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Generalized Bernstein–Durrmeyer Operators and the Associated Limit Semigroup

Antonio Attalienti

Istituto di Matematica Finanziaria, Facoltà di Economia, Università di Bari, Via Camillo Rosalba 53, 70124 Bari, Italy
E-mail: attalienti@matfin.uniba.it

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strongly continuous positive contraction semigroup on $C([0,1])$ which may be expressed in terms of powers of these operators. As a direct consequence, we are able to represent explicitly the solutions of the Cauchy problems associated with a particular class of second order differential operators.

Key Words: Beta operators; Bernstein–Durrmeyer operators; positive approximation process; semigroup of operators; Cauchy problem.

INTRODUCTION

In [15] J. L. Durrmeyer introduced a sequence $(M_n)_{n\in\mathbb{N}}$ of particular modified Bernstein polynomial operators which enjoy nice approximation properties. These operators are defined by putting for every $n\in\mathbb{N}$ and $f\in L_1([0,1])$

$$M_n(f)(x) := (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 p_{n,k}(t)\, f(t)\, dt \qquad (0\leqslant x\leqslant 1), \quad (1)$$

where $p_{n,k}(x) := \binom{n}{k} x^k(1-x)^{n-k}$ $(0\leqslant k\leqslant n,\ x\in[0,1])$.

A careful analysis of such operators, usually called Bernstein–Durrmeyer operators, was carried out for the first time by Derriennic in [11]. Subsequently Heilmann [16] studied the saturation properties in the space L_p $(1\leqslant p<\infty)$, whereas Ditzian and Ivanov [13] studied their rate of convergence and that of their derivatives in terms of the so-called Ditzian–Totik modulus of continuity. Other inverse results have been stated, for instance, in [30, 32].

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Rather recently an interesting generalization of the operators M_n in terms of the so-called Jacobi weights has been developed in [7, 8] (see also [18, 19]).

Multidimensional Bernstein–Durrmeyer operators have been introduced and studied by Derriennic in [12]; some further results in this setting may also be found, for instance, in [9, 29, 31]; particular multidimensional weighted Bernstein–Durrmeyer operators have also been considered by Sauer in [23].

The purpose of the present paper is to provide a further generalization of the operators M_n in the one-dimensional case by means of quite different techniques. More precisely, we shall use a composition argument referring, in this direction, to the obvious relationship

$$M_n = B_n \circ \mathscr{B}_n \qquad (n \in \mathbb{N}), \tag{2}$$

where for each $n \in \mathbb{N}$ B_n is the classical nth Bernstein polynomial on $C([0, 1])$ whereas $\mathscr{B}_n$ denotes the nth Beta operator introduced by Lupaş in [17, p. 37] and defined by

$$\mathscr{B}_n(f)(x) := \frac{1}{B(nx+1, n+1-nx)} \int_0^1 t^{nx}(1-t)^{n(1-x)} f(t)\, dt$$

for every $f \in C([0, 1])$ and $x \in [0, 1]$, $B(\cdot, \cdot)$ being the standard Beta function.

Our starting point is a continuous function λ: $[0, 1] \to [0, 1]$ such that $1/2 \leqslant \lambda(x) \leqslant 1$ for every $x \in [0, 1]$; we set $\gamma(x) := 2\lambda(x) - 1$ $(0 \leqslant x \leqslant 1)$ and for every $n \in \mathbb{N}$ define

$$M_{n,\lambda} := L_{n,\gamma} \circ \mathscr{B}_{n,\lambda}, \tag{3}$$

where $L_{n,\gamma}$ is the nth Lototsky–Schnabl operator on $C([0, 1])$ associated with γ (see (2.2)) and $\mathscr{B}_{n,\lambda}$ is defined in (1.1). Due to [6, formula (6.1.50), p. 399] and (1.2), the definition (3) coincides exactly with (2) in the particular case $\lambda = e_0$ (here and in the sequel e_0 denotes the continuous function on $[0, 1]$ of constant value 1) and therefore we may rightly refer to $M_{n,\lambda}$ as to the nth generalized Bernstein–Durrmeyer operator associated with λ.

The paper is split into two sections.

Section 1 is devoted to the study of the sequence of the operators $\mathscr{B}_{n,\lambda}$ defined in (1.1) and to its approximation behavior: we show that it is a positive approximation process on the Banach lattice $C([0, 1])$ and state also some estimates of the pointwise and uniform approximation in terms of the classical modulus of continuity $\omega(f, \cdot)$.

Moreover, a further uniform estimate which makes use of the Ditzian–Totik modulus of continuity is also established.

At last we prove a Voronovskaja-type formula, generalizing an analogous result proved by Lupaş in [17, Satz 2.24] for the Beta operators.

In Section 2, according to the above definition (3), we introduce the sequence of the operators $M_{n,\lambda}$ which is readily shown to be a positive approximation process on $C([0,1])$ as well.

Moreover, we state a Voronovskaja-type result and this is the key tool to prove the main theorem of the section about the existence of a strongly continuous positive contraction semigroup on $C([0,1])$ which may be expressed in terms of powers of the operators $M_{n,\lambda}$ and whose generator is a second order differential operator; consequently, in force of standard semigroup results, the solution of the associated Cauchy problem is explicitly represented in the same way as well.

1. THE OPERATORS $\mathscr{B}_{n,\lambda}$

Let $C([0,1])$ be the Banach lattice of all real-valued continuous functions on $[0,1]$ endowed with the sup-norm $\|\cdot\|$ and the natural order.

Throughout this paper, for every $p \in \mathbb{N}_0$, we shall denote by e_p the continuous function on $[0,1]$ defined by the monomials $e_p(x) := x^p$ for every $x \in [0,1]$.

As usual, for each $m \in \mathbb{N}$, $C^m([0,1])$ will denote the vector space of all real-valued m-times continuously differentiable functions on $[0,1]$, whereas $o(\cdot)$ and $O(\cdot)$ stand for the classical Landau symbols.

