



Weak approximation of G -expectations

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Abstract

We introduce a notion of volatility uncertainty in discrete time and define the corresponding analogue of Peng's G -expectation. In the continuous-time limit, the resulting sublinear expectation converges weakly to the G -expectation. This can be seen as a Donsker-type result for the G -Brownian motion.

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1. Introduction

The so-called G -expectation [12–14] is a nonlinear expectation advancing the notions of backward stochastic differential equations (BSDEs) [10] and g -expectations [11]; see also [2,16] for a related theory of second-order BSDEs. A G -expectation $\xi \mapsto \mathcal{E}^G(\xi)$ is a sublinear function which maps random variables ξ on the canonical space $\Omega = C([0, T]; \mathbb{R})$ to the real numbers. The symbol G refers to a given function $G : \mathbb{R} \rightarrow \mathbb{R}$ of the form

$$G(\gamma) = \frac{1}{2}(R\gamma^+ - r\gamma^-) = \frac{1}{2} \sup_{a \in [r, R]} a\gamma,$$

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where $0 \leq r \leq R < \infty$ are fixed numbers. More generally, the interval $[r, R]$ is replaced by a set \mathbf{D} of nonnegative matrices in the multivariate case. The extension to a random set \mathbf{D} is studied in [9].

The construction of $\mathcal{E}^G(\xi)$ runs as follows. When $\xi = f(B_T)$, where B_T is the canonical process at time T and f is a sufficiently regular function, then $\mathcal{E}^G(\xi)$ is defined to be the initial value $u(0, 0)$ of the solution of the nonlinear backward heat equation $-\partial_t u - G(u_{xx}) = 0$ with terminal condition $u(\cdot, T) = f$. The mapping \mathcal{E}^G can be extended to random variables of the form $\xi = f(B_{t_1}, \dots, B_{t_n})$ by a stepwise evaluation of the PDE and then to the completion \mathbb{L}_G^1 of the space of all such random variables. The space \mathbb{L}_G^1 consists of so-called quasi-continuous functions and contains in particular all bounded continuous functions on Ω ; however, not all bounded measurable functions are included (cf. [3]). While this setting is not based on a single probability measure, the so-called G -Brownian motion is given by the canonical process B “seen” under \mathcal{E}^G (cf. [14]). It reduces to the standard Brownian motion if $r = R = 1$ since \mathcal{E}^G is then the (linear) expectation under the Wiener measure.

In this note we introduce a discrete-time analogue of the G -expectation and we prove a convergence result which resembles Donsker’s theorem for the standard Brownian motion; the main purpose is to provide additional intuition for G -Brownian motion and volatility uncertainty. Our starting point is the dual view on G -expectation via volatility uncertainty [3,4]: We consider the representation

$$\mathcal{E}^G(\xi) = \sup_{P \in \mathcal{P}} E^P[\xi], \tag{1.1}$$

where \mathcal{P} is a set of probabilities on Ω such that under any $P \in \mathcal{P}$, the canonical process B is a martingale with volatility $d\langle B \rangle/dt$ taking values in $\mathbf{D} = [r, R]$, $P \times dt$ -a.e. Therefore, \mathbf{D} can be understood as the domain of (Knightian) volatility uncertainty and \mathcal{E}^G as the corresponding worst-case expectation. In the discrete-time case, we translate this to uncertainty about the conditional variance of the increments. Thus we define a sublinear expectation \mathcal{E}^n on the n -step canonical space in the spirit of (1.1), replacing \mathcal{P} by a suitable set of martingale laws. A natural push-forward then yields a sublinear expectation on Ω , which we show to converge weakly to \mathcal{E}^G as $n \rightarrow \infty$, if the domain \mathbf{D} of uncertainty is scaled by $1/n$ (cf. Theorem 2.2). The proof relies on (linear) probability theory; in particular, it does not use the central limit theorem for sublinear expectations [14,15]. The relation to the latter is nontrivial since our discrete-time models do not have independent increments. We remark that quite different approximations of the G -expectation (for the scalar case) can be found in discrete models for financial markets with transaction costs [8] or illiquidity [5].

The detailed setup and the main result are stated in Section 2, whereas the proofs and some ramifications are given in Section 3.

2. The main result

We fix the dimension $d \in \mathbb{N}$ and denote by $|\cdot|$ the Euclidean norm on \mathbb{R}^d . Moreover, we denote by \mathbb{S}^d the space of $d \times d$ symmetric matrices and by \mathbb{S}_+^d its subset of nonnegative definite matrices. We fix a nonempty, convex and compact set $\mathbf{D} \subseteq \mathbb{S}_+^d$; the elements of \mathbf{D} will be the possible values of our volatility processes.

The continuous-time formulation. Let $\Omega = C([0, T]; \mathbb{R}^d)$ be the space of d -dimensional continuous paths $\omega = (\omega_t)_{0 \leq t \leq T}$ with time horizon $T \in (0, \infty)$, endowed with the uniform norm $\|\omega\|_\infty = \sup_{0 \leq t \leq T} |\omega_t|$. We denote by $B = (B_t)_{0 \leq t \leq T}$ the canonical process $B_t(\omega) = \omega_t$

and by $\mathcal{F}_t := \sigma(B_s, 0 \leq s \leq t)$ the canonical filtration. A probability measure P on Ω is called a *martingale law* if B is a P -martingale and $B_0 = 0$ P -a.s. (All our martingales will start at the origin.) We set

$$\mathcal{P}_{\mathbf{D}} = \{P \text{ martingale law on } \Omega : d\langle B \rangle_t/dt \in \mathbf{D}, P \times dt\text{-a.e.}\},$$

where $\langle B \rangle$ denotes the matrix-valued process of quadratic covariations. We can then define the sublinear expectation

$$\mathcal{E}_{\mathbf{D}}(\xi) := \sup_{P \in \mathcal{P}_{\mathbf{D}}} E^P[\xi] \quad \text{for any random variable } \xi : \Omega \rightarrow \mathbb{R}$$

such that ξ is \mathcal{F}_T -measurable and $E^P|\xi| < \infty$ for all $P \in \mathcal{P}_{\mathbf{D}}$. The mapping $\mathcal{E}_{\mathbf{D}}$ coincides with the G -expectation (on its domain \mathbb{L}_G^1) if $G : \mathbb{S}^d \rightarrow \mathbb{R}$ is (half) the support function of \mathbf{D} ; i.e., $G(\Gamma) = \sup_{A \in \mathbf{D}} \text{trace}(\Gamma A)/2$. Indeed, this follows from [3] with an additional density argument as detailed in Remark 3.6 below.

