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Countably Generated Modules Over Complete Discrete Valuation Rings

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1. INTRODUCTION

Let R be a complete discrete valuation ring, and (c, R, k) the class of all countably generated reduced R modules of torsion-free rank k. The purpose of this paper is to classify the members of (c, R, k), up to isomorphism, for all R and k.

In 1933, Ulm [8] found his celebrated invariants, which classify all countable reduced primary abelian groups. But, of course, such groups can be viewed in a natural manner as reduced modules over a complete discrete valuation ring, namely, the ring of p-adic integers. Indeed, the Ulm invariants classify (c, R, k), in case $k = 0$. In 1951, Kaplansky and Mackey [5] discovered other invariants, called height equivalences in this paper, which together with the Ulm invariants classify (c, R, k), in case $k = 1$. In 1961, Rotman and Yen [7] succeeded in generalizing the technique of Kaplansky and Mackey to classify (c, R, k), in case k is finite. In this paper we are able to push this technique to the point of classifying (c, R, k) for all values of k, by proving the following theorem.

THEOREM. *Let R be a complete discrete valuation ring, and M and M' countably generated reduced R modules. Then M and M' are isomorphic if, and only if, they have the same Ulm invariants and the same height equivalences.*

Since an R module decomposes into a direct sum of its unique maximal divisible submodule and a reduced submodule (unique up to isomorphism), and since the classification of all divisible R modules is completely known, we obtain with the above theorem a classification of all countably generated modules over complete discrete valuation rings.

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2. Basic Notations and Terminology

For the most part, we will follow the usual notations and terminology developed in the study of abelian groups (see [1] or [4]). Hereafter, R denotes an arbitrary discrete valuation ring, while p is a fixed but arbitrary prime of R. Then R is simply a local principal ideal domain with a prime element, p. Hence, (p) is the unique prime, in fact, a unique maximal, ideal of R, and all primes of R are associates of p. Since p is a uniformizing parameter [2], each nonzero element $a \in R$ can be written uniquely as up^n, where u is a unit of R and n is a nonnegative integer. We write $a \equiv up^n$ to indicate this unique representation. R is said to be complete if it is complete in its p-adic topology, equivalently, if the metric d on R, induced by $d(a, b) = e^{-n}$ if $a - b \equiv up^n$, is complete. Q will denote the quotient field of R and K the residue field $R/(p)$.

We adopt the convention that the symbol ∞ is greater than any ordinal. As usual, ω is the first infinite ordinal. For the sake of brevity, we will call α a *sordinal* if α is an ordinal or the symbol ∞. Let $a = \{a_i : i < k\}$ with $k \leqslant \omega$. Then $a = \{a_0, ..., a_i, ... : i < k\}$ with the convention that a is an empty set if $k = 0$. We shall view a as an ordered set, a sequence of k terms, a row vector, or a column vector as the situation may demand. Furthermore, for any $0 \leqslant r \leqslant \omega$, $a(r)$ will denote the ordered subset $\{a_i \in a : i < r\}$. Notice that $a = a(r)$ if $k \leqslant r$, and $a(0)$ is an empty set.

Throughout this paper, M and M' will denote reduced R modules. As usual, we define, for every element $r \in R$, $M[r] = \{x \in M : rx = 0\}$ and $rM = \{rx : x \in M\}$. For each sordinal α, the submodule $p^\alpha M$ is defined inductively by $p^\alpha M = \cap \{p(p^\beta M) : \beta < \alpha\}$. (Of course $p^0 M = M$.) We write $h(x) = \alpha$, if $x \in p^\alpha M \backslash p^{\alpha+1} M$ for some ordinal α and $h(x) = \infty$, otherwise. The sordinal $h(x)$ will be called the height of x. Let ϕ be a mapping from a subset S of M into M'. Then ϕ is said to be height preserving on S if $h(x) = h[\phi(x)]$ for all $x \in S$ (where the heights are computed in M and M'. respectively). If S is a subset of M, then $[S]$ denotes the submodules generated by S. Finally, $\oplus$ indicates a direct sum.

3. The Invariants

Let R and p be as in the preceding section, and M a reduced R module. For each ordinal α, the factor module $(p^\alpha M)[p]/(p^{\alpha+1}M)[p]$ can be viewed in a natural way as a vector space over the residue field K. The uniquely determined dimension of the factor space is the α-th Ulm invariant of M, denoted by $f_M(\alpha)$; and f_M, a cardinal-valued function on the class of all ordinals, will be called the Ulm invariant of M. Clearly, f_M is an invariant of M.