Let us fix a strictly positive continuous function $\lambda: [0,1] \to [0,1]$ and for every $n \in \mathbb{N}$ consider the operator $\mathscr{B}_{n,\lambda}: C([0,1]) \to C([0,1])$ defined by

$$\mathscr{B}_{n,\lambda}(f)(x) := \frac{1}{B(nx+\lambda(x), n+\lambda(x)-nx)} \times \int_0^1 t^{nx+\lambda(x)-1}(1-t)^{n(1-x)+\lambda(x)-1} f(t)\, dt \tag{1.1}$$

for all $f \in C([0,1])$ and $x \in [0,1]$. Clearly every $\mathscr{B}_{n,\lambda}$ is positive and linear; when $\lambda = e_0$ then $\mathscr{B}_{n,e_0}$ is just the nth Beta operator $\mathscr{B}_n$ introduced by Lupaş in [17, p. 37], i.e.,

$$\mathscr{B}_{n,e_0}(f)(x) = \mathscr{B}_n(f)(x) := \frac{1}{B(nx+1, n+1-nx)} \int_0^1 t^{nx}(1-t)^{n(1-x)} f(t)\, dt. \tag{1.2}$$

Beta operators have been extensively studied in [17] and fall within a more general class of positive linear operators (sometimes called Beta-type operators) already considered in [1–3].

Our aim in this section is to carry out a detailed analysis of the approximation properties of the operators $\mathscr{B}_{n,\lambda}$ and to state some estimates of the rate of convergence.

Let us start by stating the following result.

THEOREM 1.1. *For every $f \in C([0,1])$*

$$\lim_{n\to\infty} \mathscr{B}_{n,\lambda}(f) = f \qquad \text{uniformly on } [0,1], \tag{1.3}$$

i.e., the sequence $(\mathscr{B}_{n,\lambda})_{n\in\mathbb{N}}$ is a positive approximation process on $C([0,1])$.

Proof. Indeed

$$\begin{aligned}\mathscr{B}_{n,\lambda}(e_p)(x) &= \frac{1}{B(nx+\lambda(x), n+\lambda(x)-nx)} \\ &\quad \times \int_0^1 t^{nx+p+\lambda(x)-1}(1-t)^{n(1-x)+\lambda(x)-1}\,dt \\ &= \frac{B(nx+\lambda(x)+p, n+\lambda(x)-nx)}{B(nx+\lambda(x), n+\lambda(x)-nx)} \\ &= \frac{\Gamma(nx+\lambda(x)+p)\,\Gamma(n+2\lambda(x))}{\Gamma(n+2\lambda(x)+p)\,\Gamma(nx+\lambda(x))} \\ &= \prod_{k=0}^{p-1} \frac{(nx+\lambda(x)+k)}{(n+2\lambda(x)+k)} \end{aligned} \tag{1}$$

for every $n, p \in \mathbb{N}$ and $x \in [0,1]$; in particular

$$\mathscr{B}_{n,\lambda}(e_1)(x) = \frac{nx+\lambda(x)}{n+2\lambda(x)},$$

$$\mathscr{B}_{n,\lambda}(e_2)(x) = \frac{(nx+\lambda(x))(nx+\lambda(x)+1)}{(n+2\lambda(x))(n+2\lambda(x)+1)}.$$

Since $\mathscr{B}_{n,\lambda}(e_0) = e_0$ for every $n \in \mathbb{N}$, it immediately follows that

$$\lim_{n\to\infty} \mathscr{B}_{n,\lambda}(e_i) = e_i \qquad \text{for} \quad i = 0, 1, 2$$

uniformly on $[0,1]$, which implies (1.3) on account of Korovkin's theorem (see, e.g., [6, Theorem 4.2.4, p. 214]). ∎

Now we are able to give some estimates of the rate of the pointwise and uniform convergence stated in the previous theorem. We shall use the classical modulus of continuity $\omega(f, \cdot)$ and also a weighted modulus of continuity as defined in [26, 27] and generalized in [14].

We start with some preliminary notations. For each $x \in [0, 1]$ we shall denote by ψ_x the continuous function on $[0, 1]$ defined by

$$\psi_x(t) := t - x \tag{1.4}$$

for every $t \in [0, 1]$. Applying formula (1) of the proof of Theorem 1.1 easily gives

$$\mathscr{B}_{n,\lambda}(\psi_x)(x) = \frac{\lambda(x)(1-2x)}{n+2\lambda(x)}, \tag{1.5}$$

and

$$\mathscr{B}_{n,\lambda}(\psi_x^2)(x) = \frac{nx(1-x) + \lambda(x)(2x^2-2x+1) + \lambda^2(x)(2x-1)^2}{(n+2\lambda(x))(n+2\lambda(x)+1)} \tag{1.6}$$

for every $n \in \mathbb{N}$ and $x \in [0, 1]$.

Let us denote by φ the continuous function on $[0, 1]$ defined by

$$\varphi(x) := \sqrt{x(1-x)} \tag{1.7}$$

for every $x \in [0, 1]$ and consider the following modulus of continuity

$$\omega_\varphi^2(f, \delta)_\infty := \sup_{\substack{0 \leqslant h \leqslant \delta \\ 0 \leqslant x \pm h\varphi(x) \leqslant 1}} |\Delta^2_{h\varphi(x)} f(x)| \tag{1.8}$$

for each $\delta > 0$, where $\Delta^2_{h\varphi(x)} f(x) := f(x - h\varphi(x)) - 2f(x) + f(x + h\varphi(x))$ for any $x \in [0, 1]$ such that $0 \leqslant x \pm h\varphi(x) \leqslant 1$.

THEOREM 1.2. *For every $f \in C([0, 1])$, $n \in \mathbb{N}$ and $x \in [0, 1]$ we get*

$$|\mathscr{B}_{n,\lambda}(f)(x) - f(x)| \leqslant 2\omega\left(f, \sqrt{\frac{nx(1-x) + \lambda(x)(2x^2-2x+1) + \lambda^2(x)(2x-1)^2}{(n+2\lambda(x)+1)(n+2\lambda(x))}}\right). \tag{1.9}$$

Moreover

$$\|\mathscr{B}_{n,\lambda}(f) - f\| \leqslant 2\omega\left(f, \frac{1}{\sqrt{n}}\right) \qquad \textit{for} \quad n \geqslant 3. \tag{1.10}$$

If, in addition, f is differentiable in $[0, 1]$ and $f' \in C([0, 1])$, then

$$
\begin{aligned}
&|\mathscr{B}_{n,\lambda}(f)(x) - f(x)| \\
&\quad \leqslant |f'(x)| \frac{\lambda(x)\, |(1-2x)|}{n + 2\lambda(x)} \\
&\quad + 2\sqrt{\frac{nx(1-x) + \lambda(x)(2x^2 - 2x + 1) + \lambda^2(x)(2x-1)^2}{(n + 2\lambda(x) + 1)(n + 2\lambda(x))}} \\
&\quad \times \omega\left(f', \sqrt{\frac{nx(1-x) + \lambda(x)(2x^2 - 2x + 1) + \lambda^2(x)(2x-1)^2}{(n + 2\lambda(x) + 1)(n + 2\lambda(x))}}\right). \qquad (1.11)
\end{aligned}
$$

Proof. The estimates (1.9) and (1.11) directly follow from a general result of Shisha and Mond [24] (see also [6, Theorem 5.1.2, p. 268]) on account of (1.5) and (1.6).