The discrete-time formulation. Given $n \in \mathbb{N}$, we consider $(\mathbb{R}^d)^{n+1}$ as the canonical space of d -dimensional paths in discrete time $k = 0, 1, \dots, n$. We denote by $X^n = (X_k^n)_{k=0}^n$ the canonical process defined by $X_k^n(x) = x_k$ for $x = (x_0, \dots, x_n) \in (\mathbb{R}^d)^{n+1}$. Moreover, $\mathcal{F}_k^n = \sigma(X_i^n, i = 0, \dots, k)$ defines the canonical filtration $(\mathcal{F}_k^n)_{k=0}^n$. We also introduce $0 \leq r_{\mathbf{D}} \leq R_{\mathbf{D}} < \infty$ such that $[r_{\mathbf{D}}, R_{\mathbf{D}}]$ is the spectrum of \mathbf{D} ; i.e.,

$$r_{\mathbf{D}} = \inf_{\Gamma \in \mathbf{D}} \|\Gamma^{-1}\|^{-1} \quad \text{and} \quad R_{\mathbf{D}} = \sup_{\Gamma \in \mathbf{D}} \|\Gamma\|,$$

where $\|\cdot\|$ denotes the operator norm and we set $r_{\mathbf{D}} := 0$ if \mathbf{D} has an element which is not invertible. We note that $[r_{\mathbf{D}}, R_{\mathbf{D}}] = \mathbf{D}$ if $d = 1$. Finally, a probability measure P on $(\mathbb{R}^d)^{n+1}$ is called a martingale law if X^n is a P -martingale and $X_0^n = 0$ P -a.s. Denoting by $\Delta X_k^n = X_k^n - X_{k-1}^n$ the increments of X^n , we can now set

$$\mathcal{P}_{\mathbf{D}}^n = \left\{ P \text{ martingale law on } (\mathbb{R}^d)^{n+1} : \begin{array}{l} \text{for } k = 1, \dots, n, \\ E^P[\Delta X_k^n (\Delta X_k^n)' | \mathcal{F}_{k-1}^n] \in \mathbf{D} \text{ and } d^2 r_{\mathbf{D}} \leq |\Delta X_k^n|^2 \leq d^2 R_{\mathbf{D}}, P\text{-a.s.} \end{array} \right\},$$

where prime ($'$) denotes transposition. Note that ΔX_k^n is a column vector, so $\Delta X^n (\Delta X^n)'$ takes values in \mathbb{S}_+^d . We introduce the sublinear expectation

$$\mathcal{E}_{\mathbf{D}}^n(\psi) := \sup_{P \in \mathcal{P}_{\mathbf{D}}^n} E^P[\psi] \quad \text{for any random variable } \psi : (\mathbb{R}^d)^{n+1} \rightarrow \mathbb{R}$$

such that ψ is \mathcal{F}_n^n -measurable and $E^P|\psi| < \infty$ for all $P \in \mathcal{P}_{\mathbf{D}}^n$, and we think of $\mathcal{E}_{\mathbf{D}}^n$ as a discrete-time analogue of the G -expectation.

Remark 2.1. The second condition in the definition of $\mathcal{P}_{\mathbf{D}}^n$ is motivated by the desire to generate the volatility uncertainty by a *small* set of scenarios; we remark that the main results remain true if, e.g., the lower bound $r_{\mathbf{D}}$ is omitted and the upper bound $R_{\mathbf{D}}$ replaced by any other condition yielding tightness. Our bounds are chosen such that

$$\mathcal{P}_{\mathbf{D}}^n = \{P \text{ martingale law on } (\mathbb{R}^d)^{n+1} : \Delta X^n (\Delta X^n)' \in \mathbf{D}, P\text{-a.s.}\} \quad \text{if } d = 1.$$

The continuous-time limit. To compare our objects from the two formulations, we shall extend any discrete path $x \in (\mathbb{R}^d)^{n+1}$ to a continuous path $\hat{x} \in \Omega$ by linear interpolation. More precisely,

we define the interpolation operator

$$\begin{aligned} \widehat{\cdot} : (\mathbb{R}^d)^{n+1} &\rightarrow \Omega, & x = (x_0, \dots, x_n) &\mapsto \widehat{x} = (\widehat{x}_t)_{0 \leq t \leq T}, & \text{where} \\ \widehat{x}_t &:= ([nt/T] + 1 - nt/T)x_{[nt/T]} + (nt/T - [nt/T])x_{[nt/T]+1} \end{aligned}$$

and $[y] := \max\{m \in \mathbb{Z} : m \leq y\}$ for $y \in \mathbb{R}$. In particular, if X^n is the canonical process on $(\mathbb{R}^d)^{n+1}$ and ξ is a random variable on Ω , then $\xi(\widehat{X}^n)$ defines a random variable on $(\mathbb{R}^d)^{n+1}$. This allows us to define the following push-forward of $\mathcal{E}_{\mathbf{D}}^n$ to a continuous-time object:

$$\widehat{\mathcal{E}}_{\mathbf{D}}^n(\xi) := \mathcal{E}_{\mathbf{D}}^n(\xi(\widehat{X}^n)) \quad \text{for } \xi : \Omega \rightarrow \mathbb{R}$$

which is suitably integrable.

