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Involutions on graded matrix algebras

Yuri Bahturin [a,b,*,1], Mikhail Zaicev [b,2]

[a] *Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, NL A1C5S7, Canada*
[b] *Department of Algebra, Faculty of Mathematics and Mechanics, Moscow State University, Moscow, 119992, Russia*

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Abstract

In this paper we describe graded antiautomorphisms of finite order on matrix algebras endowed with a group gradings by a finite abelian group over an arbitrary algebraically closed field of characteristic different from 2.

Keywords: Matrix algebras; Graded algebras; Involutions; Automorphisms and antiautomorphisms

1. Introduction

This paper is devoted to the correction of an error in the paper [5] in which the classification of involution gradings on matrix algebras was derived from the fact that in the decomposition of a graded matrix algebra as the tensor product of an elementary and a fine component, these components remain invariant under the involution.

* Corresponding author at: Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, NL A1C5S7, Canada.

E-mail addresses: yuri@math.mun.ca (Y. Bahturin), zaicev@mech.math.msu.su (M. Zaicev).

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2. Some notation and simple facts

Let F be an arbitrary field, A a not necessarily associative algebra over an F and G a group. We say that A is a G-graded algebra, if there is a vector space sum decomposition

$$A=\bigoplus_{g\in G} A_g, \tag{1}$$

such that

$$A_g A_h \subset A_{gh} \quad \text{for all } g,h\in G. \tag{2}$$

A subspace $V\subset A$ is called *graded* (or *homogeneous*) if $V=\bigoplus_{g\in G}(V\cap A_g)$. An element $a\in R$ is called *homogeneous of degree* g if $a\in A_g$. We also write $\deg a=g$. The *support* of the G-grading is a subset

$$\operatorname{Supp} A=\{g\in G \mid A_g\neq 0\}.$$

Suppose now that F is of characteristic different from 2. If A is an associative algebra with involution $*$ and, in addition to (2), one has

$$(A_g)^* = A_g \quad \text{for all } g\in G \tag{3}$$

then we say that (1) is an *involution preserving* grading or simply an *involution* grading. In this case, given a graded subspace $B\subset A$ we set

$$H(B,*)=\bigl\{b\in B \mid b^*=b\bigr\}, \quad \text{the set of symmetric elements of } B \tag{4}$$

and

$$K(B,*)=\bigl\{b\in B \mid b^*=-b\bigr\}, \quad \text{the set of skew-symmetric elements of } B. \tag{5}$$

If B is an associative subalgebra of A then $B^{(-)}$ is a Lie subalgebra of A, that is, with respect to $[x,y]=xy-yx$ while $B^{(+)}$ is a Jordan subalgebra of A, that is, with respect to $x\circ y=xy+yx$. We always have $B=B^{(-)}\oplus B^{(+)}$.

3. Reminder: Group gradings on matrix algebras

Below we briefly recall the results of [4], where the complete description of abelian group gradings on the full matrix algebra has been given. For non-commutative gradings see [3].

A grading $R=\bigoplus_{g\in G} R_g$ on the matrix algebra $R=M_n(F)$ is called *elementary* if there exists an n-tuple $(g_1,\dots,g_n)\in G^n$ such that the matrix units E_{ij}, $1\leqslant i,j\leqslant n$ are homogeneous and $E_{ij}\in R_g \Leftrightarrow g=g_i^{-1}g_j$.

A grading is called *fine* if $\dim R_g = 1$ for any $g \in \operatorname{Supp} R$. A particular case of fine gradings is the so-called ε-grading where ε is nth primitive root of 1. Let $G = \langle a\rangle_n \times \langle b\rangle_n$ be the direct product of two cyclic groups of order n and

$$X_a = \begin{pmatrix} \varepsilon^{n-1} & 0 & \dots & 0 \\ 0 & \varepsilon^{n-2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix}, \qquad X_b = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \\ 1 & 0 & \dots & 0 \end{pmatrix}. \tag{6}$$

Then

$$X_a X_b X_a^{-1} = \varepsilon X_b, \qquad X_a^n = X_b^n = I \tag{7}$$

and all $X_a^i X_b^j$, $1 \leqslant i, j \leqslant n$, are linearly independent. Clearly, the elements $X_a^i X_b^j$, $i, j = 1, \dots, n$, form a basis of R and all the products of these basis elements are uniquely defined by (7).

Now for any $g \in G$, $g = a^i b^j$, we set $X_g = X_a^i X_b^j$ and denote by R_g a one-dimensional subspace

$$R_g = \langle X_a^i X_b^j \rangle. \tag{8}$$

Then from (7) it follows that $R = \bigoplus_{g\in G} R_g$ is a G-grading on $M_n(F)$ which is called an ε-grading.

Now let $R = M_n(F)$ be the full matrix algebra over F graded by an abelian group G. The following result has been proved in [4, Section 4, Theorems 5, 6] and [2, Subsection 2.2, Theorem 6, Subsection 2.3, Theorem 8].