The uniform estimate (1.10) is a consequence of (1.9) since

$$\sup_{0 \leqslant x \leqslant 1} \mathscr{B}_{n,\lambda}(\psi_x^2)(x) \leqslant \frac{1}{n} \qquad \text{for} \quad n \geqslant 3,$$

as can be easily seen by using (1.6). ∎

A further uniform estimate is indicated in the next theorem.

THEOREM 1.3. *For every $f \in C([0, 1])$ and $n \in \mathbb{N}$ we get*

$$\|\mathscr{B}_{n,\lambda}(f) - f\| \leqslant K\left(\omega_\varphi^2\left(f, \frac{1}{\sqrt{n}}\right)_\infty + \omega\left(f, \frac{1}{n}\right)\right), \qquad (1.12)$$

where the constant $K > 0$ is independent of f and n and $\omega_\varphi^2(f, \cdot)_\infty$ is the modulus of continuity defined in (1.8).

Proof. Let us choose $f \in C([0, 1])$ and consider the Bernstein polynomial $g := B_n(f)$; for a fixed $x \in [0, 1]$ we have

$$g(t) - g(x) = g'(x)(t - x) + \int_x^t g''(s)(t - s)\, ds, \qquad t \in [0, 1],$$

and therefore, on account of (1.5) and (1.6), we easily get

$$
\begin{aligned}
\|\mathscr{B}_{n,\lambda}(g) - g\| &\leqslant C\left(\frac{1}{n}\|g'\| + \frac{1}{n}\|\varphi^2 g''\| + \frac{1}{n^2}\|g''\|\right) \\
&\leqslant C\left(\frac{1}{n}\|B_n'(f)\| + \frac{1}{n}\|\varphi^2 B_n''(f)\|\right), \qquad (1)
\end{aligned}
$$

$C>0$ being an absolute constant. On the other hand

$$\frac{1}{n}\,\|B_n'(f)\| \leqslant C\,\omega\left(f, \frac{1}{n}\right), \qquad \frac{1}{n}\,\|\varphi^2 B_n''(f)\| \leqslant C\,\omega_\varphi^2\left(f, \frac{1}{\sqrt{n}}\right)_\infty,$$

which, together with (1), yields

$$\|\mathscr{B}_{n,\lambda}(g)-g\| \leqslant C\left(\omega_\varphi^2\left(f, \frac{1}{\sqrt{n}}\right)_\infty + \omega\left(f, \frac{1}{n}\right)\right).$$

Hence for a suitable constant $K>0$

$$\begin{aligned}
\|\mathscr{B}_{n,\lambda}(f)-f\| &\leqslant \|g-f\| + \|\mathscr{B}_{n,\lambda}(g)-g\| + \|\mathscr{B}_{n,\lambda}(f-g)\| \\
&\leqslant K\left(\|B_n(f)-f\| + \omega_\varphi^2\left(f, \frac{1}{\sqrt{n}}\right)_\infty + \omega\left(f, \frac{1}{n}\right)\right) \\
&\leqslant K\left(\omega_\varphi^2\left(f, \frac{1}{\sqrt{n}}\right)_\infty + \omega\left(f, \frac{1}{n}\right)\right),
\end{aligned}$$

as required. ∎

Now we are going to establish a Voronovskaja-type result for the operators $\mathscr{B}_{n,\lambda}$. Henceforth we shall denote by σ the continuous function on $[0, 1]$ defined by

$$\sigma(x) := x(1-x) \tag{1.13}$$

for every $x \in [0, 1]$.

THEOREM 1.4. *If $u \in C^2([0, 1])$, then*

$$\lim_{n\to\infty} n(\mathscr{B}_{n,\lambda}(u)(x)-u(x)) = \frac{\sigma(x)}{2}\,u''(x) + \sigma'(x)\,\lambda(x)\,u'(x) \tag{1.14}$$

uniformly with respect to $x \in [0, 1]$.

Proof. We shall apply a result by Mamedov [20]. First of all note that from (1.5) and (1.6) it is immediate to conclude that

$$\lim_{n\to\infty} n\mathscr{B}_{n,\lambda}(\psi_x)(x) = \sigma'(x)\,\lambda(x), \qquad \lim_{n\to\infty} n\mathscr{B}_{n,\lambda}(\psi_x^2)(x) = \sigma(x)$$

uniformly with respect to $x \in [0, 1]$ and, obviously,

$$\sup_{0\leqslant x\leqslant 1} n\mathscr{B}_{n,\lambda}(\psi_x^2)(x) < +\infty \qquad (n \in \mathbb{N}).$$

Therefore, by virtue of the above quoted result of Mamedov, to accomplish (1.14) it suffices to show that

$$\lim_{n\to\infty} n\mathscr{B}_{n,\lambda}(\psi_x^4)(x)=0 \tag{1}$$

uniformly with respect to $x\in[0, 1]$.

To this purpose, we look for a good estimate of $n\mathscr{B}_{n,\lambda}(\psi_x^4)$ and this is the crucial part of the proof.