Our main result states that this sublinear expectation with discrete-time volatility uncertainty converges to the G -expectation as the number n of periods tends to infinity, if the domain of volatility uncertainty is scaled as $\mathbf{D}/n := \{n^{-1}\Gamma : \Gamma \in \mathbf{D}\}$.

Theorem 2.2. *Let $\xi : \Omega \rightarrow \mathbb{R}$ be a continuous function satisfying $|\xi(\omega)| \leq c(1 + \|\omega\|_\infty)^p$ for some constants $c, p > 0$. Then $\widehat{\mathcal{E}}_{\mathbf{D}/n}^n(\xi) \rightarrow \mathcal{E}_{\mathbf{D}}(\xi)$ as $n \rightarrow \infty$; that is,*

$$\sup_{P \in \mathcal{P}_{\mathbf{D}/n}^n} E^P[\xi(\widehat{X}^n)] \rightarrow \sup_{P \in \mathcal{P}_{\mathbf{D}}} E^P[\xi]. \tag{2.1}$$

We shall see that all expressions in (2.1) are well defined and finite. Moreover, we will show in Theorem 3.8 that the result also holds true for a “strong” formulation of volatility uncertainty.

Remark 2.3. Theorem 2.2 cannot be extended to the case where ξ is merely in \mathbb{L}_G^1 , which is defined as the completion of $C_b(\Omega; \mathbb{R})$ under the norm $\|\xi\|_{\mathbb{L}_G^1} := \sup\{E^P|\xi|, P \in \mathcal{P}_{\mathbf{D}}\}$. This is because $\|\cdot\|_{\mathbb{L}_G^1}$ “does not see” the discrete-time objects, as illustrated by the following example. Assume for simplicity that $0 \notin \mathbf{D}$ and let $A \subset \Omega$ be the set of paths with finite variation. Since $P(A) = 0$ for any $P \in \mathcal{P}_{\mathbf{D}}$, we have $\xi := 1 - \mathbf{1}_A = 1$ in \mathbb{L}_G^1 and the right hand side of (2.1) equals one. However, the trajectories of \widehat{X}^n lie in A , so that $\xi(\widehat{X}^n) \equiv 0$ and the left hand side of (2.1) equals zero.

In view of the previous remark, we introduce a smaller space \mathbb{L}_*^1 , defined as the completion of $C_b(\Omega; \mathbb{R})$ under the norm

$$\|\xi\|_* := \sup_{Q \in \mathcal{Q}} E^Q|\xi|, \quad \mathcal{Q} := \mathcal{P}_{\mathbf{D}} \cup \{P \circ (\widehat{X}^n)^{-1} : P \in \mathcal{P}_{\mathbf{D}/n}^n, n \in \mathbb{N}\}. \tag{2.2}$$

If ξ is as in Theorem 2.2, then $\xi \in \mathbb{L}_*^1$ by Lemma 3.4 below and so the following is a generalization of Theorem 2.2.

Corollary 2.4. *Let $\xi \in \mathbb{L}_*^1$. Then $\widehat{\mathcal{E}}_{\mathbf{D}/n}^n(\xi) \rightarrow \mathcal{E}_{\mathbf{D}}(\xi)$ as $n \rightarrow \infty$.*

Proof. This follows from Theorem 2.2 by approximation, using that the two norms $\|\xi\|_*$ and $\sup\{E^P|\xi| : P \in \mathcal{P}_{\mathbf{D}}\} + \sup\{E^P|\xi(\widehat{X}^n)| : P \in \mathcal{P}_{\mathbf{D}/n}^n, n \in \mathbb{N}\}$ are equivalent. \square

3. Proofs and ramifications

In the next two subsections, we prove separately two inequalities that jointly imply Theorem 2.2 and a slightly stronger result, reported in Theorem 3.8.

3.1. The first inequality

In this subsection we prove the first inequality of (2.1), namely that

$$\limsup_{n \rightarrow \infty} \sup_{P \in \mathcal{P}_{\mathbf{D}}^n/n} E^P[\xi(\widehat{X}^n)] \leq \sup_{P \in \mathcal{P}_{\mathbf{D}}} E^P[\xi]. \tag{3.1}$$

The essential step in this proof is a stability result for the volatility (see Lemma 3.3(ii) below); the necessary tightness follows from the compactness of \mathbf{D} ; i.e., from $R_{\mathbf{D}} < \infty$. We shall denote $\lambda \mathbf{D} = \{\lambda \Gamma : \Gamma \in \mathbf{D}\}$ for $\lambda \in \mathbb{R}$.

Lemma 3.1. *Given $p \in [1, \infty)$, there exists a universal constant $K > 0$ such that for all $0 \leq k \leq l \leq n$ and $P \in \mathcal{P}_{\mathbf{D}}^n$,*

- (i) $E^P[\sup_{k=0, \dots, n} |X_k^n|^{2p}] \leq K(nR_{\mathbf{D}})^p$,
- (ii) $E^P|X_l^n - X_k^n|^4 \leq KR_{\mathbf{D}}^2(l - k)^2$,
- (iii) $E^P[(X_l^n - X_k^n)(X_l^n - X_k^n)' | \mathcal{F}_k^n] \in (l - k)\mathbf{D}$ *P*-a.s.

Proof. We set $X := X^n$ to simplify the notation.

