\psi_j^{(k)}(x) + P^{(k)}(x)$$

is an approximation of $f^{(k)}(x)$ on $[x_0, x_N]$, such that

$$|(\mathcal{L}_{\mathcal{E}} f)^{(k)}(x) - f^{(k)}(x)| \le \mathcal{O}(h^{\frac{2}{k+1}}).$$

The another way is by polynomial extension. We extend the function $f(x)$ from $[x_0, x_N]$ to domain $(-\infty, \infty)$ according to boundary conditions by Taylor's expansion:

$$f^*(x) = \begin{cases} P_1(x), & \text{if } x \in (-\infty, x_0), \\ f(x), & \text{if } x \in [x_0, x_N], \\ P_2(x), & \text{if } x \in (x_N, \infty) \end{cases}$$

where $P_1(x)$ and $P_2(x)$ are polynomials satisfying $P_1^{(l)}(x_0) = f^{(l)}(x_0)$ and $P_2^{(l)}(x_N) = f^{(l)}(x_N)$ for $0 \le l \le k+1$ respectively (At this time $f^*(x) \notin C^{k+2}(R)$, however the order of the convergence is still valid). Apply the operator $\mathcal{L}$ to $f^*(x)$ by using $x_j = x_0 + jh_1$ for $j < 0$ and $x_j = x_N + (j-N)h_1$ for $j > N$, then

$$\mathcal{L}f^*(x) = \sum_{j=-\infty}^{0} P_1(x_j)\psi_j(x) + \sum_{j=1}^{N-1} f(x_j)\psi_j(x) + \sum_{j=N}^{\infty} P_2(x_j)\psi_j(x).$$

For the part $j \notin (0, N)$, let $h_1 \to 0$ and then $c \to 0$, through a simple calculation we have

$$\begin{aligned} \mathcal{L}_{\mathcal{F}}{}^{(k)}f(x) :=& f(x_0)\frac{\phi_1^{(k)}(x) - \phi_0^{(k)}(x)}{2(x_1 - x_0)} - \sum_{l=1}^{k-1} f^{(l)}(x_0)\phi_0^{(k-l+1)}(x) + \sum_{j=1}^{N-1} f(x_j)\psi_j^{(k)}(x) \\ & - f(x_N)\frac{\phi_N^{(k)}(x) - \phi_{N-1}^{(k)}(x)}{2(x_N - x_{N-1})} + \sum_{l=1}^{k-1} f^{(l)}(x_N)\phi_N^{(k-l+1)}(x) \\ \sim\ & (\mathcal{L}f^*)^{(k)}(x) \end{aligned}$$

is an approximation of $f^{(k)}(x)$ on $[x_0, x_N]$, such that

$$|(\mathcal{L}_{\mathcal{F}} f)^{(k)}(x) - f^{(k)}(x)| \le \mathcal{O}(h^{\frac{2}{k+1}}).$$

Remark 1. If the given data is only the function values $\{f(x_0), \ldots, f(x_N)\}$, we can use the divided differences (e.g. $[x_0, \ldots, x_l]f \sim f^{(l)}(x_0)/l!$) near the boundary, instead of the derivatives, to construct the polynomials $P(x), P_1(x), P_2(x)$ as above respectively.

Remark 2. We should mention that the multivariate problems can be solved by using the method in [15] or the method shown in [13].

3. Numerical examples

In this section, some numerical experiments are presented to demonstrate the accuracy and performance of the proposed multiquadric quasi-interpolation schemes. Since many works on the approximation order of multiquadric quasi-interpolation schemes to functions have been already done [4,6,11,12,5], we concentrate on the approximation order of the multiquadric quasi-interpolation schemes to derivatives, which were presented in the previous section. Now consider the approximand function $f(x) = \cos x + \exp(2x^2)$, $x \in [0, 1]$. For simplicity, we take $k = 2$ as an example. Given function values $\{f(x_j) : j = 0, 1, \ldots, N\}$, using the method introduced in Remark 1, we want to construct the scheme $\mathcal{L}_{\mathcal{E}}$. Firstly, one can easily get a polynomial $P(x)$ of degree 9 satisfying the boundary condition

$$\begin{aligned} &P(x_0) = f(x_0), \ldots, P(x_4) = f(x_4), \\ &P(x_N) = f(x_N), \ldots, P(x_{N-4}) = f(x_{N-4}). \end{aligned}$$

Then

$$(\mathcal{L}f)''(x) = \sum_{j=5}^{N-5} [f(x_j) - P(x_j)] \cdot \psi_j''(x) + P''(x),$$

Table 1
Posteriori error estimation.

h	N	Absolute error	Posteriori error estimation
0.04	26	0.0038385	
0.02	51	0.0023497	0.70807
0.01	101	0.0013912	0.75615