Formula (1) of the proof of Theorem 1.1 allows us to get an explicit expression of $\mathscr{B}_{n,\lambda}(\psi_x^4)(x)$ for an arbitrary $n\in\mathbb{N}$ and $x\in[0, 1]$. Hence, since $0<\lambda(x)\leqslant 1$ for any $x\in[0, 1]$, a direct computation shows that

$$n\mathscr{B}_{n,\lambda}(\psi_x^4)(x)\leqslant f_n(x), \tag{2}$$

where

$$f_n(x):=\frac{p(n, \lambda_0)\, x^4+q(n, \lambda_0)\, x^3+r(n, \lambda_0)\, x^2+s(n, \lambda_0)\, x+24}{n^3},$$

with λ_0, $p(n, \lambda_0)$, $q(n, \lambda_0)$, $r(n, \lambda_0)$, and $s(n, \lambda_0)$ defined as

$$\begin{aligned}
\lambda_0 &:= \min_{0\leqslant x\leqslant 1} \lambda(x),\\
p(n, \lambda_0) &:= 3n^2-18n-44n\lambda_0-24n\lambda_0^2+120,\\
q(n, \lambda_0) &:= -6n^2+172n-24\lambda_0-88\lambda_0^2-96\lambda_0^3-32\lambda_0^4,\\
r(n, \lambda_0) &:= 3n^2-24n-58n\lambda_0-30n\lambda_0^2+240,\\
s(n, \lambda_0) &:= 26n-24\lambda_0-52\lambda_0^2-36\lambda_0^3-8\lambda_0^4.
\end{aligned}$$

We readily get

$$\lim_{n\to\infty} f_n(x)=0 \tag{6}$$

uniformly with respect to $x\in[0, 1]$ which implies (1) by virtue of (2) and the proof is complete. ∎

Remark 1.5. In the particular case $\lambda=e_0$, Theorem 1.4 states a Voronovskaja-type result for the classical Beta operators $\mathscr{B}_n$ which has already been proved in [17, Satz 2.24].

2. THE OPERATORS $M_{n,\lambda}$

In this section we introduce and study a sequence of positive linear operators on $C([0,1])$ which are obtained by a suitable composition of the operators $\mathscr{B}_{n,\lambda}$ defined in (1.1) together with the so-called Lototsky–Schnabl operators (see (2.2)).

Our starting point is a continuous function $\lambda\colon [0,1]\to[0,1]$ such that $1/2\leqslant\lambda(x)\leqslant 1$ for every $x\in[0,1]$. Moreover let us set

$$\gamma(x) := 2\lambda(x)-1 \tag{2.1}$$

for every $x\in[0,1]$. Then γ is a continuous function on $[0,1]$ satisfying $0\leqslant\gamma\leqslant e_0$, and therefore for each $n\in\mathbb{N}$ we may consider the nth Lototsky–Schnabl operator on $C([0,1])$ associated with γ, i.e., the positive linear operator $L_{n,\gamma}\colon C([0,1])\to C([0,1])$ defined by

$$L_{n,\gamma}(f)(x) := \sum_{k=0}^{n}\sum_{h=0}^{k}\binom{n}{k}\binom{k}{h}\gamma(x)^k\,(1-\gamma(x))^{n-k}\,x^h(1-x)^{k-h}$$
$$\times f\left(\frac{h}{n}+\left(1-\frac{k}{n}\right)x\right) \tag{2.2}$$

for all $f\in C([0,1])$ and $x\in[0,1]$.

Such operators have been introduced and studied by Altomare [4, 5]; in addition, a rather detailed analysis of their main properties may be found, for instance, in [6, Chap. 6] together with some results concerning the possibility of representing explicitly the solutions of particular Cauchy problems in terms of powers of these and other operators.

Now, for the reader's convenience, we summarize those properties of the operators $L_{n,\gamma}$ which shall be used in the sequel, referring to [6, Chap. 6] for further details.

For each $n\in\mathbb{N}$ one has

$$L_{n,\gamma}(e_0)=e_0, \tag{2.3}$$

and consequently

$$\|L_{n,\gamma}\| = \|L_{n,\gamma}(e_0)\| = 1. \tag{2.4}$$

Moreover for every $f\in C([0,1])$

$$\lim_{n\to\infty} L_{n,\gamma}(f)=f \qquad \text{uniformly on } [0,1], \tag{2.5}$$

i.e., the sequence $(L_{n,\gamma})_{n\in\mathbb{N}}$ is a positive approximation process on $C([0,1])$.

In addition, the following uniform estimate in terms of the first modulus of continuity is also available

$$\|L_{n,\gamma}(f)-f\| \leqslant 2\omega\left(f, \frac{1}{\sqrt{n}}\right) \tag{2.6}$$

for every $n \in \mathbb{N}$ and $f \in C([0,1])$. At last we remind a Voronovskaja-type result

$$\lim_{n\to\infty} n(L_{n,\gamma}(u)(x)-u(x)) = \frac{\sigma(x)}{2}\,\gamma(x)\,u''(x), \tag{2.7}$$

which holds uniformly with respect to $x \in [0,1]$ for every $u \in C^2([0,1])$.

After these preliminaries we can proceed further and for each $n \in \mathbb{N}$ we consider the linear operator $M_{n,\lambda}\colon C([0,1]) \to C([0,1])$ defined by

$$M_{n,\lambda} := L_{n,\gamma} \circ \mathscr{B}_{n,\lambda}, \tag{2.8}$$

where γ satisfies (2.1) and $L_{n,\gamma}$ and $\mathscr{B}_{n,\lambda}$ are defined in (2.2) and (1.1), respectively. Note that, explicitly, for every $n \in \mathbb{N}$ we have

$$\begin{aligned} M_{n,\lambda}(f)(x) &= \sum_{k=0}^{n}\sum_{h=0}^{k}\binom{n}{k}\binom{k}{h}\gamma(x)^k\,(1-\gamma(x))^{n-k}\,x^h(1-x)^{k-h} \\ &\quad\times \frac{\int_0^1 t^{\alpha_{\lambda,n}^{k,h}(x)-1}(1-t)^{\beta_{\lambda,n}^{k,h}(x)-1}\,f(t)\,dt}{B(\alpha_{\lambda,n}^{k,h}(x), \beta_{\lambda,n}^{k,h}(x))} \end{aligned} \tag{2.9}$$

for every $f \in C([0,1])$ and $x \in [0,1]$, where $\alpha_{\lambda,n}^{k,h}(x) := h + x(n-k) + \lambda(h/n + (1-k/n)\,x)$ and $\beta_{\lambda,n}^{k,h}(x) := n-h-x(n-k)+\lambda(h/n+(1-k/n)\,x)$ $(k, h \in \mathbb{N},\ 0 \leqslant h \leqslant k \leqslant n,\ x \in [0,1])$.