Theorem 1. *Let F be an algebraically closed field of characteristic zero. Then as a G-graded algebra R is isomorphic to the tensor product*

$$R^{(0)} \otimes R^{(1)} \otimes \dots \otimes R^{(k)}$$

where $R^{(0)} = M_{n_0}(F)$ has an elementary G-grading, $\operatorname{Supp} R^{(0)} = S$ *is a finite subset of G, $R^{(i)} = M_{n_i}(F)$ has the ε_i grading, ε_i being a primitive n_i*th *root of* 1, $\operatorname{Supp} R^{(i)} = H_i \cong \mathbb{Z}_{n_i} \times \mathbb{Z}_{n_i}$, *$i = 1, \dots, k$. Also $H = H_1 \cdots H_k \cong H_1 \times \dots \times H_k$ and $S \cap H = \{e\}$ in G.*

Remark 1. It follows from a very general lemma in [4] that the support T of a fine grading on $R = M_n$ is a subgroup of the grading group G. Thus we have $R = \bigoplus_{t\in T} R_t$ and $R_t = \langle X_t\rangle$, for a non-degenerate matrix X_t. Let us also recall that the product in $R = M_n(F)$ with fine grading as above is defined by a *bicharacter* $\alpha : T \times T \to F^*$ as follows: $X_t X_u = \alpha(t, u) X_{tu}$, for any $t, u \in T$. The commutation relations in R take the form $X_t X_u = \beta(t, u) X_u X_t$ where $\beta(t, u) = \alpha(t, u)/\alpha(u, t)$ is a *skew-symmetric bicharacter* on T (see [1]).

Let us recall that any involution $*$ of $R = M_n$ can always be written as

$$X^* = \Phi^{-1}\left({}^t X\right)\Phi \tag{9}$$

where Φ is a non-degenerate matrix which is either symmetric or skew-symmetric and $X \mapsto {}^t X$ is the ordinary transpose map. In the case where Φ is symmetric, we call $*$ a transpose involution.

If Φ is skew-symmetric, $*$ is called a symplectic involution. Before we formulate the theorem describing involution gradings on M_n in the case where the elementary and fine components are invariant under the involution, we need three (slightly modified) lemmas from [5]. The general restriction in [5] zero characteristic was not used in the proof of these particular lemmas. The first two deal with elementary involution gradings while the last with certain fine involution gradings. If R has an involution $*$ then by $R^{(\pm)}$ we denote the space of symmetric (respectively skew-symmetric) matrices in R under $*$.

The next lemma handles the case of an elementary grading compatible with an involution defined by a symmetric non-degenerate bilinear form.

Lemma 1. *Let $R = M_n(F)$, n a natural number, be a matrix algebra with involution $*$ defined by a symmetric non-degenerate bilinear form. Let G be an abelian group and let R be equipped with an elementary involution G-grading defined by an n-tuple $(g_1, \dots, g_n)$. Then, after a renumbering, $g_1^2 = \cdots = g_m^2 = g_{m+1}g_{m+l+1} = \cdots = g_{m+l}g_{m+2l}$ for some $0 \leqslant l \leqslant \frac{n}{2}$ and $m + 2l = n$. The involution $*$ acts as $X^* = (\Phi^{-1})^t X\Phi$ where*

$$\Phi = \begin{pmatrix} I_m & 0 & 0 \\ 0 & 0 & I_l \\ 0 & I_l & 0 \end{pmatrix},$$

where I_s is the $s \times s$ identity matrix. Moreover, $R^{(-)}$ consists of all matrices of the type

$$\begin{pmatrix} P & S & T \\ -{}^tT & A & B \\ -{}^tS & C & -{}^tA \end{pmatrix}, \tag{10}$$

where ${}^tP = -P$, ${}^tB = -B$, ${}^tC = -C$ and

$$P \in M_m(F), \quad A, B, C, D \in M_l(F), \quad S, T \in M_{m\times l}(F)$$

while $R^{(+)}$ consists of all matrices of the type

$$\begin{pmatrix} P & S & T \\ {}^tT & A & B \\ {}^tS & C & {}^tA \end{pmatrix}, \tag{11}$$

where ${}^tP = P$, ${}^tB = B$, ${}^tC = C$ and

$$P \in M_m(F), \quad A, B, C, D \in M_l(F), \quad S, T \in M_{m\times l}(F).$$

The next lemma deals with the case of an elementary grading compatible with an involution defined by a skew-symmetric non-degenerate bilinear form.

Lemma 2. *Let $R = M_n(F)$, $n = 2k$, be the matrix algebra with involution $*$ defined by a skew-symmetric non-degenerate bilinear form. Let G be an abelian group and let R be equipped with* *an elementary involution G-grading defined by an n-tuple $(g_1, \dots, g_n)$. Then, after a renumbering, $g_1 g_{k+1} = \dots = g_k g_{2k}$, the involution $*$ acts as $X^* = (\Phi^{-1})^t X \Phi$ where*

$$\Phi = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

I is the $k \times k$ identity matrix, $R^{(-)}$ consists of all matrices of the type

$$\begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix}, \quad A, B, C \in M_k(F), \quad {}^tB = B, \quad {}^tC = C \tag{12}$$

while $R^{(+)}$ consists of all matrices of the type

$$\begin{pmatrix} A & B \\ C & {}^tA \end{pmatrix}, \quad A, B, C \in M_k(F), \quad {}^tB = -B, \quad {}^tC = -C. \tag{13}$$

Lemma 3. *Let $R = M_2(F)$ be a 2×2 matrix algebra endowed with an involution $*: R \to R$ corresponding to a symmetric or skew-symmetric non-degenerate bilinear form with the matrix Φ. The (-1)-grading of M_2 by $G = \langle a \rangle_2 \times \langle b \rangle_2$ is an involution grading if and only if one of the following holds*:

(1) *Φ is skew-symmetric,*

$$\Phi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad K(R,*) = \operatorname{Span}\{X_a, X_b, X_{ab}\}, \quad H(R,*) = \operatorname{Span}\{X_e\};$$

(2) *Φ is symmetric,*

$$\Phi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad K(R,*) = \operatorname{Span}\{X_a\}, \quad H(R,*) = \operatorname{Span}\{X_e, X_b, X_{ab}\};$$

(3) *Φ is symmetric,*

$$\Phi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad K(R,*) = \operatorname{Span}\{X_{ab}\}, \quad H(R,*) = \operatorname{Span}\{X_e, X_a, X_b\};$$