We now define another kind of invariant. Let k be a fixed but arbitrary nonzero ordinal not exceeding ω. We let R^k designate the direct sum of k copies of R. Every element $a \in R^k$ is a k-dimensional vector of the form $\{a_i : i < k\}$. Of course $a_i = 0$ for almost all i, in case $k = \omega$. Next, define $m^k(Q)$ to be the collection of all $k \times k$ row-finite matrices over the field Q. A matrix $c \in m^k(Q)$ is said to be integral if all components of c are elements of R. Identity, diagonal and nonsingular matrices are defined as usual even in case $k = \omega$.

Let $f(R^k)$ be the class of all sordinal-valued functions on R^k, and suppose $g, g' \in f(R^k)$. Define $g \sim g'$ to mean that there is a fixed nonsingular integral matrix $c \in m^k(Q)$ and a fixed nonsingular diagonal integral matrix $d \in m^k(Q)$ such that $g(ac) = g'(ad)$ for all $a \in R^k$. It is routine to prove that the relation "$\sim$" is an equivalence relation on $f(R^k)$.

A torsion-free basis (or simply basis) of an R module, M, is a maximal linearly independent subset of M. A basis of M is not necessarily unique, but all bases of M have the same cardinality which is called the torsion-free rank (or simply rank) of M, and is denoted by $r(M)$. The rank of M is precisely the dimension of the vector space $Q \otimes_R M$ over Q. Suppose that $r(M)$ is non-zero and at most countable, and set $k = r(M)$. Let $y = \{y_i : i < k\}$ be an ordered basis of M. The basis y defines a sordinal-valued function g on R^k by $g(a) = h(ay) = h(\sum \{a_i y_i : i < k\})$ for all $a \in R^k$. We will call g the *height function* of the basis y. Clearly, $g \in f(R^k)$. Let $y' = \{y_i' : i < k\}$ be another ordered basis of M, and g' its height function. Utilizing the maximality of bases, we can show that $g \sim g'$. Thus, M determines uniquely an equivalence class of $f(R^k)$, which we will call the *height equivalence* of M and denote by $h(M)$. In addition, we adopt the convention that $h(M) = \infty$ if $r(M) = 0$. Obviously, $h(M)$ is an invariant of M. For later use we state the following fact whose proof is a trivial generalization of Lemma 4.2 in [6]. Let M and M' be countably generated reduced R modules having the same ranks. Then M and M' have the same height equivalences if, and only if, there are ordered bases of y and y' of M and M', respectively, with a height preserving isomorphism π from the submodule $[y]$ onto the submodule $[y']$, such that $\pi(y_i) = y_i'$ for all $y_i \in y$ and $y_i' \in y'$.

4. The Extension Lemma

In this section we consider an extension lemma that is crucial in the proof of our theorem. Hereafter, we assume that R is complete. Rotman and Yen [7] stated and utilized an incorrect fact that an arbitrary sequence in R either contains a convergent subsequence or contains a subsequence whose terms are of the form $u_i p^n$, where n is a fixed nonnegative integer and u_i's are incongruent units of R (that is, the difference of any two u_i's is also a unit). This is true if R is, for example, a ring of p-adic integers, but generally untrue. However, it can be proved that an arbitrary sequence in R either contains a convergent subsequence or contains a subsequence whose terms are of the form $a_{i+1} = a_0 + b_i p^n$ such that, for all $i < \omega$, $b_{i+1} - b_i$ is a unit of R and n is a fixed nonnegative integer.

Let M be a reduced R module and N a submodule of M. Then, following Hill [3], we say that N is nice in M if for each element $x \in M \backslash N$ the coset $x + N$ contains an element t such that $h(s) \leqslant h(t)$ for all $s \in x + N$. In this case, t is said to be proper with respect to N. We can now establish the following important lemma which was first proved in the rank one case by Kaplansky and Mackey ([5], Lemma 2) and then generalized incorrectly by Rotman and Yen ([7], Lemma 2).

LEMMA 4.1. *Let M be a reduced R module. If N is a finitely generated submodule of M, then N is nice in M.*

Proof. Since N is a finitely generated module over a principal ideal domain R, N is a direct sum of finitely many cyclic modules. Hence, we can write $N = \bigoplus \{[t_i] : i < s + 1\}$ for some nonnegative integer $s + 1$. We use induction on $s + 1$. First, suppose that $s + 1 = 0$. Then N is nice in M, since $N = 0$. Next, assuming the induction hypothesis, we suppose that $s + 1$ is a positive integer. Take any element $x \in M \backslash N$. It suffices to show that there is an element $t \in x + N$ that is proper with respect to N. If only finitely many different heights occur in $x + N$, let t be simply an element in the coset $x + N$ of the greatest height. Assume then that infinitely many different heights occur in the coset $x + N$. It is easy to see that no more than countably many different heights can occur. Hence, there is a sequence $\{z_i : i < \omega\}$ in $x + N$, such that $\{\alpha_i = h(z_i) : i < \omega\}$ is a nondecreasing sequence of ordinals and for each height α that occurs in $x + N$, there is an m such that $\alpha \leqslant \alpha_m$. Write $z_i = x + \sum \{a_{ij} t_j : j < s + 1\}$ for all $i < \omega$.

