Fundamental results for pointfree convex geometry

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Abstract

Inspired by locale theory, we propose “pointfree convex geometry”. We introduce the notion of convexity algebra as a pointfree convexity space. There are two notions of a point for convexity algebra: one is a chain-prime meet-complete filter and the other is a maximal meet-complete filter. In this paper we show the following: (1) the former notion of a point induces a dual equivalence between the category of “spatial” convexity algebras and the category of “sober” convexity spaces as well as a dual adjunction between the category of convexity algebras and the category of convexity spaces; (2) the latter notion of point induces a dual equivalence between the category of “m-spatial” convexity algebras and the category of “m-sober” convexity spaces. We finally argue that the former notion of a point is more useful than the latter one from a category theoretic point of view and that the former notion of a point actually represents a polytope (or generic point) and the latter notion of a point properly represents a point. We also remark on the close relationships between pointfree convex geometry and domain theory.

1. Introduction

Can you see any “point” in space? The answer will be no. The notions of topological space and convexity space (explained below) presuppose that of a point, which seems to be epistemologically ideal (see [19,33,34]). From the viewpoints of duality theory and algebraic geometry (see [21,17]), we notice that a point amounts to a prime ideal (or a model in logical terms), which is an infinite entity, and we need some indeterministic principle such as (a weaker form of) the axiom of choice in order to show the existence of it, and therefore the notion of a point is very ideal. On the other hand, we can actually see “regions” of space in some sense and, from the viewpoints of duality theory and algebraic geometry, a (basic) region can be identified with an algebraic formula, which is a finite entity. Hence, the notion of a region seems to be epistemologically more certain than that of a point (where it is supposed that one notion is epistemologically more certain than another if the former precedes the latter in human knowledge).

This leads us to the notion of region-based pointfree space. There are several ways to realize this notion in a mathematical form. In mathematics we often encounter the following phenomenon: a space is recovered from the function algebra on it (a space has the same information as the function algebra on it). Such spaces include manifolds, compact Hausdorff spaces, and affine schemes (see [17,21]). Moreover, geometric notions can often be translated into algebraic ones via the correspondence between space and algebra (for example, in algebraic geometry, the dimension of an algebraic variety corresponds to the...
Krull dimension of the coordinate ring of it). These facts give us the idea “Algebra itself is a space”. This idea has already been pursued in several areas of mathematics such as non-commutative geometry (see [5]) and we also follow it in this paper.

Locale theory can be considered as an algebraic theory of topological structures which does not presuppose the notion of a point and is primarily based on that of a region, since locale theory studies the lattice structure of open sets in an algebraic way; i.e., a “space” in locale theory is a join-complete lattice with finite meets that distribute over arbitrary joins, which is called a frame (for locale theory, see [21,23,24,26,27,31,32]). Usually, localic versions of theorems in the ordinary topology do not need non-constructive principles such as the law of excluded middle or the axiom of choice, and so locale theory can also be seen as constructive topology (see [1,4,9,7]). There are two fundamental results of locale theory (see [21,1,13]): (i) a dual adjunction between the category of frames and the category of topological spaces; (ii) a dual equivalence between the category of spatial frames and the category of sober topological spaces, which is sometimes called the Isbell duality (see [3]).

Along with topology, convex geometry has been studied extensively from different perspectives (see [6,15,30]). Among many results of convex geometry, Helly-type theorems (see [12]) are important, especially for combinatorial convex geometry. They characterize the dimension of a Euclidean space and so seem to be significant from a philosophical as well as a mathematical point of view. Helly-type theorems can be extended to the case of convexity spaces (see [6]), which are defined as a set $S$ equipped with a subset $C \subseteq 2^S$, a convexity, satisfying some conditions (see Definition 2.1). These results contribute to our understanding of the notion of dimension, clarifying the convexity theoretical meaning of it. There have been many more studies on convexity spaces than mentioned above (see [6,22,30]). Note that the notion of topological space in topology corresponds to that of convexity space in convex geometry.