Clearly every $M_{n,\lambda}$ is positive and $\|M_{n,\lambda}\| = \|M_{n,\lambda}(e_0)\| = 1$. The next theorem indicates some approximation properties of the sequence $(M_{n,\lambda})_{n\in\mathbb{N}}$.

THEOREM 2.1. *For every* $f \in C([0,1])$

$$\lim_{n\to\infty} M_{n,\lambda}(f) = f \qquad \textit{uniformly on } [0,1], \tag{2.10}$$

i.e., the sequence $(M_{n,\lambda})_{n\in\mathbb{N}}$ *is a positive approximation process on* $C([0,1])$. *Moreover*

$$\|M_{n,\lambda}(f)-f\| \leqslant 4\omega\left(f, \frac{1}{\sqrt{n}}\right) \qquad \textit{for} \quad n \geqslant 3. \tag{2.11}$$

Proof. Indeed, for an arbitrary $f \in C([0, 1])$ and $n \in \mathbb{N}$ we have

$$\begin{aligned}\|M_{n,\lambda}(f)-f\| &\leqslant \|L_{n,\gamma}(\mathscr{B}_{n,\lambda}(f))-L_{n,\gamma}(f)\| + \|L_{n,\gamma}(f)-f\| \\ &\leqslant \|\mathscr{B}_{n,\lambda}(f)-f\| + \|L_{n,\gamma}(f)-f\|\end{aligned}$$

due to (2.4). The assertion immediately follows by virtue of (1.3), (1.10), (2.5), and (2.6). ∎

Remark 2.2. It would be desiderable to go deeper into the approximation properties of the operators $M_{n,\lambda}$ and to achieve a further uniform estimate similar to that one stated in Theorem 1.3 for the operators $\mathscr{B}_{n,\lambda}$. However, up to the present, this seems hard enough to get and remains an open (and, perhaps, interesting) problem.

The operators $M_{n,\lambda}$ enjoy a Voronovskaja-type property, as shown in the next theorem.

THEOREM 2.3. *Let σ be defined as in* (1.13). *Then for every $u \in C^2([0, 1])$ we get*

$$\lim_{n\to\infty} n(M_{n,\lambda}(u)(x)-u(x)) = \lambda(x)(\sigma(x)\,u'(x))' \tag{2.12}$$

uniformly with respect to $x \in [0, 1]$.

Proof. Let us consider the operators $A_1\colon C^2([0, 1] \to C([0, 1])$ and $A_2\colon C^2([0, 1]) \to C([0, 1])$ defined by

$$A_1(u)(x) := \frac{\sigma(x)}{2}\,\gamma(x)\,u''(x),$$

$$A_2(u)(x) := \frac{\sigma(x)}{2}\,u''(x) + \sigma'(x)\,\lambda(x)\,u'(x)$$

for every $u \in C^2([0, 1])$ and $x \in [0, 1]$. Taking (2.1) into account, a simple computation shows that, in order to accomplish (2.12), we have to prove that

$$\lim_{n\to\infty} \|n(M_{n,\lambda}(u)-u)-(A_1+A_2)(u)\| = 0 \tag{1}$$

for every $u \in C^2([0, 1])$. Indeed, for an arbitrary $n \in \mathbb{N}$ and $u \in C^2([0, 1])$, we easily get

$$\begin{aligned}&\|n(M_{n,\lambda}(u)-u)-(A_1+A_2)(u)\| \\ &\quad\leqslant \|n(L_{n,\gamma}(u)-u)-A_1(u)\| + \|L_{n,\gamma}(A_2(u))-A_2(u)\| \\ &\qquad + \|L_{n,\gamma}\|\;\|n(\mathscr{B}_{n,\lambda}(u)-u)-A_2(u)\|,\end{aligned}$$

where each member on the right hand side tends to zero as $n \to \infty$ by virtue of (1.14), (2.4), (2.5) and (2.7). Therefore (1) holds true and the proof is complete. ∎

Remark 2.4. It is worth while underlining that, in the particular case $\lambda = e_0$, the results of Theorems 2.1 and 2.3 are to be referred to the operators M_{n, e_0}, i.e., to the classical Bernstein–Durrmeyer operators. These same results have already been stated in [11, Théorème II.2 and Théorème II.5], in which a uniform estimate equal to (2.11) up to a constant is established.

We are now in a position to state the main theorem of this section about the existence of a strongly continuous positive contraction semigroup on $C([0, 1])$ which can be expressed in terms of powers of the operators $M_{n, \lambda}$. However, we need first to recall briefly some classical results about the theory of strongly continuous semigroups on a Banach space, as may be found, e.g., in [21, 22].

If $A: D(A) \to E$ is a closed linear operator defined on a dense subspace $D(A)$ of a Banach space $(E, \|\cdot\|)$ over the field $\mathbb{K}$ of real or complex numbers, we say that a subspace D_0 of $D(A)$ is a core for A if it is dense in $D(A)$ for the graph-norm

$$\|u\|_A := \|u\| + \|A(u)\| \qquad (u \in D(A)). \tag{2.13}$$

If $A_1: D(A_1) \to E$ and $A_2: D(A_2) \to E$ are closed operators and $D_0 \subset D(A_1) \cap D(A_2)$ is a core for A_1 and A_2, then $D(A_1) = D(A_2)$ and $A_1 = A_2$ provided $A_1 = A_2$ on D_0.

If A is closed and $zI - A$ is invertible for some $z \in \mathbb{K}$ (I being the identity operator on E), then a subspace D_0 of $D(A)$ is a core for A if and only if $(zI - A)(D_0)$ is dense in E. If $(A, D(A))$ is the generator of a strongly continuous semigroup $(T(t))_{t \geqslant 0}$ on E, then a dense subspace D_0 of $D(A)$ is a core for A if $T(t)(D_0) \subset D_0$ for every $t \geqslant 0$.