Case 1

A sequence $\{a_{ij} : i < \omega\}$ contains no convergent subsequence.

Without loss of generality, we may assume that $\{a_{is} : i < \omega\}$ contains no convergent subsequence. For brevity, let us write a_i for a_{is} for all $i < \omega$. From our remark at the beginning of this section, we can write, dropping to a suitable subsequence of $\{z_i : i < \omega\}$ if necessary, $a_{i+1} = a_0 + b_i p^n$ in such a way that, for all $i < \omega$, n is a fixed nonnegative integer and $b_{i+1} - b_i$ is a unit of R. For all $i < \omega$ define $y_i = (b_{i+1} - b_i)^{-1} (b_{i+1} z_i - b_i z_{i+1})$. Notice that $h(y_i) \geqslant \alpha_i$ and that $\{y_i : i < \omega\}$ is a sequence in the coset $x' + N'$, where $x' = x + a_0 t_s$ and $N' = \bigoplus \{[t_i] : i < s\}$. By our induction hypothesis,

$x' + N'$ contains an element t that is proper with respect to N'. Clearly, $t \in x + N$ and $h(t) \geqslant \alpha_i$ for all i.

Case 2

There is a sequence $\{z_i : i < \omega\}$ such that each sequence $\{a_{ij} : < \omega\}$, $j < s + 1$, converges.

Suppose that $a_{ij} \to a_j$ as $i \to \omega$ for each $j < s + 1$. Let

$$t = x + \sum \{a_j t_j : j < s + 1\}.$$

For the verification that t is proper with respect to N, see [7], Lemma 2. It is, however, in this case that the completeness of R is essential.

The following is a powerful extension lemma for height preserving isomorphisms that has been the backbone of the classification techniques developed by Ulm, Kaplansky and Mackey, Rotman and Yen, Megibben, Hill, and others. It will also play an indispensable role in this paper. For its proof, which depends on our Lemma 4.1, readers are referred to [5], Lemma 3.

LEMMA 4.2. *Let M and M' be reduced R modules having the same Ulm invariants, N and N' finitely generated submodules of M and M', respectively, and ϕ a height preserving isomorphism from N onto N'. If x is an element of $M \backslash N$ such that $px \in N$, then ϕ can be extended to a height preserving isomorphism, say θ, from* $[N, x]$ *onto* $[N', \theta(x)]$.

5. PROOF OF THE THEOREM

Let M and M' be as in the statement of our theorem in Section 1. The necessity of the conditions of the theorem is, of course, obvious. Since M and M' are countably generated and have the same rank, we may write $r(M) = r(M') = k$ for some ordinal $k \leqslant \omega$. Although we need only prove the theorem in case $k = \omega$, the proof we give is valid for all values of $k \leqslant \omega$.

Since M and M' have the same height equivalences, there are bases $y = \{y_i : i < k\}$ and $y' = \{y_i' : i < k\}$ of M and M', respectively, with a height preserving isomorphism π from $[y]$ onto $[y']$, such that $\pi(y_i) = y_i'$ for all $i < k$. Obviously, there are countable subsets x and x' of M and M', respectively, such that $M = [x, y]$ and $M' = [x', y']$, with $px_i \in [x(i), y(i)]$ and $px_i' \in [x'(i), y'(i)]$.

Let A_0 and A_0' be the zero submodules of M and M', respectively, and ϕ_0 the map from A_0 onto A_0'. Assume that, for a positive integer $n + 1$, we have constructed $n + 1$ mappings $\{\phi_i : i < n + 1\}$, such that

(1*) each ϕ_i is a height preserving isomorphism from A_i onto A_i', where A_i and A_i' are submodules of M and M', respectively;

(2*) A_i and A_i' are given by

$$A_i = [x(i), y(i), \phi_i^{-1}[x'(i) \cup y'(i)]],$$
$$A_i' = [x'(i), y'(i), \phi_i[x(i) \cup y(i)]];$$

(3*) each ϕ_{i+1} is an extension of ϕ_i, that is, $\phi_0 \leqslant \cdots \leqslant \phi_n$; and

(4*) there exists a nonnegative integer $r(n)$, depending on n, such that $p^{r(n)}A_n \subseteq [y(n)]$ and $p^{r(n)}A_n' \subseteq [y'(n)]$, and, furthermore, $\phi_n = \pi$, as height preserving isomorphisms from $p^{r(n)}A_n$ onto $p^{r(n)}A_n'$.