(i) Let $p \in [1, \infty)$. By the Burkholder–Davis–Gundy (BDG) inequalities there exists a universal constant $C = C(p, d)$ such that

$$E^P \left[\sup_{k=0, \dots, n} |X_k^n|^{2p} \right] \leq CE^P \|[X]_n\|^p.$$

In view of $P \in \mathcal{P}_{\mathbf{D}}^n$, we have $\|[X]_n\| = \|\sum_{i=1}^n \Delta X_i(\Delta X_i)'\| \leq nd^2R_{\mathbf{D}}$ *P*-a.s.

(ii) The BDG inequalities yield a universal constant C such that

$$E^P|X_l - X_k|^4 \leq CE^P\|[X]_l - [X]_k\|^2.$$

Similarly to in (i), $P \in \mathcal{P}_{\mathbf{D}}^n$ implies that $\|[X]_l - [X]_k\| \leq (l - k)d^2R_{\mathbf{D}}$ *P*-a.s.

(iii) The orthogonality of the martingale increments yields that

$$E^P[(X_l - X_k)(X_l - X_k)' | \mathcal{F}_k^n] = \sum_{i=k+1}^l E^P[\Delta X_i(\Delta X_i)' | \mathcal{F}_k^n].$$

Since $E^P[\Delta X_i(\Delta X_i)' | \mathcal{F}_{i-1}^n] \in \mathbf{D}$ *P*-a.s. and since \mathbf{D} is convex,

$$E^P[\Delta X_i(\Delta X_i)' | \mathcal{F}_k^n] = E^P[E^P[\Delta X_i(\Delta X_i)' | \mathcal{F}_{i-1}^n] | \mathcal{F}_k^n]$$

again takes values in \mathbf{D} . It remains to observe that if $\Gamma_1, \dots, \Gamma_m \in \mathbf{D}$, then $\Gamma_1 + \dots + \Gamma_m \in m\mathbf{D}$ by convexity. \square

The following lemma shows in particular that all expressions in Theorem 2.2 are well defined and finite.

Lemma 3.2. *Let $\xi : \Omega \rightarrow \mathbb{R}$ be as in Theorem 2.2. Then $\|\xi\|_* < \infty$; that is,*

$$\sup_{n \in \mathbb{N}} \sup_{P \in \mathcal{P}_{\mathbf{D}}^n/n} E^P|\xi(\widehat{X}^n)| < \infty \quad \text{and} \quad \sup_{P \in \mathcal{P}_{\mathbf{D}}} E^P|\xi| < \infty. \tag{3.2}$$

Proof. Let $n \in \mathbb{N}$ and $P \in \mathcal{P}_{\mathbf{D}/n}^n$. By the assumption on ξ , there exist constants $c, p > 0$ such that

$$E^P |\xi(\widehat{X}^n)| \leq c + cE^P \left[\sup_{0 \leq t \leq T} |\widehat{X}_t^n|^p \right] \leq c + cE^P \left[\sup_{k=0, \dots, n} |X_k^n|^p \right].$$

Hence Lemma 3.1(i) and the observation that $R_{\mathbf{D}/n} = R_{\mathbf{D}}/n$ yield that $E^P |\xi(\widehat{X}^n)| \leq KR_{\mathbf{D}}^{p/2}$ and the first claim follows. The second claim similarly follows from the estimate that $E^P [\sup_{0 \leq t \leq T} |B_t|^p] \leq C_p$ for all $P \in \mathcal{P}_{\mathbf{D}}$, which is obtained from the BDG inequalities by using that \mathbf{D} is bounded. \square

We can now prove the key result of this subsection.

Lemma 3.3. For each $n \in \mathbb{N}$, let $\{M^n = (M_k^n)_{k=0}^n, \tilde{P}^n\}$ be a martingale with law $P^n \in \mathcal{P}_{\mathbf{D}/n}^n$ on $(\mathbb{R}^d)^{n+1}$ and let Q^n be the law of \widehat{M}^n on Ω . Then,

- (i) the sequence (Q^n) is tight on Ω ,
- (ii) any cluster point of (Q^n) is an element of $\mathcal{P}_{\mathbf{D}}$.

Proof. (i) Let $0 \leq s \leq t \leq T$. As $R_{\mathbf{D}/n} = R_{\mathbf{D}}/n$, Lemma 3.1(ii) implies that

$$E^{Q^n} |B_t - B_s|^4 = E^{\tilde{P}^n} |\widehat{M}_t^n - \widehat{M}_s^n|^4 \leq C|t - s|^2$$

for a constant $C > 0$. Hence (Q^n) is tight by the moment criterion.

(ii) Let Q be a cluster point; then B is a Q -martingale as a consequence of the uniform integrability implied by Lemma 3.1(i) and it remains to show that $d(B)_t/dt \in \mathbf{D}$ holds $Q \times dt$ -a.e. It will be useful to characterize \mathbf{D} by scalar inequalities: given $\Gamma \in \mathbb{S}^d$, the separating hyperplane theorem implies that

$$\Gamma \in \mathbf{D} \quad \text{if and only if} \quad \ell(\Gamma) \leq C_{\mathbf{D}}^\ell := \sup_{A \in \mathbf{D}} \ell(A) \quad \text{for all } \ell \in (\mathbb{S}^d)^*, \tag{3.3}$$

where $(\mathbb{S}^d)^*$ is the set of all linear functionals $\ell : \mathbb{S}^d \rightarrow \mathbb{R}$.