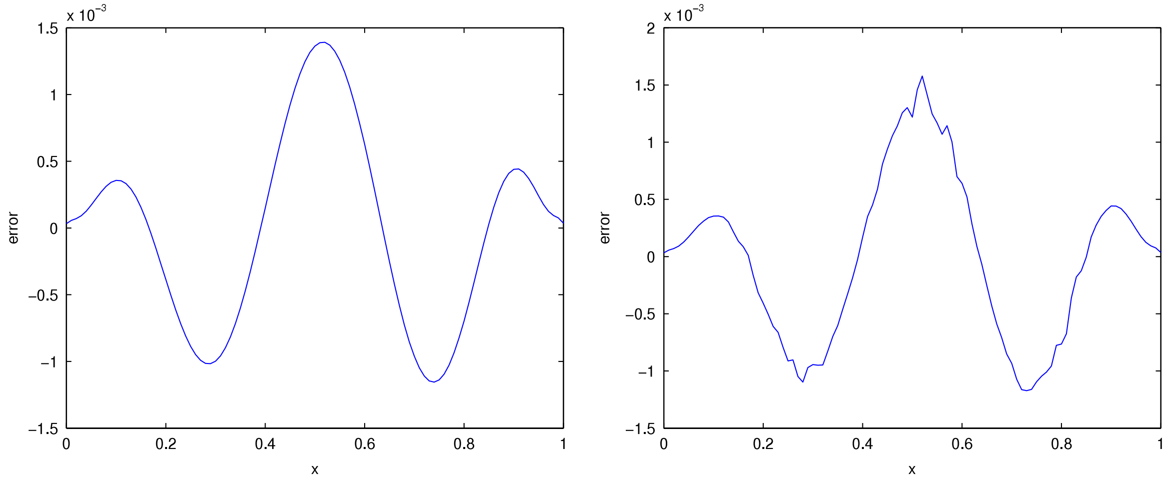

Fig. 1. Plot the error function for uniform data (left) and scattered data (right) using multiquadric quasi-interpolation with $h = 0.01$ and parameter $c = 0.1h^{1/3}$.

in which $x \in [x_0, x_N]$. We should mention that, by introducing the polynomial $P(x)$, the terms $0 \le j \le 4$ and $N-4 \le j \le N$ disappeared in the scheme. To back up the presented convergence results, we chose $h = 0.04, 0.02, 0.01$ respectively. Table 1 gives the posteriori error estimation (>0.7) to support our convergence theories. It seems that the numerical posteriori error estimation is better than the theoretical ones (which is about $2/3$).

In Fig. 1, the error functions are plotted for uniform data (left) and scattered data (right) using multiquadric quasi-interpolation with $h = 0.01$ and $c = 0.1h^{1/3}$. From the figure, we conclude that the presented scheme is still valid for scattered data.

4. Conclusion

In this paper we consider an approximation $(\mathcal{L}f)(x)$ to a function $\{f(x) : x \in [x_0, x_N]\}$ from the space that is spanned by the multiquadrics $\{\phi(x-x_j)\}_{j=0}^N$ and by polynomials, the data points $\{x_j\}_{j=0}^N$ being given distinct points in the interval $[x_0, x_N]$. The coefficients of the approximation depend on the function values $\{f(x_j)\}_{j=0}^N$. Theoretical proof and numerical examples show that $f^{(k)}(x) \approx (\mathcal{L}f)^{(k)}(x)$ holds for $k = 0, 1, 2, \ldots$ with a satisfactory approximation ability. It is a very difficult problem to deal with the boundary in the quasi-interpolation scheme. We handle this problem by introducing a polynomial, which is showed to be an effective technique in our multiquadric quasi-interpolation schemes. From the theoretical proof and numerical examples, we conclude that our scheme gives good accuracy even if the distribution of the data points in $[x_0, x_N]$ is very irregular.

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References

[1] R.L. Hardy, Multiquadric equations of topography and other irregular surfaces, Journal of Geophysical Research 76 (1971) 1905–1915.
[2] R. Franke, Scattered data interpolation: Tests of some methods, Mathematics of Computation 38 (1982) 181–200.
[3] C.A. Micchelli, Interpolation of scattered data: Distance matrix and conditionally positive definite functions, Constructive Approximation 2 (1986) 11–22.
[4] R.K. Beatson, M.J.D. Powell, Univariate multiquadric approximation: Quasi-interpolation to scattered data, Constructive Approximation 8 (1992) 275–288.
[5] Z.M. Wu, R. Schaback, Shape preserving properties and convergence of univariate multiquadric quasi-interpolation, Acta Mathematicae Applicatae Sinica 10 (1994) 441–446.

[6] R.K. Beatson, N. Dyn, Multiquadric B-splines, Journal of Approximation Theory 87 (1996) 1–24.
[7] E.J. Kansa, Multiquadrics-a scattered data approximation scheme with applications to computational fluid dynamics I, Computers and Mathematics with Applications 19 (1990) 127–145.
[8] Y.C. Hon, Z.M. Wu, An Quasi-interpolation Method for solving stiff ordinary difference equations, International Journal for Numerical Methods in Engineering 48 (2000) 1187–1197.
[9] Z.M. Wu, Dynamical knot and shape papameter setting for simulating shock wave by suing multi-quadric quasi-interpolation, Engineering analysis with boundary elements 29 (2005) 354–358.
[10] G.E. Farin, Curves and Surfaces for Computer-Aided Geometric Design, Academic Press, 1997.
[11] M.D. Buhmann, Radial Basis Functions: Theory and Implementations, Cambridge University Press, 2003.
[12] M.J.D. Powell, Univariate multiquadric approximation: Reproduction of linear poly-nomials, in: W. Haussman, K. Jetter (Eds.), Multivariate Approximation and Interpolation, Birkhäuser Verlag, Basel, 1990, pp. 227–240.
[13] Z.M. Wu, J.P. Liu, Generalized Strang-Fix condition for scattered data quasi-interpolation, Advances in Computational Mathematics 23 (2005) 201–214.
[14] R. Schaback, Z.M. Wu, Construction techniques for highly accurate quasi-interpolation operators, Journal of Approximation Theory 91 (1997) 115–124.
[15] L. Ling, Multivariate quasi-interpolation schemes for dimension-splitting multiquadric, Applied Mathematics and Computation 161 (2005) 195C209.