Clearly, $\{\phi_0, A_0, A_0', r(0) = 0\}$ satisfies conditions (1*)-(4*). Suppose that we can construct, for each nonnegative integer n, a mapping ϕ_n satisfying the above four conditions. Set $\phi = \sup\{\phi_i : i < \omega\}$. ϕ will be an isomorphism from M onto M' and the proof will be fineshed. Thus, our task in the remainder of this paper is to construct a mapping ϕ_{n+1} satisfying conditions (1*)-(4*), assuming that $\phi_0, \ldots, \phi_n$ with the desired properties exist.

Suppose that there is $\{\theta_m, B_m, B_m'\}$ for some nonnegative integer m, such that

(1) θ_m is a height preserving isomorphism from B_m onto B_m', where B_m and B_m' are submodules of M and M', respectively;

(2) B_m and B_m' are given by

$$B_m = [A_n, p^m y_n], \qquad B_m' = [A_n', p^m y_n'];$$

(3) θ_m is an extension of ϕ_n; and

(4) $p^{r(n)}B_m \subseteq [y(n+1)]$, $p^{r(n)}B_m' \subseteq [y'(n+1)]$, and θ_m agrees with π on $p^{r(n)}B_m$.

Then by applying Lemma 4.2 at most $2(m+1)$ times, possibly $m+1$ times forward and $m+1$ times backward, we can extend θ_m to ϕ_{n+1} satisfying conditions (1*)-(4*), with $r(n+1) = r(n) + m + 1$. Thus, our task reduces now to finding a θ_m as described above.

In case $k \leqslant n < \omega$, already $y_n \in A_n$ and $y_n' \in A_n'$. Hence, in this case, our task is finished by setting $\theta_m = \phi_n$, $B_m = A_n$, and $B_m' = A_n'$. Assume then that $n < k$. For convenience of notation, let us write t and t' for y_n and y_n', respectively, and, u and v, with or without subscripts, for units of R.

For each nonnegative integer m, define $B_m = [A_n, p^m t]$ and $B_m' = [A_n, 'p^m t']$. Obviously, $B_m = A_n \oplus [p^m t]$ and $B_m' = A_n' \oplus [p^m t']$, and $B_0 \supseteq B_1 \supseteq \cdots$ and $B_0' \supseteq B_1' \supseteq \cdots$. Hence, we can write every element $z \in B_m$ in a unique way as $z = a + up^j t$ with $a \in A_n$. We shall write

$z = a \dotplus up^j t$ to indicate the uniqueness of this representation and shall follow the same convention for elements z' of B_m'. Define a mapping θ_m on B_m by $\theta_m(a + up^j t) = \phi_n(a) + up^j t'$ for all $a \dotplus up^j t \in B_m$. Notice that $\theta_0 \geqslant \cdots \geqslant \theta_m \geqslant \cdots$. Clearly, for each nonnegative integer m, $\{\theta_m, B_m, B_m'\}$ satisfies the conditions (1^-), (2), (3), and (4), where the condition (1^-) is the condition (1) without the words "height preserving." Hence, our task will be completed if there is a mapping θ_m that is also height preserving. For the sake of brevity, let us write B for B_0, B' for B_0', and θ for θ_0.

Let us call $z \in B$ *good* if $h(z) = h[\theta(z)]$ and bad, otherwise. It is clear, then, that we need only establish the following lemma.

LEMMA 5.1. *$\{j : a \dotplus up^j t$ is bad$\}$ is bounded above.*

Proof. Since A_n is a finitely generated module over the principal ideal domain R, we can write $A_n = \bigoplus\{[t_i] : i < s + 1\}$ for some nonnegative integer $s + 1$. Since ϕ_n is an isomorphism from A_n onto A_n', we can also write $A_n' = \bigoplus\{[t_i'] : i < s + 1\}$, where $\theta(t_i) = t_i'$ for all $i < s + 1$. Let us deny the assertion of the lemma and show that contradictions arise. Observe that the lemma is false if, and only if, there exists a sequence $\{z_i : i < \omega\}$ of bad elements, such that

$$z_i = \sum \{a_{ij} t_j : j < s + 1\} \dotplus u_i p^{n(i)} t,$$
$$\{n(i) : i < \omega\} \text{ is unbounded above.}$$

Case 1

There is an ordinal $i < \omega$ such that $p^{r(n)}$ is a common divisor of the coefficients $\{a_{ij} : j < s + 1\}$.