Let $H : [0, T] \times \Omega \rightarrow [0, 1]$ be a continuous and adapted function and let $\ell \in (\mathbb{S}^d)^*$. We fix $0 \leq s < t \leq T$ and define $\Delta_{s,t}Y := Y_t - Y_s$ for a process $Y = (Y_u)_{0 \leq u \leq T}$. Let $\varepsilon > 0$ and let $\tilde{\mathbf{D}}$ be any neighborhood of \mathbf{D} ; then for n sufficiently large,

$$E^{\tilde{P}^n} \left[(\Delta_{s,t}\widehat{M}^n)(\Delta_{s,t}\widehat{M}^n)' | \sigma(\widehat{M}_u^n, 0 \leq u \leq s - \varepsilon) \right] \in (t - s)\tilde{\mathbf{D}} \quad \tilde{P}^n\text{-a.s.}$$

as a consequence of Lemma 3.1(iii). Since $\tilde{\mathbf{D}}$ was arbitrary, it follows by (3.3) that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} E^{Q^n} \left[H(s - \varepsilon, B) \left\{ \ell((\Delta_{s,t}B)(\Delta_{s,t}B)') - C_{\mathbf{D}}^\ell(t - s) \right\} \right] \\ &= \limsup_{n \rightarrow \infty} E^{\tilde{P}^n} \left[H(s - \varepsilon, \widehat{M}^n) \left\{ \ell((\Delta_{s,t}\widehat{M}^n)(\Delta_{s,t}\widehat{M}^n)') - C_{\mathbf{D}}^\ell(t - s) \right\} \right] \leq 0. \end{aligned}$$

Using (3.2) with $\xi(\omega) = \|\omega\|_\infty^2$, we may pass to the limit and conclude that

$$E^Q \left[H(s - \varepsilon, B) \ell((\Delta_{s,t}B)(\Delta_{s,t}B)') \right] \leq E^Q \left[H(s - \varepsilon, B) C_{\mathbf{D}}^\ell(t - s) \right]. \tag{3.4}$$

Since $H(s - \varepsilon, B)$ is \mathcal{F}_s -measurable and

$$E^Q [(\Delta_{s,t}B)(\Delta_{s,t}B)' | \mathcal{F}_s] = E^Q [B_t B_t' - B_s B_s' | \mathcal{F}_s] = E^Q [(B)_t - (B)_s | \mathcal{F}_s]$$

as B is a square-integrable Q -martingale, (3.4) is equivalent to

$$E^Q[H(s - \varepsilon, B) \ell(\langle B \rangle_t - \langle B \rangle_s)] \leq E^Q[H(s - \varepsilon, B) C_{\mathbf{D}}^\ell(t - s)].$$

Using the continuity of H and dominated convergence as $\varepsilon \rightarrow 0$, we obtain

$$E^Q[H(s, B) \ell(\langle B \rangle_t - \langle B \rangle_s)] \leq E^Q[H(s, B) C_{\mathbf{D}}^\ell(t - s)]$$

and then it follows that

$$E^Q\left[\int_0^T H(t, B) \ell(d\langle B \rangle_t)\right] \leq E^Q\left[\int_0^T H(t, B) C_{\mathbf{D}}^\ell dt\right].$$

By an approximation argument, this inequality extends to functions H which are measurable instead of continuous. It follows that $\ell(d\langle B \rangle_t/dt) \leq C_{\mathbf{D}}^\ell$ holds $Q \times dt$ -a.e., and since $\ell \in (\mathbb{S}^d)^*$ was arbitrary, (3.3) shows that $d\langle B \rangle_t/dt \in \mathbf{D}$ holds $Q \times dt$ -a.e. \square

We can now deduce the first inequality of Theorem 2.2 as follows.

Proof of (3.1). Let ξ be as in Theorem 2.2 and let $\varepsilon > 0$. For each $n \in \mathbb{N}$ there exists an ε -optimizer $P^n \in \mathcal{P}_{\mathbf{D}/n}^n$; i.e., if Q^n denotes the law of \widehat{X}^n on Ω under P_n , then

$$E^{Q^n}[\xi] = E^{P^n}[\xi(\widehat{X}^n)] \geq \sup_{P \in \mathcal{P}_{\mathbf{D}/n}^n} E^P[\xi(\widehat{X}^n)] - \varepsilon.$$

By Lemma 3.3, the sequence (Q^n) is tight and any cluster point belongs to $\mathcal{P}_{\mathbf{D}}$. Since ξ is continuous and (3.2) implies $\sup_n E^{Q^n}|\xi| < \infty$, tightness yields that $\limsup_n E^{Q^n}[\xi] \leq \sup_{P \in \mathcal{P}_{\mathbf{D}}} E^P[\xi]$. Therefore,

$$\limsup_{n \rightarrow \infty} \sup_{P \in \mathcal{P}_{\mathbf{D}/n}^n} E^P[\xi(\widehat{X}^n)] \leq \sup_{P \in \mathcal{P}_{\mathbf{D}}} E^P[\xi] + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that (3.1) holds. \square

Finally, we also prove the statement preceding Corollary 2.4.

Lemma 3.4. *Let $\xi : \Omega \rightarrow \mathbb{R}$ be as in Theorem 2.2. Then $\xi \in \mathbb{L}_{*}^1$.*

Proof. We show that $\xi^m := (\xi \wedge m) \vee m$ converges to ξ in the norm $\|\cdot\|_*$ as $m \rightarrow \infty$, or equivalently, that the upper expectation $\sup\{E^Q[\cdot] : Q \in \mathcal{Q}\}$ is continuous along the decreasing sequence $|\xi - \xi^m|$, where \mathcal{Q} is as in (2.2). Indeed, \mathcal{Q} is tight by (the proof of) Lemma 3.3. Using that $\|\xi\|_* < \infty$ by Lemma 3.2, we can then argue as in the proof of [3, Theorem 12] to obtain the claim. \square

3.2. The second inequality

The main purpose of this subsection is to show the second inequality “ \geq ” of (2.1). Our proof will yield a more precise version of Theorem 2.2. Namely, we will include “strong” formulations of volatility uncertainty both in discrete and in continuous time; i.e., consider laws generated by integrals with respect to a fixed random walk and Brownian motion, respectively. In the financial interpretation, this means that the uncertainty can be generated by complete market models.