(4) *Φ is symmetric,*

$$\Phi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad K(R,*) = \operatorname{Span}\{X_b\}, \quad H(R,*) = \operatorname{Span}\{X_e, X_a, X_{ab}\}.$$

Notice that the involution in each case is already defined, say, in case (1) one has

$$(\alpha X_e + \beta X_a + \gamma X_b + \delta X_{ab})^* = \alpha X_e - \beta X_a - \gamma X_b - \delta X_{ab}.$$

Remark 2. If $R = M_n$ has a fine grading by a group G whose support is an elementary abelian 2-subgroup T then it is immediate from the previous lemma and a Remark 1 after Theorem 1 that R has a basis $\{X_t \mid t \in T\}$ such that $X_t X_u = \alpha(t,u) X_{tu}$ where $\alpha(t,u) = \pm 1$ and for each $u \in T$ we have $X_u^{-1} = {}^tX_u = \alpha(u,u) X_u$.

We can now formulate the most general result available earlier, which describes gradings on a matrix algebra with involution (a weaker form of [5, Theorem 2], which is not true). With our additional assumption that the involution respects the fine and the elementary components of the grading, the proof of [5] works without changes. We remark here that this condition is always satisfied provided that the Sylow 2-subgroup of G is cyclic.

Theorem 2. *Let $R = M_n(F) = \bigoplus_{g\in G} R_g$ be a matrix algebra over an algebraically closed field of characteristic zero graded by the group G and* Supp R *generates G. Suppose that $*: R \to R$ is a graded involution. Then G is abelian, and R as a G-graded algebra is isomorphic to the tensor product $R^{(0)} \otimes R^{(1)} \otimes \cdots \otimes R^{(k)}$ of a matrix subalgebra $R^{(0)}$ with elementary grading and $R^{(1)} \otimes \cdots \otimes R^{(k)}$ a matrix subalgebra with fine grading. Suppose further that both these subalgebras are invariant under the involution. Then $n = 2^k m$ and*

(1) *$R^{(0)} = M_m(F)$ is as in Lemma* 2 *if $*$ is symplectic on $R^{(0)}$ or as in Lemma* 1 *if $*$ is transpose on R^0;*
(2) *$R^{(1)} \otimes \cdots \otimes R^{(k)}$ is a $T = T_1 \times \cdots \times T_k$-graded algebra and any $R^{(i)}, 1 \leqslant i \leqslant k$, is $T_i \cong \mathbb{Z}_2 \times \mathbb{Z}_2$-graded algebra as in Lemma* 3.
(3) *A graded basis of R is formed by the elements $Y \otimes X_{t_1} \otimes \cdots \otimes X_{t_k}$, where Y is an element of a graded basis of $R^{(0)}$ and the elements X_{t_i} are of the type* (8)*, with $n = 2$, $t_i \in T_i$. The involution on these elements is given canonically by*

$$(Y \otimes X_{t_1} \otimes \cdots \otimes X_{t_k})^* = Y^* \otimes X_{t_1}^* \otimes \cdots \otimes X_{t_k}^* = \operatorname{sgn}(t)\big(Y^* \otimes X_{t_1} \otimes \cdots \otimes X_{t_k}\big),$$

where $Y \in R^{(0)}$, X_{t_i} are the elements of the basis of the canonical (-1)-grading of M_2, $i = 1, \ldots, k$, $t = t_1 \cdots t_k \in T$, $\operatorname{sgn}(t) = \pm 1$, *depending on the cases in Lemma* 3.

In the next two sections we describe the antiautomorphisms of graded matrix algebras in the general case, including that now we only assume that the base field is algebraically closed of characteristic different from 2.

4. Antiautomorphisms of graded matrix algebras

We start this section with a result about the structure of fine gradings of $R = M_n$ compatible with an antiautomorphism. This result is a far going generalization of [5, Lemma 2]. Any antiautomorphism φ of $R = M_n$ can always be written as

$$\varphi * X = \Phi^{-1}\big({}^t X\big)\Phi \tag{14}$$

where Φ is a non-degenerate matrix and $X \mapsto {}^t X$ is the ordinary transpose map. It is well known that φ is an involution if and only if Φ is either symmetric or skew-symmetric. Recall that in the case where Φ is symmetric, φ is called a *transpose involution* and if Φ is skew-symmetric then φ is called a *symplectic involution*.

Lemma 4. *Let $R = M_n(F) = \bigoplus_{t\in T} R_t$ be the $n \times n$-matrix algebra with an ε-grading, $T = \langle a\rangle_n \times \langle b\rangle_n$. Let also $\varphi : R \to R$ be an antiautomorphism of R defined by $\varphi * X = \Phi^{-1\,t}X\Phi$. If $\varphi * R_t = R_t$ for all $t \in T$ then $n = 2$, Φ coincides with the scalar multiple one of the matrices I, X_a, X_b or X_{ab} (see* (6)*).*

Proof. First we consider the φ-action on X_a. Since R_a is stable under φ,

$$\Phi^{-1\,t}X_a\Phi = \Phi^{-1}X_a\Phi = \alpha X_a$$

for some scalar $\alpha \neq 0$. Then

$$X_a\Phi X_a^{-1} = \alpha\Phi. \tag{15}$$

Since $X_a^n = I$, we obtain $\alpha^n = 1$, so that $\alpha = \varepsilon^j$ for some $0 \leqslant j \leqslant n-1$.