As usual, for a given $m \in \mathbb{N}$, the power $M^m_{n,\lambda}$ of order m of $M_{n,\lambda}$ is defined by

$$M^m_{n,\lambda} := \begin{cases} M_{n,\lambda}, & \text{if} \quad m = 1, \\ M^{m-1}_{n,\lambda} \circ M_{n,\lambda}, & \text{if} \quad m \geqslant 2. \end{cases}$$

THEOREM 2.5. *Let λ be a continuous function on $[0, 1]$ such that $1/2 \leqslant \lambda(x) \leqslant 1$ for every $x \in [0, 1]$ and let σ be defined as in* (1.13). *There exists a strongly continuous positive contraction semigroup $(T_\lambda(t))_{t \geqslant 0}$ on $C([0, 1])$ whose generator is the operator $A_\lambda: D(A_\lambda) \to C([0, 1])$ defined by*

$$A_\lambda(u)(x) := \begin{cases} \lambda(0)\, u'(0), & \text{if} \quad x=0, \\ \lambda(x)(\sigma(x)\, u'(x))', & \text{if} \quad 0<x<1, \\ -\lambda(1)\, u'(1), & \text{if} \quad x=1, \end{cases} \tag{2.14}$$

for every u in

$$D(A_\lambda) := \{v \in C^1([0,1]) \cap C^2(]0,1[) \mid \lim_{x\to 0^+} \lambda(x)\, \sigma(x)\, v''(x) = \lim_{x\to 1^-} \lambda(x)\, \sigma(x)\, v''(x) = 0\}$$

such that for every $t \geqslant 0$ and for every sequence $(k(n))_{n\in\mathbb{N}}$ of positive integers satisfying $\lim_{n\to\infty} k(n)/n = t$, one gets

$$T_\lambda(t) = \lim_{n\to\infty} M_{n,\lambda}^{k(n)} \qquad \textit{strongly on } C([0,1]). \tag{2.15}$$

In particular, for every $t \geqslant 0$

$$T_\lambda(t) = \lim_{n\to\infty} M_{n,\lambda}^{[nt]} \qquad \textit{strongly on } C([0,1]), \tag{2.16}$$

$[nt]$ being the integer part of nt.

Proof. Let us consider the linear operator $A\colon D(A) \to C([0,1])$ defined by

$$A(u)(x) := \begin{cases} u'(0), & \text{if} \quad x=0, \\ (\sigma(x)\, u'(x))', & \text{if} \quad 0<x<1, \\ -u'(1), & \text{if} \quad x=1, \end{cases} \tag{1}$$

for every u in

$$D(A) := \{v \in C^1([0,1]) \cap C^2(]0,1[) \mid \lim_{x\to 0^+} \sigma(x)\, v''(x) = \lim_{x\to 1^-} \sigma(x)\, v''(x) = 0\}.$$

From [10, Theorem 2.3, Proposition 3.8 and Theorem 5.5] it follows that $(A, D(A))$ is the generator of a strongly continuous semigroup on $C([0,1])$ and that $C^2([0,1])$ is a core for A. Since clearly

$$A_\lambda = \lambda A \qquad \text{and} \qquad D(A_\lambda) = D(A), \tag{2}$$

$(A_\lambda, D(A_\lambda))$ is a generator, too, due to a result by Dorroh (see, e.g., [21, B-II, Theorem 1.20, p. 131]).

Moreover, for every $u \in D(A_\lambda)$ we have

$$\|u\|_{A_\lambda} = \|u\| + \|A_\lambda(u)\| = \|u\| + \|\lambda A(u)\| \leqslant \|u\|_A, \tag{3}$$

and, therefore, $C^2([0,1])$, which is dense in $D(A)$ for the graph-norm $\|\cdot\|_A$, is also dense in $D(A_\lambda)$ for the graph-norm $\|\cdot\|_{A_\lambda}$, i.e., $C^2([0,1])$ is a core for A_λ, too. Accordingly

$$(I - A_\lambda)(C^2([0,1])) \text{ is dense in } C([0,1]), \tag{4}$$

I being the identity operator on $C([0,1])$.

Now let us consider the linear operator $Z_\lambda \colon D(Z_\lambda) \to C([0,1])$ defined by

$$Z_\lambda(f) := \lim_{n\to\infty} n(M_{n,\lambda}(f) - f) \tag{5}$$

for every $f \in D(Z_\lambda) := \{g \in C([0,1]) \mid \lim_{n\to\infty} n(M_{n,\lambda}(g) - g)$ exists in $C([0,1])\}$

Applying Theorem 2.3 yields

$$C^2([0,1]) \subset D(A_\lambda) \cap D(Z_\lambda) \qquad \text{and} \qquad Z_\lambda = A_\lambda \text{ on } C^2([0,1]). \tag{6}$$

In particular $D(Z_\lambda)$ is dense in $C([0,1])$.

Moreover the range $R(I - Z_\lambda) := (I - Z_\lambda)(D(Z_\lambda))$ of $I - Z_\lambda$ is dense in $C([0,1])$ due to (4) and (6). Since $\|M_{n,\lambda}\| = 1$ for every $n \in \mathbb{N}$, we may apply a result by Trotter [28] (see, also, [22, Chap. 3, Theorem 6.7, p. 96]) and conclude that the operator $(Z_\lambda, D(Z_\lambda))$ defined in (5) is closable and that its closure $(\bar{Z}_\lambda, D(\bar{Z}_\lambda))$ is the generator of a strongly continuous contraction semigroup $(T_\lambda(t))_{t\geqslant 0}$ on $C([0,1])$ satisfying (2.15).

Obviously every $T_\lambda(t)$ is positive. Now, to accomplish the proof, we have only to show that $(\bar{Z}_\lambda, D(\bar{Z}_\lambda)) = (A_\lambda, D(A_\lambda))$. Indeed, since $(\bar{Z}_\lambda, D(\bar{Z}_\lambda))$ is an extension of $(Z_\lambda, D(Z_\lambda))$, we already know that $D(Z_\lambda) \subset D(\bar{Z}_\lambda)$ and that $\bar{Z}_\lambda = Z_\lambda$ on $D(Z_\lambda)$. This, together with (6), implies that

$$C^2([0,1]) \subset D(\bar{Z}_\lambda) \qquad \text{and} \qquad \bar{Z}_\lambda = A_\lambda \text{ on } C^2([0,1]).$$

Consequently, by virtue of (4),

$$(I - \bar{Z}_\lambda)(C^2([0,1])) \text{ is dense in } C([0,1]),$$

i.e., $C^2([0,1])$ is a core for $\bar{Z}_\lambda$. Therefore $D(\bar{Z}_\lambda) = D(A_\lambda)$ and $\bar{Z}_\lambda = A_\lambda$ and this completes the proof. ∎

Remark 2.6. Obviously, all the results stated in the previous theorem still hold true when $\lambda = e_0$. In this case one has only to replace the operators $M_{n,\lambda}$ by the classical Bernstein–Durrmeyer operators M_n.