The strong formulation in continuous time. Here we shall consider Brownian martingales: with P_0 denoting the Wiener measure, we define

$$\mathcal{Q}_{\mathbf{D}} = \left\{ P_0 \circ \left(\int f(t, B) dB_t \right)^{-1} : f \in C([0, T] \times \Omega; \sqrt{\mathbf{D}}) \text{ adapted} \right\},$$

where $\sqrt{\mathbf{D}} = \{\sqrt{\Gamma} : \Gamma \in \mathbf{D}\}$. (For $\Gamma \in \mathbb{S}_+^d$, $\sqrt{\Gamma}$ denotes the unique square root in \mathbb{S}_+^d .) We note that $\mathcal{Q}_{\mathbf{D}}$ is a (typically strict) subset of $\mathcal{P}_{\mathbf{D}}$. The elements of $\mathcal{Q}_{\mathbf{D}}$ with nondegenerate f have the predictable representation property; i.e., they correspond to a complete market in the terminology of mathematical finance. We have the following density result; the proof is deferred to the end of the section.

Proposition 3.5. *The convex hull of $\mathcal{Q}_{\mathbf{D}}$ is a weakly dense subset of $\mathcal{P}_{\mathbf{D}}$.*

We can now deduce the connection between $\mathcal{E}_{\mathbf{D}}$ and the G -expectation associated with \mathbf{D} .

Remark 3.6. (i) Proposition 3.5 implies that

$$\sup_{P \in \mathcal{Q}_{\mathbf{D}}} E^P[\xi] = \sup_{P \in \mathcal{P}_{\mathbf{D}}} E^P[\xi], \quad \xi \in C_b(\Omega; \mathbb{R}). \tag{3.5}$$

In [3, Section 3] it is shown that the G -expectation as introduced in [12,13] coincides with the mapping $\xi \mapsto \sup_{P \in \mathcal{Q}_{\mathbf{D}}^*} E^P[\xi]$ for a certain set $\mathcal{Q}_{\mathbf{D}}^*$ satisfying $\mathcal{Q}_{\mathbf{D}} \subseteq \mathcal{Q}_{\mathbf{D}}^* \subseteq \mathcal{P}_{\mathbf{D}}$. In particular, we deduce that the right hand side of (3.5) is indeed equal to the G -expectation, as claimed in Section 2.

(ii) A result similar to Proposition 3.5 can also be deduced from [17, Proposition 3.4.], which relies on a PDE-based verification argument of stochastic control. We include a (possibly more enlightening) probabilistic proof at the end of the section.

The strong formulation in discrete time. For fixed $n \in \mathbb{N}$, we consider

$$\Omega_n := \{\omega = (\omega_1, \dots, \omega_n) : \omega_i \in \{1, \dots, d + 1\}, i = 1, \dots, n\}$$

equipped with its power set and let $P_n := \{(d + 1)^{-1}, \dots, (d + 1)^{-1}\}^n$ be the product probability associated with the uniform distribution. Moreover, let ξ_1, \dots, ξ_n be an i.i.d. sequence of \mathbb{R}^d -valued random variables on Ω_n such that $|\xi_k| = d$ and such that the components of ξ_k are orthonormal in $L^2(P_n)$, for each $k = 1, \dots, n$. Let $Z_k = \sum_{l=1}^k \xi_l$ be the associated random walk. Then, we consider martingales M^f which are discrete-time integrals of Z of the form

$$M_k^f = \sum_{l=1}^k f(l - 1, Z) \Delta Z_l,$$

where f is measurable and adapted with respect to the filtration generated by Z ; i.e., $f(l, Z)$ depends only on $Z|_{\{0, \dots, l\}}$. We define

$$\mathcal{Q}_{\mathbf{D}}^n = \left\{ P_n \circ (M^f)^{-1}; f : \{0, \dots, n - 1\} \times (\mathbb{R}^d)^{n+1} \rightarrow \sqrt{\mathbf{D}} \text{ measurable, adapted} \right\}.$$

To see that $\mathcal{Q}_{\mathbf{D}}^n \subseteq \mathcal{P}_{\mathbf{D}}^n$, we note that $\Delta_k M^f = f(k - 1, Z) \xi_k$ and the orthonormality property of ξ_k yield

$$E^{P_n} [\Delta_k M^f (\Delta_k M^f)' | \sigma(Z_1, \dots, Z_{k-1})] = f(k - 1, Z)^2 \in \mathbf{D} \quad P_n\text{-a.s.},$$

while $|\xi_k| = d$ and $f^2 \in \mathbf{D}$ imply that

$$\left\| \Delta_k M^f (\Delta_k M^f)' \right\| = |f(k - 1, Z) \xi_k|^2 \in [d^2 r_{\mathbf{D}}, d^2 R_{\mathbf{D}}] \quad P_n\text{-a.s.}$$

Remark 3.7. We recall from [7] that such ξ_1, \dots, ξ_n can be constructed as follows. Let A be an orthogonal $(d + 1) \times (d + 1)$ matrix whose last row is $((d + 1)^{-1/2}, \dots, (d + 1)^{-1/2})$ and let $v_l \in \mathbb{R}^d$ be column vectors such that $[v_1, \dots, v_{d+1}]$ is the matrix obtained from A by deleting the last row. Setting $\xi_k(\omega) := (d + 1)^{1/2}v_{\omega_k}$ for $\omega = (\omega_1, \dots, \omega_n)$ and $k = 1, \dots, n$, the above requirements are satisfied.

We can now formulate a result which includes [Theorem 2.2](#).