Since $\{\theta, B, B', r(n)\}$ satisfies condition (4), $\theta(z_i) = \pi(z_i)$. Hence, $h(z_i) = h[\theta(z_i)]$, that is, z_i is good. Thus, z_i is bad as well as good, which is a contradiction.

Case 2

Without loss of generality, dropping to a subsequence if necessary, we may assume that $\{a_i = a_{is} : i < \omega\}$ is of the form $a_i \equiv u_i p^m$, where $m < r(n)$ is a fixed nonnegative integer for all i. Let us use induction on $s + 1$. Firstly, in case $s + 1 = 0$, clearly all z_i's are good, which is a contradiction. Secondly, in case $s + 1$ is a positive integer, with the induction hypothesis, we proceed as follows. For notational convenience, we write z' for the image $\theta(z)$, α_i for $h(z_i)$, and α_i' for $h(z_i')$.

Subcase a.

$$\alpha_0 < \cdots < \alpha_i < \cdots, \qquad \alpha_0' < \cdots < \alpha_i' < \cdots,$$
$$\alpha_i < \alpha_i' \qquad \text{for all} \qquad i < \omega.$$

Define $\zeta_i = (u_{i+1})^{-1} z_{i+1} - (u_i)^{-1} z_i$ and $\zeta' = (u_{i+1})^{-1} z'_{i+1} - (u_i)^{-1} z_i'$ for all $i < \omega$. It is clear that $h(\zeta_i) = \alpha_i$, $h(\zeta_i') = \alpha_i'$, and $\theta(\zeta_i) = \zeta_i'$. Thus, $\{\zeta_i : i < \omega\}$ is a sequence of bad elements of the form

$$\zeta_i = \sum \{b_{ij} t_j : j < s\} \dotplus v_i p^{m(i)} t,$$
$$\{m(i) : i < \omega\} \text{ is unbounded above.}$$

This, however, contradicts the induction hypothesis.

Subcase b.

$$\alpha_0 = \cdots = \alpha_i = \cdots, \qquad \alpha_0' < \cdots < \alpha_i' < \cdots,$$
$$\alpha_i' < \alpha_0 \qquad \text{for all} \qquad i < \omega.$$

Construct sequences $\{\zeta_i : i < \omega\}$ and $\{\zeta_i' : i < \omega\}$ in the same manner as in the preceding subcase. Then $h(\zeta_i) \geqslant \alpha_i$, $h(\zeta_i') = \alpha_i'$, and $\theta(\zeta_i) = \zeta_i'$ for all i. But then we have a contradiction precisely as in Subcase a.

Subcase c.

$$\alpha_0 = \cdots = \alpha_i = \cdots, \qquad \alpha_0' = \cdots = \alpha_i' = \cdots,$$
$$\alpha_0 < \alpha_0'.$$

Since A_n is a finitely generated submodule of M, by Lemma 4.1, A_n is nice in M. Let $z = p^{n(0)} t$. Since $z \in M \backslash A_n$, the coset $z + A_n$ contains an element $\zeta = a + z$ that is proper with respect to A_n. Define $\zeta_i = z_i - u_i p^{n(i)-n(0)} \zeta$ for all i, such that $0 < i < \omega$. It is clear that $\zeta_i \in A_n$, $\zeta_i' \in A_n'$, $h(\zeta_i) = \alpha_0$, $h(\zeta_i') > \alpha_0$, and $\phi_n(\zeta_i) = \theta(\zeta_i) = \zeta_i'$. These observations however contradict the fact that ϕ_n is height preserving.

Subcase d.

$$\alpha_0 = \cdots = \alpha_i = \cdots, \qquad \alpha_0' < \cdots < \alpha_i' < \cdots,$$
$$\alpha_0 < \alpha_0' \qquad \text{for all } i.$$

Construct sequences $\{\zeta_i : i < \omega\}$ and $\{\zeta_i' : i < \omega\}$ exactly as in Subcase c. Notice that $h(\zeta_i) = \alpha_i$ and $h(\zeta_i') > \alpha_i$ for all i, such that $0 < i < \omega$. This is again a contradiction.

It is easily seen now that all possibilities under Case 2 reduce to one of the four subcases above by passing to subsequences and exploiting the symmetry between B and B'. Our proof is, therefore, complete.

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