Denote by P the linear span of $I, X_a, \dots, X_a^{n-1}$. Then $R = P \oplus X_bP \oplus \cdots \oplus X_b^{n-1}P$ as a vector space and the conjugation by X_a acts on X_b^iP as the multiplication by ε^i. In particular, all eigenvectors with eigenvalue ε^j are in X_b^jP. It follows that $\Phi \in X_b^jP$, that is, $\Phi = X_b^jQ$ for some $Q \in P$.

Now we consider the action of φ on X_b:

$$\varphi * X_b = \Phi^{-1\,t}X_b\Phi = \Phi^{-1}X_b^{-1}\Phi = \gamma X_b,$$

that is, $X_b\Phi X_b = \mu\Phi$ with $\mu = \gamma^{-1} \neq 0$. If we write $Q = \sum \alpha_i X_a^i$ then

$$X_b\Phi X_b = X_b^j\sum_i \alpha_i X_bX_a^iX_b = X_b^j\sum_i \alpha_i' X_a^iX_b^2 = \mu\Phi = \mu X_b^j\sum_i \alpha_i X_a^i. \tag{16}$$

In this case $X_b^j\sum_i \alpha_i' X_a^iX_b^2 = \mu X_b^j\sum_i \alpha_i X_a^i$ where the scalars α_i' can be explicitly computed using (15). Since the degrees in X_a, X_b define the degrees in the T-grading, we can see that (16) immediately implies $X_b^2 = I$, i.e. $n = 2$.

As we have shown before, (15) implies $\Phi = X_b^jQ$ with $Q = \alpha_0I + \alpha_1X_a$. Since $n = 2$, the argument following (15) applies if we change a and b places so that $\Phi = X_a^k(\beta_0I + \beta_1X_b)$. Comparing these two expressions we obtain that Φ must be one of I, X_a, X_b, or X_{ab}, up to a scalar multiple. $\square$

Now we make few remarks about the structure of elementary gradings on $M_n(F)$. Recall that a grading $M_n = R = \bigoplus_{g\in G} R_g$ is elementary if there exists an n-tuple $\tau = (g_1, \dots, g_n) \in G^n$ such that the matrix units E_{ij}, $1 \leqslant i, j \leqslant n$ are homogeneous and $E_{ij} \in R_g \Leftrightarrow g = g_i^{-1}g_j$. Elementary gradings arise from the gradings on vector spaces. Let $V = \mathrm{Span}\{v_1, \dots, v_n\}$ be a graded vector space and $\{v_1, \dots, v_n\}$ is a graded basis such that $\deg v_i = g_i^{-1}$. Then any E_{ij} is a homogeneous linear transformation of V and $\deg E_{ij} = g_i^{-1}g_j$. Any permutation $v_i \mapsto v_{\sigma(i)}$ of basis elements induces a graded automorphism of $M_n = \mathrm{End}\, V$ and the corresponding permutation on the n-tuple $\tau = (g_1, \dots, g_n)$. Hence we may permute the components of τ. Now suppose τ has the form

$$\tau = (\underbrace{t_1, \dots, t_1}_{p_1}, \dots, \underbrace{t_m, \dots, t_m}_{p_m})$$

with $t_1, \dots, t_m$ pairwise distinct. In this case the identity component R_e is isomorphic to $A_1 \oplus \cdots \oplus A_m$ where $A_i \cong M_{p_i}$, for any $i = 1, \dots, m$ and consists of all block-diagonal matrices

$$X = \mathrm{diag}\{X_1, X_2, \dots, X_m\}$$

where X_j is a $p_j \times p_j$-matrix. Moreover, for any $i \neq j$ the subspace $A_i R A_j$ is graded and all $X \in A_i R A_j$ are of degree $t_i^{-1} t_j$ in the G-grading. As an easy consequence of this realization we obtain

Lemma 5. *Let $R = M_n = \bigoplus_{g \in G} R_g$ be a matrix algebra with an elementary G-grading. If R_e is simple then the grading is trivial. If R_e is the sum of two simple components, $R_e = A_1 \oplus A_2$, then there exists $g \in G$, $g \neq e$, such that $A_1 R A_2 \subseteq R_g$.*

Now we consider a matrix algebra $R = M_n$ with an involution $* : R \to R$ preserving R_e. Permuting $t_1, \dots, t_m$ in τ we may assume that for any $1 \leqslant j \leqslant m-1$ either $A_j^* = A_j$ or $A_j^* = A_{j+1}$, $A_{j+1}^* = A_j$. In the first case A_j is simple and $A_j R A_j = A_j$. In the second case $B = A_j \oplus A_{j+1}$ is not simple but $*$-simple and $A_j \simeq A_{j+1}$, i.e. A_j and A_{j+1} are matrix algebras of the same size s. It is convenient to consider the subalgebra BRB as a subset of all matrices

$$\mathrm{diag}\{0, \dots, 0, X, 0, \dots, 0\}$$

where X is $2s \times 2s$-matrix on the respective position.

Next we consider a general G-graded matrix algebra $R = M_n$. According to Theorem 1, $R = C \otimes D$ where $C \otimes I$ is a matrix algebra with elementary grading while $I \otimes D$ is an algebra with fine grading.