Now we establish the next result.

THEOREM 2.7. *For a given $f \in C([0,1])$ the following statements are equivalent*:

(i) $\|M_{n,\lambda}(f) - f\| = o(1/n)$, $n \to \infty$.

(ii) $\|T_\lambda(t)\, f - f\| = o(t)$, $t \to 0^+$.

(iii) *f is constant.*

Proof. (i) $\Rightarrow$ (ii). Fix $t > 0$ and consider a sequence $(k(n))_{n \in \mathbb{N}}$ of positive integers such that $\lim_{n \to \infty} k(n)/n = t$. After setting $\alpha_n := n\, \|M_{n,\lambda}(f) - f\|$ $(n \in \mathbb{N})$, it is easily seen that

$$\|M_{n,\lambda}^{k(n)}(f) - f\| \leqslant \frac{k(n)}{n}\, \alpha_n \qquad \text{for every } n \in \mathbb{N}$$

because $\|M_{n,\lambda}\| = 1$.

But $\lim_{n \to \infty} \alpha_n = 0$ by assumption and therefore, passing to the limit as $n \to \infty$ in both members and taking the representation formula (2.15) into account, we obtain $T_\lambda(t)\, f = f$ which obviously implies (ii).

(ii) $\Rightarrow$ (iii). Statement (ii) equals to $f \in D(A_\lambda)$ and $A_\lambda(f) = 0$ (let us recall that the operator A_λ, defined in (2.14), is the generator of the semigroup $(T_\lambda(t))_{t \geqslant 0}$). Consequently

$$f'(0) = f'(1) = 0 \tag{1}$$

and there exists a constant K such that

$$f'(x) = \frac{K}{\sigma(x)} \qquad (0 < x < 1). \tag{2}$$

Combining (1) and (2) easily gives $K = 0$ because $f \in C^1([0,1])$ and $\sigma(0) = \sigma(1) = 0$ (see (1.13)). But then $f' \equiv 0$ in $[0,1]$, i.e., f is constant, as required.

The implication (iii) $\Rightarrow$ (i) is straightforward, because $M_{n,\lambda}(e_0) = e_0$ for every $n \in \mathbb{N}$. ∎

Remark 2.8. Consider the linear operator B: $D(B) \to C([0,1])$ defined as

$$B(u)(x) := \begin{cases} \beta(0)\, u'(0), & \text{if} \quad x = 0, \\ \alpha(x)\, u''(x) + \beta(x)\, u'(x), & \text{if} \quad 0 < x < 1, \\ \beta(1)\, u'(1), & \text{if} \quad x = 1, \end{cases} \tag{2.17}$$

for every u in

$$D(B) := \{v \in C^1([0,1]) \cap C^2(]0,1[) \mid \lim_{x \to 0^+} \alpha(x)\, v''(x) = \lim_{x \to 1^-} \alpha(x)\, v''(x) = 0\},$$

where α and β are continuous functions on $[0,1]$ enjoying the following properties:

(i) α vanishes only at the endpoints 0 and 1 and is here differentiable with $\alpha'(0) \neq 0 \neq \alpha'(1)$.

(ii) β vanishes only at the midpoint 1/2 and is here differentiable with $\beta'(1/2) \neq 0$.

(iii) $\alpha/\sigma = \beta/\sigma'$, where σ is defined in (1.13) and α/σ and β/σ' still denote the continuous extensions to the whole interval $[0,1]$ of the functions α/σ and β/σ' defined on $]0,1[$ and $[0,1] - \{1/2\}$, respectively.

(iv) There exists a positive constant c such that

$$\frac{c}{2} \leqslant h(x) \leqslant c \qquad (0 \leqslant x \leqslant 1),$$

where, by definition, $h := \alpha/\sigma = \beta/\sigma'$.

If we define

$$\lambda := \frac{h}{c}, \tag{2.18}$$

we easily get $1/2 \leqslant \lambda(x) \leqslant 1$ for every $x \in [0,1]$. In addition

$$\alpha(x) = c\lambda(x)\, \sigma(x), \qquad \beta(x) = c\lambda(x)\, \sigma'(x) \text{ for every } x \in [0,1], \tag{2.19}$$

and therefore the operator B defined in (2.17) and its domain $D(B)$ may be rewritten in an apparently different way as

$$B(u)(x) := \begin{cases} c\lambda(0)\, u'(0), & \text{if } \ x = 0, \\ c\lambda(x)(\sigma(x)\, u'(x))' & \text{if } \ 0 < x < 1, \\ -c\lambda(1)\, u'(1), & \text{if } \ x = 1, \end{cases} \tag{2.20}$$

for every u in

$$D(B) := \{v \in C^1([0,1]) \cap C^2(]0,1[) \mid \lim_{x \to 0^+} c\lambda(x)\, \sigma(x)\, v''(x) = \lim_{x \to 1^-} c\lambda(x)\, \sigma(x)\, v''(x) = 0\}.$$

Consequently, from Theorem 2.5 and the general theory of strongly continuous semigroups (see, e.g., [21, 22]), it follows that the problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \alpha(x)\dfrac{\partial^2 u}{\partial x^2}(x,t) + \beta(x)\dfrac{\partial u}{\partial x}(x,t), & 0 < x < 1, \quad t \geqslant 0, \\ \lim\limits_{x\to 0^+} \alpha(x)\dfrac{\partial^2 u}{\partial x^2}(x,t) = \lim\limits_{x\to 1^-} \alpha(x)\dfrac{\partial^2 u}{\partial x^2}(x,t) = 0, & t \geqslant 0, \\ u(x,0) = u_0(x), \qquad 0 \leqslant x \leqslant 1, & \\ u_0 \in C^1([0,1]) \cap C^2(]0,1[), & \lim\limits_{x\to 0^+} \alpha(x)\, u_0''(x) = \lim\limits_{x\to 1^-} \alpha(x)\, u_0''(x) = 0, \end{cases} \tag{2.21}$$

has a unique (classical) solution given by

$$u(x,t) = T_\lambda(ct)(u_0)(x) = \lim_{n\to\infty} M_{n,\lambda}^{[cnt]}(u_0)(x) \tag{2.22}$$

for every $x \in [0,1]$ and $t \geqslant 0$. Moreover (2.22) holds uniformly with respect to $x \in [0,1]$.