Theorem 3.8. *Let $\xi : \Omega \rightarrow \mathbb{R}$ be as in [Theorem 2.2](#). Then*

$$\begin{aligned} \lim_{n \rightarrow \infty} \sup_{P \in \mathcal{Q}_{\mathbf{D}/n}^n} E^P[\xi(\widehat{X}^n)] &= \lim_{n \rightarrow \infty} \sup_{P \in \mathcal{P}_{\mathbf{D}/n}^n} E^P[\xi(\widehat{X}^n)] \\ &= \sup_{P \in \mathcal{Q}_{\mathbf{D}}} E^P[\xi] \\ &= \sup_{P \in \mathcal{P}_{\mathbf{D}}} E^P[\xi]. \end{aligned} \tag{3.6}$$

Proof. Since $\mathcal{Q}_{\mathbf{D}/n}^n \subseteq \mathcal{P}_{\mathbf{D}/n}^n$ for each $n \geq 1$, the inequality [\(3.1\)](#) yields that

$$\limsup_{n \rightarrow \infty} \sup_{P \in \mathcal{Q}_{\mathbf{D}/n}^n} E^P[\xi(\widehat{X}^n)] \leq \sup_{P \in \mathcal{P}_{\mathbf{D}}} E^P[\xi].$$

As the equality in [\(3.6\)](#) follows from [Proposition 3.5](#), it remains to show that

$$\liminf_{n \rightarrow \infty} \sup_{P \in \mathcal{Q}_{\mathbf{D}/n}^n} E^P[\xi(\widehat{X}^n)] \geq \sup_{P \in \mathcal{Q}_{\mathbf{D}}} E^P[\xi].$$

To this end, let $P \in \mathcal{Q}_{\mathbf{D}}$; i.e., P is the law of a martingale of the form

$$M = \int f(t, W) dW_t,$$

where W is a Brownian motion and $f \in C([0, T] \times \Omega; \sqrt{\mathbf{D}})$ is an adapted function. We shall construct martingales $M^{(n)}$ whose laws are in $\mathcal{Q}_{\mathbf{D}/n}^n$ and tend to P .

For $n \geq 1$, let $Z_k^{(n)} = \sum_{l=1}^k \xi_l$ be the random walk on (Ω_n, P_n) as introduced before [Remark 3.7](#). Let

$$W_t^{(n)} := n^{-1/2} \sum_{k=1}^{[nt/T]} \xi_k, \quad 0 \leq t \leq T$$

be the piecewise constant càdlàg version of the scaled random walk and let $\widehat{W}^{(n)} := n^{-1/2} \widehat{Z}^{(n)}$ be its continuous counterpart obtained by linear interpolation. It follows from the central limit theorem that

$$(W^{(n)}, \widehat{W}^{(n)}) \Rightarrow (W, W) \quad \text{on } D([0, T]; \mathbb{R}^{2d}),$$

the space of càdlàg paths equipped with the Skorohod topology. Moreover, since f is continuous, we also have that

$$(W^{(n)}, f([nt/T]T/n, \widehat{W}^{(n)})) \Rightarrow (W, f(t, W)) \quad \text{on } D([0, T]; \mathbb{R}^{d+d^2}).$$

Thus, if we introduce the discrete-time integral

$$M_k^{(n)} := \sum_{l=1}^k f((l-1)T/n, \hat{W}^{(n)}) \left(\hat{W}_{lT/n}^{(n)} - \hat{W}_{(l-1)T/n}^{(n)} \right),$$

it follows from the stability of stochastic integrals (see [6, Theorem 4.3 and Definition 4.1]) that

$$\left(M_{\lfloor nt/T \rfloor}^{(n)} \right)_{0 \leq t \leq T} \Rightarrow M \quad \text{on } D([0, T]; \mathbb{R}^d).$$

Moreover, since the increments of $M^{(n)}$ uniformly tend to 0 as $n \rightarrow \infty$, it also follows that

$$\hat{M}^{(n)} \Rightarrow M \quad \text{on } \Omega.$$

As f^2/n takes values in \mathbf{D}/n , the law of $M^{(n)}$ is contained in $\mathcal{Q}_{\mathbf{D}/n}^n$ and the proof is complete. \square

It remains to give the proof of Proposition 3.5, which we will obtain by a randomization technique. Since similar arguments, at least for the scalar case, can be found elsewhere (e.g., [8, Section 5]), we shall be brief.

Proof of Proposition 3.5. We may assume without loss of generality that

$$\text{there exists an invertible element } \Gamma_* \in \mathbf{D}. \tag{3.7}$$

Indeed, using that \mathbf{D} is a convex subset of \mathbb{S}_+^d , we observe that (3.7) is equivalent to $K = \{0\}$ for $K := \bigcap_{\Gamma \in \mathbf{D}} \ker \Gamma$. If $k = \dim K > 0$, a change of coordinates brings us to the situation where K corresponds to the last k coordinates of \mathbb{R}^d . We can then reduce all considerations to \mathbb{R}^{d-k} and thereby recover the situation of (3.7).

1. *Regularization.* We first observe that the set

$$\{P \in \mathcal{P}_{\mathbf{D}} : d\langle B \rangle_t/dt \geq \varepsilon \mathbb{1}_d P \times dt\text{-a.e. for some } \varepsilon > 0\} \tag{3.8}$$

is weakly dense in $\mathcal{P}_{\mathbf{D}}$. (Here $\mathbb{1}_d$ denotes the unit matrix.) Indeed, let M be a martingale whose law is in $\mathcal{P}_{\mathbf{D}}$. Recall (3.7) and let N be an independent continuous Gaussian martingale with $d\langle N \rangle_t/dt = \Gamma_*$. For $\lambda \uparrow 1$, the law of $\lambda M + (1 - \lambda)N$ tends to the law of M and is contained in the set (3.8), since \mathbf{D} is convex.