Lemma 6. *Let $R = C \otimes D = \bigoplus_{g \in G} R_g$ be a G-graded matrix algebra with an elementary grading on C and a fine grading on D. Let $\varphi : R \to R$ be an antiautomorphism on R preserving G-grading. Let also φ acts as an involution on the identity component R_e i.e. $\varphi^2|_{R_e} = \mathrm{Id}$. Then*

(1) *$C_e \otimes I$ is φ-stable where I is the unit of D and hence φ induces an involution $*$ on C_e;*
(2) *there are subalgebras $B_1, \dots, B_k \subseteq C_e$ such that $C_e = B_1 \oplus \dots \oplus B_k$, $B_1 \otimes I, \dots, B_k \otimes I$ are φ-stable and all $B_1, \dots, B_k$ are $*$-simple algebras;*
(3) *φ acts on $R_e = C_e \otimes I$ as $\varphi * X = S^{-1t} X S$ where $S = S_1 \otimes I + \dots + S_k \otimes I$, $S_i \in B_i C B_i$ and $S_i = I_{p_i}$ if B_i is $p_i \times p_i$-matrix algebra with transpose involution, $S_i = \begin{pmatrix} 0 & I_{p_i} \\ -I_{p_i} & 0 \end{pmatrix}$ if B_i is $2p_i \times 2p_i$-matrix algebra with symplectic involution or $S_i = \begin{pmatrix} 0 & I_{p_i} \\ I_{p_i} & 0 \end{pmatrix}$ if $B_i \simeq M_{p_i} \oplus M_{p_i}$;*
(4) *the centralizer of $R_e = C_e \otimes I$ in R can be decomposed as $Z_1 D_1 \oplus \dots \oplus Z_k D_k$ where $D_1, \dots, D_k$ are φ-stable graded subalgebras of R isomorphic to D and $Z_i = Z_i' \otimes I$ where Z_i' is the center of B_i;*
(5) *D as a graded algebra is isomorphic to $M_2 \otimes \dots \otimes M_2$ where any factor M_2 has the fine (-1)-grading.*

Proof. From Theorem 1 it follows that the identity component R_e equals to $C_e \otimes I$. Since R_e is φ-stable and $\varphi^2 = \mathrm{Id}$ on R_e, the φ-action induces an involution $*$ on C_e. Since C_e is semisimple it is a direct sum of $*$-simple algebras,

$$C_e = B_1 \oplus \dots \oplus B_k. \tag{17}$$

Now (1), (2) and (3) follows from the classification of involution simple algebras [6].

Denote by $e_1, \dots, e_k$ the units of $B_1, \dots, B_k$, respectively. Clearly, the centralizer Z of C_e in C is equal to $Z_1' \oplus \dots \oplus Z_k'$ where Z_i' is the center of B_i and the centralizer of R_e in R coincides with $Z \otimes D = Z_1 D_1 \oplus \cdots \oplus Z_k D_k$ where $Z_i = Z_i' \otimes I$ and $D_i = e_i \otimes D$. Obviously the map $e_i \otimes d \mapsto d \in D$ is an isomorphism of graded algebras. Hence for proving (4) we only need to check that all $D_1, \ldots, D_k$ are φ-stable.

We fix $1 \leqslant i \leqslant k$ and consider $R' = (e_i \otimes I)R(e_i \otimes I) = C' \otimes D$ where $C' = e_i C e_i$ and $D_i = e_i \otimes D$ is a graded subalgebra of R'. Then R' is φ-stable since $\varphi * (e_i \otimes I) = e_i \otimes I$. Also $R'_e = C'_e \otimes I$ with $C'_e = B_i$. If B_i is simple then $C' = C'_e$ by Lemma 5. In this case D_i is φ-stable since φ preserves R'_e and D_i is the centralizer of R'_e in R'.

Now suppose $B_i = A_1 \oplus A_2$ is the sum of two matrix algebras. First we will show that $C' \otimes I$ is a φ-stable graded subalgebra of R. Denote by f_1, f_2 the units of A_1 and A_2 respectively. Then $f_1, f_2 \in R_e$ and φ permutes f_1, f_2. Moreover, $f_1 C' f_2 \otimes I$ and $f_2 C' f_1 \otimes I$ are graded subspaces. Since $\varphi * f_1 \otimes I = f_2 \otimes I, \varphi * f_2 \otimes I = f_1 \otimes I$ we have

$$\varphi * \left(f_1 C' f_2 \otimes I\right) \subseteq (f_1 \otimes I)R(f_2 \otimes I) = f_1 C' f_2 \otimes D.$$

On the other hand, since by Lemma 5 there is $g \in G$ such that $f_1 C' f_2 \subseteq C'_g$ for some $g \in G$ it follows that

$$\varphi * \left(f_1 C' f_2 \otimes I\right) \subseteq R_g.$$

Suppose now that $x \in f_1 C' f_2$, $y \in D$, x and y are homogeneous, $\deg x = g, \deg y = h$. Then $\deg(x \otimes y) = g$ if and only if $h = e$ that is $y = \lambda I$, for some scalar λ. It follows that $f_1 C' f_2 \otimes I$ is a φ-stable subspace. Similarly, $\varphi * f_2 C' f_1 \otimes I \subseteq f_2 C' f_1 \otimes I$, hence $C' \otimes I$ is φ-stable.

Now from the decomposition $R' = C' \otimes D$ it follows that $D_i = e_i \otimes D$ is a φ-stable graded subalgebra.

For proving (5) we remark that D is isomorphic to, say, D_1 as a G-graded algebra and D_1 is φ-stable. So, it is enough to prove that D_1 is the tensor product of several copies of M_2. We decompose D_1 as the tensor product

$$D_1 \simeq R_1 \otimes \cdots \otimes R_m$$

where each R_i is a matrix algebra M_{n_i} with a fine ε_i-grading. Recall that $H = \operatorname{Supp} D_1 = H_1 \times \cdots \times H_m$ where $H_i \simeq \mathbb{Z}_{n_i} \times \mathbb{Z}_{n_i} = \operatorname{Supp} R_i$, $1 \leqslant i \leqslant m$. Now since the G-grading on D_1 is φ-stable and

$$R_i = \bigoplus_{h \in H_i} (D_1)_h$$

it follows that $\varphi * R_i = R_i$. Since any antiautomorphism φ on a matrix algebra acts as $\varphi * X = \Phi^{-1} X \Phi$, we can apply Lemma 4. Now the proof of our lemma is complete. □