In particular, if $\alpha = \sigma$ and $\beta = \sigma'$, then properties (i)–(iv) are obviously fulfilled with $c = 1$ and $\lambda = e_0$ and, on account of Remark 2.6, (2.22) reads like

$$u(x,t) = T_{e_0}(t)(u_0)(x) = \lim_{n\to\infty} M_n^{[nt]}(u_0)(x) \tag{2.23}$$

for every $x \in [0,1]$ and $t \geqslant 0$, i.e., in this case the solution of problem (2.21) is expressed in terms of powers of the Bernstein–Durrmeyer operators, as already announced in [10], after Proposition 5.9.

Remark 2.9. If $\alpha = \sigma/2$ and $\beta = \sigma'$, the result stated in Remark 2.8 may not be applied since property (iii) is not fulfilled. Nevertheless we are able to establish the following theorem.

THEOREM 2.10. *Let σ be defined as in* (1.13). *There exists a strongly continuous positive contraction semigroup $(S(t))_{t\geqslant 0}$ on $C([0,1])$ whose generator is the operator $B_1\colon D(B_1) \to C([0,1])$ defined by*

$$B_1(u)(x) := \frac{\sigma(x)}{2}\, u''(x) + \sigma'(x)\, u'(x) \qquad (0 \leqslant x \leqslant 1) \tag{2.24}$$

for every $u \in D(B_1) := \{v \in C([0,1]) \cap C^2(]0,1[) \mid B_1(v) \in C([0,1])\}$ such that for every $t \geqslant 0$ and for every sequence $(k(n))_{n\in\mathbb{N}}$ of positive integers satisfying $\lim_{n\to\infty} k(n)/n = t$, one gets

$$S(t) = \lim_{n\to\infty} \mathscr{B}_n^{k(n)} \qquad \textit{strongly on } C([0,1]), \tag{2.25}$$

where, for every $n \in \mathbb{N}$, $\mathscr{B}_n^{k(n)}$ *denotes the power of order* $k(n)$ *of the* n*th Beta operator* $\mathscr{B}_n$ *(see* (1.2)).

In particular, for every $t \geqslant 0$

$$S(t) = \lim_{n \to \infty} \mathscr{B}_n^{[nt]} \qquad \textit{strongly on } C([0,1]), \tag{2.26}$$

$[nt]$ *being the integer part of* nt.

Proof. Let us first prove that $(B_1, D(B_1))$ is a generator and, at this purpose, we shall apply a theorem by Timmermans [25]. For the reader's convenience we shall use the same notations of that paper.

Let us choose $x_0 = 1/2$ and compute

$$W(x) := \exp\left(-\int_{1/2}^{x} \frac{2\sigma'(t)}{\sigma(t)}\, dt\right) = \frac{1}{16x^2(1-x)^2} \qquad (0 < x < 1). \tag{1}$$

Consequently

$$\begin{aligned} R(x) &:= W(x) \int_{1/2}^{x} \left(\frac{\sigma(t)}{2}\, W(t)\right)^{-1} dt \\ &= \frac{1}{x^2(1-x)^2}\left(-\frac{2}{3}\, x^3 + x^2 - \frac{1}{6}\right) \qquad (0 < x < 1), \end{aligned} \tag{2}$$

and therefore

$$-\int_{0}^{1/2} R(x)\, dx = \int_{1/2}^{1} R(x)\, dx = +\infty. \tag{3}$$

It follows that $(B_1, D(B_1))$ is the generator of a strongly continuous semigroup on $C([0,1])$ by virtue of [25, Theorem 3]. Now let us consider the linear operator $Q: D(Q) \to C([0,1])$ defined by

$$Q(f) := \lim_{n \to \infty} n(\mathscr{B}_n(f) - f) \tag{4}$$

for every $f \in D(Q) := \{g \in C([0,1]) \mid \lim_{n \to \infty} n(\mathscr{B}_n(g) - g)$ exists in $C([0,1])\}$.

Applying Theorem 1.4 and the subsequent remark yields

$$C^2([0,1]) \subset D(B_1) \cap D(Q) \qquad \text{and} \qquad Q = B_1 \text{ on } C^2([0,1]). \tag{5}$$

In particular $D(Q)$ is dense in $C([0,1])$.

On the other hand $\|\mathscr{B}_n\| = 1$ for every $n \in \mathbb{N}$ and

$$\mathscr{B}_n(P_m([0,1])) \subset P_m([0,1]) \tag{6}$$

for every $n, m \in \mathbb{N}$, where $P_m([0,1])$ denotes the space of all polynomials on $[0,1]$ of degree $\leqslant m$ (see formula (1) of the proof of Theorem 1.1).

Since $\bigcup_{m=1}^{+\infty} P_m([0,1]) = P([0,1])$ is dense in $C([0,1])$ ($P([0,1])$ being the subalgebra of all polynomials on $[0,1]$), we may apply a result by Schnabl (see, e.g., [6, Theorem 1.6.8, p. 68]) and conclude that the operator $(Q, D(Q))$ defined in (4) is closable and that its closure $(\bar{Q}, D(\bar{Q}))$ is the generator of a strongly continuous contraction semigroup $(S(t))_{t \geqslant 0}$ on $C([0,1])$ satisfying (2.25).

Clearly every $S(t)$ is positive. Now the proof will be complete if we show that $(\bar{Q}, D(\bar{Q})) = (B_1, D(B_1))$. Indeed, since $(\bar{Q}, D(\bar{Q}))$ is an extension of $(Q, D(Q))$, we already know that $D(Q) \subset D(\bar{Q})$ and that $\bar{Q} = Q$ on $D(Q)$. This, together with (5), implies that

$$\bar{Q} = B_1 \qquad \text{on } P([0,1]). \tag{7}$$

Furthermore, from (6) and (2.25) it follows that

$$S(t)(P([0,1])) \subset P([0,1]) \qquad \text{for every} \quad t \geqslant 0,$$

i.e., $P([0,1])$ is a core for $\bar{Q}$.

Therefore,

$$(I - \bar{Q})(P([0,1])) \text{ is dense in } C([0,1])$$

and, consequently,

$$(I - B_1)(P([0,1])) \text{ is dense in } C([0,1])$$

on account of (7). But then $P([0,1])$ is a core for B_1, too, and this accomplishes the proof. ∎

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