2. *Discretization.* Next, we reduce to martingales with piecewise constant volatility. Let M be a martingale whose law belongs to (3.8). We have

$$M = \int \sigma_t dW_t \quad \text{for } \sigma_t := \sqrt{d\langle M \rangle/dt} \quad \text{and} \quad W := \int \sigma_t^{-1} dM_t,$$

where W is a Brownian motion by Lévy’s theorem. For $n \geq 1$, we introduce $M^{(n)} = \int \sigma_t^{(n)} dW_t$, where $\sigma^{(n)}$ is an \mathbb{S}_+^d -valued piecewise constant process satisfying

$$(\sigma_t^{(n)})^2 = \Pi_{\mathbf{D}} \left[\left(\frac{n}{T} \int_{(k-1)T/n}^{kT/n} \sigma_s ds \right)^2 \right], \quad t \in (kT/n, (k+1)T/n]$$

for $k = 1, \dots, n-1$, where $\Pi_{\mathbf{D}}: \mathbb{S}^d \rightarrow \mathbf{D}$ is the Euclidean projection. On $[0, T/n]$ one can take, e.g., $\sigma^{(n)} := \sqrt{T_*}$. We then have

$$E \left\| \left\langle M - M^{(n)} \right\rangle_T \right\| = E \int_0^T \left\| \sigma_t - \sigma_t^{(n)} \right\|^2 dt \rightarrow 0$$

and in particular $M^{(n)}$ converges weakly to M .

3. *Randomization.* Consider a martingale of the form $M = \int \sigma_t dW_t$, where W is a Brownian motion on some given filtered probability space and σ is an adapted $\sqrt{\mathbf{D}}$ -valued process which is piecewise constant; i.e.,

$$\sigma = \sum_{k=0}^{n-1} \mathbf{1}_{[t_k, t_{k+1})} \sigma(k) \quad \text{for some } 0 = t_0 < t_1 < \dots < t_n = T$$

and some $n \geq 1$. Consider also a second probability space carrying a Brownian motion \tilde{W} and a sequence U^1, \dots, U^n of $\mathbb{R}^{d \times d}$ -valued random variables such that the components $\{U^k_{ij} : 1 \leq i, j \leq d; 1 \leq k \leq n\}$ are i.i.d. uniformly distributed on $(0, 1)$ and independent of \tilde{W} .

Using the existence of regular conditional probability distributions, we can construct functions $\Theta_k : C([0, t_k]; \mathbb{R}^d) \times (0, 1)^{d^2} \times \dots \times (0, 1)^{d^2} \rightarrow \sqrt{\mathbf{D}}$ such that the random variables $\tilde{\sigma}(k) := \Theta_k(\tilde{W}|_{[0, t_k]}, U^1, \dots, U^k)$ satisfy

$$\{\tilde{W}, \tilde{\sigma}(0), \dots, \tilde{\sigma}(n-1)\} = \{W, \sigma(0), \dots, \sigma(n-1)\} \quad \text{in law.} \tag{3.9}$$

We can then consider the volatility corresponding to a fixed realization of U^1, \dots, U^n . Indeed, for $u = (u^1, \dots, u^n) \in (0, 1)^{nd^2}$, let

$$\tilde{\sigma}(k; u) := \Theta_k(\tilde{W}|_{[0, t_k]}, u^1, \dots, u^k)$$

and consider $\tilde{M}^u = \int \tilde{\sigma}_t^u d\tilde{W}_t$, where $\tilde{\sigma}^u := \sum_{k=0}^{n-1} \mathbf{1}_{[t_k, t_{k+1})} \tilde{\sigma}(k; u)$. For any $F \in C_b(\Omega; \mathbb{R})$, the equality (3.9) and Fubini’s theorem yield that

$$\begin{aligned} E[F(M)] &= E[F(\tilde{M}^{(U^1, \dots, U^n)})] = \int_{(0,1)^{nd^2}} E[F(\tilde{M}^u)] du \\ &\leq \sup_{u \in (0,1)^{nd^2}} E[F(\tilde{M}^u)]. \end{aligned}$$

Hence, by the Hahn–Banach theorem, the law of M is contained in the weak closure of the convex hull of the laws of $\{\tilde{M}^u : u \in (0, 1)^{nd^2}\}$. We note that \tilde{M}^u is of the form $\tilde{M}^u = \int g(t, \tilde{W}) d\tilde{W}_t$ with a measurable, adapted, $\sqrt{\mathbf{D}}$ -valued function g , for each fixed u .

4. *Smoothing.* As \mathcal{Q}_D is defined through continuous functions, it remains to approximate g by a continuous function f . Let $g : [0, T] \times \Omega \rightarrow \sqrt{\mathbf{D}}$ be a measurable adapted function and $\delta > 0$. By standard density arguments there exists $\tilde{f} \in C([0, T] \times \Omega; \mathbb{S}^d)$ such that

$$E \int_0^T \|\tilde{f}(t, \tilde{W}) - g(t, \tilde{W})\|^2 dt \leq \delta.$$

Let $f(t, x) := \sqrt{I_{\mathbf{D}}(\tilde{f}(t, x)^2)}$. Then $f \in C([0, T] \times \Omega; \sqrt{\mathbf{D}})$ and

$$\|f - g\|^2 \leq \|f^2 - g^2\| \leq \|\tilde{f}^2 - g^2\| \leq (\|\tilde{f}\| + \|g\|)\|\tilde{f} - g\| \leq 2\sqrt{R_{\mathbf{D}}}\|\tilde{f} - g\|$$

(see [1, Theorem X.1.1] for the first inequality). By Jensen’s inequality we conclude that $E \int_0^T \|f(t, \tilde{W}) - g(t, \tilde{W})\|^2 dt \leq 2\sqrt{TR_{\mathbf{D}}}\delta$, which, in view of the above steps, completes the proof. \square

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