In what follows we discuss the canonical form of the involution φ on the whole of R. As mentioned, the φ-action on R is defined by

$$\varphi * A = \Phi^{-1\,t} A \Phi$$

for some matrix Φ. First let $A \in R_e$. Consider the decomposition $C_e = B_1 \oplus \cdots \oplus B_k$ found in Lemma 6. Then $A = A_1 \otimes I + \cdots + A_k \otimes I$ with $A_i \in B_i$, $1 \leqslant i \leqslant k$. By Lemma 6 φ acts on A as

$$\varphi * A = S^{-1t}AS.$$

Hence the matrix ΦS^{-1} commutes with ${}^tA$ for any $A \in R_e$, that is ΦS^{-1} is an element of the centralizer of R_e in R. Applying claim (4) of Lemma 6 we obtain

$$\Phi = S_1Y_1 \otimes Q_1 + \cdots + S_kY_k \otimes Q_k \tag{18}$$

where $Q_i \in D$, $Y_i \in Z_i'$, $1 \leqslant i \leqslant k$. Compute now the action of φ^2 on an arbitrary $A \in R$:

$$\varphi^2 * A = \varphi * \left(\Phi^{-1t}A\Phi\right) = \Phi^{-1t}\left(\Phi^{-1t}A\Phi\right)\Phi = \left({}^t\Phi^{-1}\Phi\right)^{-1}A\left({}^t\Phi^{-1}\Phi\right).$$

Set $P = {}^t\Phi^{-1}\Phi$. Note that for any $T_i, T_i' \in B_iCB_i$ and $Q_i, Q_i' \in D$, $i = 1, \dots, k$, the relation

$$\left(\sum_i T_i \otimes Q_i\right)\left(\sum_i T_i' \otimes Q_i'\right) = \sum_i T_iT_i' \otimes Q_iQ_i'$$

holds.

We compute the value of P:

$$P = {}^t\Phi^{-1}\Phi = \sum_{i=1}^{k} {}^t(S_iY_i)^{-1}S_iY_i \otimes {}^tQ_i{}^{-1}Q_i = \sum_i {}^tS_i{}^{-1t}Y_i{}^{-1}S_iY_i \otimes {}^tQ_i{}^{-1}Q_i. \tag{19}$$

Lemma 7. *All Q_i in* (19) *satisfy* ${}^tQ_i{}^{-1}Q_i = \pm I$.

Proof. Obviously it is sufficient to prove the relation

$$e_i \otimes {}^tQ_i^{-1}Q_i = \pm e_i \otimes I$$

in $D_i = e_i \otimes D$. Recall that D_i is φ-stable (see Lemma 6) and φ acts on $e_i \otimes X$, $X \in D$ as

$$\varphi * (e_i \otimes X) = \Phi^{-1t}(e_i \otimes X)\Phi = (S_iY_i)^{-1}(e_i)(S_iY_i) \otimes Q_i^{-1t}XQ_i = e_i \otimes Q_i^{-1t}XQ_i$$

i.e. φ-action induces an antiautomorphism $e_i \otimes X \mapsto e_i \otimes Q_i^{-1t}XQ_i$ on D_i. By Lemma 6(5) D_i is the tensor product $M_2^{(1)} \otimes \cdots \otimes M_2^{(r)}$ of 2×2-matrix algebras with fine grading. As in the proof of Lemma 6(5) we remark that all factors are φ-stable. Fix a factor $M_2^{(j)}$ and consider the action of φ on $M_2^{(j)}$. Then

$$\varphi * Y = T_j^{-1t}YT_j$$

and by Lemma 4 $T_j = I, X_a, X_b$ or X_{ab}. In particular, ${}^tT_j{}^{-1}T_j = \pm I_2$ where I_2 is 2×2 identity matrix. Since $e_i \otimes Q_i^{-1}{}^tXQ_i = T^{-1}(e_i \otimes {}^tX)T$ for all $X \in D$ where $T = T_1 \otimes \cdots \otimes T_r$ it follows that $e_i \otimes Q_i = \lambda T$ for some non-zero scalar λ. Hence $e_i \otimes Q_i$ satisfies a similar relation $e_i \otimes {}^tQ_i^{-1}Q_i = \pm I$. $\square$

We summarize what was done in this section as follows.

Proposition 1. *Suppose $R = M_n(F)$ is the full matrix algebra over an algebraically closed field of characteristic different from 2, graded by a finite abelian group G. Let φ be a G-graded antiautomorphism of R whose restriction to the identity component R_e is of order two. Then φ can be given as $\varphi * X = \Phi^{-1}{}^tX\Phi$ where*

$$\Phi = S_1Y_1 \otimes Q_1 + \cdots + S_kY_k \otimes Q_k \tag{20}$$

where S_i and Y_i are described in Lemma 6 and each $Q_i \in e_i \otimes D$ is such that ${}^tQ_i^{-1}Q_i = \pm I$.

5. Involutions on group graded matrix algebras

In this section we preserve the notation introduced earlier except that we write $\varphi * X = X^*$. Our aim is to describe involutions on group graded matrix algebras. We will start with Eq. (18), in which we additionally know from Lemma 7 that ${}^tQ_i{}^{-1}Q_i = \pm I$. Let $g^{(p)}$ mean $\underbrace{g, \ldots, g}_{q}$. Our aim is to prove the following.

Theorem 3. *Let $\varphi : X \to \Phi^{-1}{}^tX\Phi$ be an involution compatible with a grading of a matrix algebra R by a finite abelian group G. Then, after a G-graded conjugation, we can reduce Φ to the form*

$$\Phi = S_1 \otimes X_{t_1} + \cdots + S_k \otimes X_{t_k} \tag{21}$$

where S_i is one of the matrices I, $\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$, or $\left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$ and each X_{t_i} is a matrix spanning D_{t_i}, $t_i \in T$. The defining tuple of the elementary grading on C should satisfy the following condition. We assume that the first l of summands in (21) correspond to those B_i in (17) which are simple and the remaining $k - l$ to B_i which are not simple. Let the dimension of a simple B_i be equal to p_i^2 and that of a non-simple B_j to $2p_j^2$. Then the defining tuple has the form

$$\left(g_1^{(p_1)}, \ldots, g_l^{(p_l)}, \left(g'_{l+1}\right)^{(p_{l+1})}, \left(g''_{l+1}\right)^{(p_{l+1})}, \ldots, \left(g'_k\right)^{(p_k)}, \left(g''_k\right)^{(p_k)}t\right), \tag{22}$$

$$g_1^2t_1 = \cdots = g_l^2t_l = g'_{l+1}g''_{l+1}t_{l+1} = \cdots = g'_kg''_kt_k. \tag{23}$$

Additionally, if φ is a transpose involution then each i, S_i is symmetric (skew-symmetric) at the same time as X_{t_i}, for any $i = 1, \ldots, k$. If φ is a symplectic involution, then each S_i is symmetric (skew-symmetric) if and only if the respective X_{t_i} is skew-symmetric (symmetric), $i = 1, \ldots, k$.

Conversely, if we have a grading by a group G on a matrix algebra R defined by a tuple as in (22), for the component C with elementary grading, and by an elementary abelian 2-subgroup T as the support of the component D with fine grading and all of the above conditions are satisfied then (21) defines a graded involution on R.

Proof. Choose $X_u \in D_u$, $u \in T$, and consider $(I \otimes X_u)^*$. Since $\mathrm{Span}\{e_1, \dots, e_k\} \otimes D$ is invariant with respect to φ we must have $(I \otimes X_u)^* = \alpha_1 e_1 \otimes X_u + \dots + \alpha_k e_k \otimes X_u$, for some scalars $\alpha_1, \dots, \alpha_k$. Now by Proposition 1 we have

$$
\begin{aligned}
(I \otimes X_u)^* &= \Phi^{-1t}(I \otimes X_u)\Phi \\
&= \left(Y_1^{-1}S_1^{-1} \otimes Q_1^{-1} + \dots + Y_k^{-1}S_k^{-1} \otimes Q_k^{-1}\right) \\
&\quad \times \left(e_1 \otimes {}^tX_u + \dots + e_k \otimes {}^tX_u\right)(S_1Y_1 \otimes Q_1 + \dots + S_kY_k \otimes Q_k) \\
&= e_1 \otimes Q_1^{-1t}X_u Q_1 + \dots + e_k \otimes Q_k^{-1t}X_u Q_k \\
&= e_1 \otimes \alpha_1 X_u + \dots + e_k \otimes \alpha_k X_u.
\end{aligned}
$$

It follows then that for each $i = 1, \dots, k$ we must have $Q_i^{-1t}X_u Q_i = \alpha_i X_u$ for all $X_u \in e_i \otimes D_u$. As a result, the mapping $X \to Q_i^{-1t}X Q_i$ is a graded involution of an algebra with fine grading $e_i \otimes D$. By Lemma 4 this mapping must have the form $X_u \mapsto X_{t_i}^{-1}X_u X_{t_i}$, for some $t_i \in T$. This allows us to conclude that our matrix Φ can be chosen in the form

$$\Phi = S_1Y_1 \otimes X_{t_1} + \dots + S_kY_k \otimes X_{t_k} \tag{24}$$

where each Y_i is in the center of B_i. We have that $Y_i = \lambda_i e_i$ in every case where B_i is simple and $Y_i = \xi_i e_i' + \zeta_i e_i''$ in every case where B_i is not simple. Here e_i', e_i'' are the identities of simple components of B_i and $e_i = e_i' + e_i''$. Also any X_{t_i} is either symmetric or skew-symmetric.

Now let us check the conditions (23). This is done on case-by-case basis. If $U = e_iUe_j \in C$, $1 \leqslant i, j \leqslant l$ then $\deg U = g_i^{-1}g_j$. Also, using (18) we obtain that $U^* = e_jY_j^{-1}S_j^{-1t}US_iY_ie_i \otimes X_{t_j}^{-1}X_{t_i}$ which is of degree $g_j^{-1}g_it_jt_i$ (we recall that by Lemma 4 all elements in T are of order 1 or 2). Therefore, we have an equality $g_i^2t_i = g_j^2t_j$. If $U = e_iUe_j'$, $1 \leqslant i \leqslant l$, $l+1 \leqslant j \leqslant k$, then $\deg U = g_i^{-1}g_j'$. We also have that $U^* = e_j''Y_j^{-1}S_j^{-1t}US_iY_ie_i \otimes X_{t_j}^{-1}X_{t_i}$, which is of degree $(g_j'')^{-1}g_it_it_j$. It follows then that $g_j'g_j''t_j = g_i^2t_i$, also in accordance with (23). Finally, if $U = e_i'Ue_j''$, $l+1 \leqslant i, j \leqslant k$, then $\deg U = (g_i')^{-1}g_j''$ while $U^* = e_j'Y_j^{-1}S_j^{-1t}US_iY_ie_i'' \otimes X_{t_j}^{-1}X_{t_i}$. Therefore, $\deg U^* = (g_j')^{-1}g_i''t_it_j$. It then follows that $g_j'g_j''t_j = g_i'g_i''t_i$, as required. The remaining three cases are in symmetry with the previous ones and produce the same results. By the way, these calculations also show that if a mapping is given by (14) where Φ is as in (21) satisfying (23) that this mapping is G-graded.

Now we need to eliminate $Y_1, \dots, Y_k$ from the formula for Φ. Recall the decomposition $R_e = B_1 \oplus \dots \oplus B_k$ from Lemma 6. Each summand in (24) correspond to one of subalgebras B_i. Notice that if we apply an inner automorphism to R then Φ is changed as a matrix of a bilinear form. If this automorphism is a conjugation by a matrix P with identity grading then it is an isomorphism of graded algebras. In this isomorphic copy of $R = M_n$ the matrix of the involution φ will take the form of $\Phi' = {}^tP\Phi P$. We build P as $P = P_1 \otimes I + \dots + P_k \otimes I$ where $P_i \in B_iCB_i$, for each $i = 1, \dots, k$. If B_i is simple then $Y_i = \xi_i I$. If B_i is not simple then $Y_i = \zeta_i e_i' + \xi_i e_i''$. Here e_i', e_i'' are the identities of simple components of B_i and $e_i = e_i' + e_i''$. In the matrix form, $Y_i = \left(\begin{smallmatrix} \zeta_i I_{p_i} & 0 \\ 0 & \xi_i I_{p_i} \end{smallmatrix}\right)$. Also, $S_i = \left(\begin{smallmatrix} 0 & I_{p_i} \\ I_{p_i} & 0 \end{smallmatrix}\right)$.

Notice that since φ is an involution, ${}^t\Phi^{-1}\Phi = \omega I$ where $\omega = \pm 1$. In other words, $\Phi = \omega {}^t\Phi$. Now

$$\begin{aligned}{}^t\Phi &= Y_1{}^tS_1 \otimes {}^tX_{t_1} + \cdots + Y_k{}^tS_k \otimes {}^tX_{t_k} \\ &= Y_1{}^tS_1 \otimes \alpha(t_1, t_1)X_{t_1} + \cdots + Y_k{}^tS_k \otimes \alpha(t_k, t_k)X_{t_k}.\end{aligned}$$

Let us set $P_i = \frac{1}{\sqrt{\xi_i}} e_i$. If B_i is simple then

$$S_i' = {}^tP_i S_i Y_i P_i = P_i S_i Y_i P_i = S_i.$$

If B_i is not simple then it follows from ${}^t\Phi = \omega\Phi$ that $\zeta_i = \xi_i \omega\alpha(t_i, t_i)$. Then

$$S_i' = {}^tP_i S_i Y_i P_i = P_i S_i Y_i P_i = \frac{1}{\xi_i}\begin{pmatrix} 0 & \xi_i I_{p_i} \\ \xi_i \omega\alpha(t_i, t_i) I_{p_i} & 0 \end{pmatrix} = \begin{pmatrix} 0 & I_{p_i} \\ \omega\alpha(t_i, t_i) I_{p_i} & 0 \end{pmatrix}.$$

For example, if φ is a transpose involution, that is, Φ is symmetric, then $\omega = 1$ and the conjugation by P as above reduces Φ to the form

$$\Phi = S_1' \otimes X_{t_1} + \cdots + S_k' \otimes X_{t_k}$$

with

$${}^t\Phi = {}^tS_1' \otimes \alpha(t_1, t_1)X_{t_1} + \cdots + {}^tS_k' \otimes \alpha(t_k, t_k)X_{t_k}$$

so that, according to Remark 2, each S_i' is symmetric if and only if X_{t_i} is symmetric, as claimed. If φ is a symplectic involution then $\omega = -1$ and using the same equations implies that S_i' is symmetric if and only if X_{t_i} is skew-symmetric.

The converse in the above theorem is immediate. □

Now, for the determination of the gradings on simple matrix Jordan and Lie algebras, it is important to be able to compute the sets of symmetric and skew-symmetric elements of $R = M_n(F)$ under the involution just computed. A very simple remark is as follows:

$$\begin{aligned} H(R, *) &= \operatorname{Span}\{A + A^* \mid A \text{ from a spanning set of } R\}, \\ K(R, *) &= \operatorname{Span}\{A - A^* \mid A \text{ from a spanning set of } R\}. \end{aligned}$$

If $A = e_i U e_j \otimes X_u$ then $A^* = e_j S_j^{-1}{}^tU S_i e_i \otimes {}^tX_{t_j}{}^tX_u X_{t_i}$. If we perform obvious calculations we obtain the sets of symmetric and skew-symmetric elements of φ in the following form

$$H(R, *) = \operatorname{Span}\{e_i U e_j \otimes X_u + e_j S_j{}^tU S_i e_i \otimes X_{t_j}{}^tX_u X_{t_i}\} \tag{25}$$

where $1 \leqslant i, j \leqslant k$, $u \in T$, and $U = e_i U e_i \in C$.

Quite similarly,

$$K(R, *) = \operatorname{Span}\{e_i U e_j \otimes X_u - e_j S_j{}^tU S_i e_i \otimes X_{t_j}{}^tX_u X_{t_i}\} \tag{26}$$

where $1 \leqslant i, j \leqslant k$, $u \in T$, and $U = e_i U e_i \in C$. Here we simultaneously replaced S_j^{-1} and $X_{t_j}^{-1}$ by S_j and X_{t_j}.

Incidentally, this gives a canonical form for the simple graded Jordan algebras of the types $H(M_n, *)$ where $*$ is either transpose or symplectic involution (formula (25)), or a simple Lie algebra of the type B_l, $l \geqslant 2$, C_l, $l \geqslant 3$, or D_l, $l \geqslant 5$ (formula (26)), of which the forms suggested in [5] are a particular case.

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