Inspired by locale theory, we propose “pointfree convex geometry”, toward which this paper takes a first step (for related categorical work, see [20]). Pointfree convex geometry is an algebraic theory of convex structures which does not presuppose the notion of a point and is primarily based on that of a region. Pointfree convex geometry studies the lattice structure of convex sets in an algebraic way; i.e., a “space” in pointfree convex geometry is a meet-complete poset with joins of chains that distribute over arbitrary meets, which we call a convexity algebra. The notion of a point does not appear explicitly, but, if we want, we can consider points of pointfree spaces and recover “all” points under certain assumptions.

We emphasize that there are two ways to recover points. One way is to consider a chain-prime meet-complete filter (cp-mc filter, for short) as a point. The other way is to consider a maximal meet-complete filter (m-mc filter, for short) as a point. These two views on the notion of a point induce two kinds of categorical dualities between some convexity algebras and some convexity spaces. The following are the main results in this paper:

- there is a dual adjunction between the category of convexity algebras and homomorphisms and the category of convexity spaces and convexity-preserving maps (Theorem 3.9);
- there is a dual equivalence between the category of spatial convexity algebras and homomorphisms and the category of sober convexity spaces and convexity-preserving maps (Theorem 4.18);
- there is a dual equivalence between the category of m-spatial convexity algebras and m-homomorphisms and the category of m-sober convexity spaces and convexity-preserving maps (Theorem 5.18), where note that convexity-preserving maps between m-sober convexity spaces correspond to m-homomorphisms, not homomorphisms.

These results are considered as fundamental for pointfree convex geometry as (i) and (ii) above are for locale theory. These results clarify the categorical relationships between pointfree spaces and pointset spaces, or epistemological and ontological aspects of the notion of space (they might be considered to be almost equivalent from a mathematical point of view). For more discussion of these results, we refer the reader to Section 6.

Our investigation in this paper proceeds as follows. In Section 2, we first review the concept of convexity space and then introduce the concept of convexity algebra as a pointfree analogue of a convexity space and related concepts of a filter. In Section 3, we obtain a dual adjunction between the category of convexity algebras and the category of convexity spaces, which is based on the view that a point is a cp-mc filter. In Section 4, by introducing the concepts of spatiality and sobriety, we obtain a duality between the category of spatial convexity algebras and the category of sober convexity spaces, which is based on the view that a point is a cp-mc filter. An algebraic characterization of spatiality is also provided. We remark that Euclidean spaces are sober topological spaces and are not sober convexity spaces (the same holds also for the other spaces in Example 2.2), whence the sobriety of convexity seems to be considerably different from the sobriety of topology. We also show that a convexity algebra is filter closed iff there is no infinite descending chain in it and that an analogue of the prime filter theorem for distributive lattices holds for filter-closed convexity algebras. In Section 5, by introducing the concepts of m-homomorphism, m-spatiality and m-sobriety, we obtain a duality between the category of m-spatial convexity algebras and m-homomorphisms and the category of m-sober convexity spaces and convexity-preserving maps, which is based on the view that a point is an m-mc filter. We also give an algebraic characterization of m-spatiality. We remark that many ordinary convexity spaces such as Euclidean spaces are m-sober even if they are not sober. In Section 6, we discuss the questions “Which notion of a point is better?” and “Which notion of a point is the proper one?”, and also the relationships between pointfree geometry and Hilbert’s instrumentalism or Husserl’s phenomenology. In this section we
also remark that pointfree convex geometry is closely related to domain theory (in some sense it coincides with the theory of continuous lattices).

2. Convexity spaces and convexity algebras

In this section we review the basics of convexity spaces and introduce the notion of convexity algebra with related basic concepts and propositions.

2.1. Convexity spaces

We first review the notion of convexity space. For more detailed exposition, see the books [6,30]. Convexity spaces are sometimes called aligned spaces, as in [6].

Let \( 2^X \) denote the powerset of \( X \).

**Definition 2.1** ([6,30]). For a set \( S \) and a subset \( C \) of \( 2^S \), \( (S, C) \) is a convexity space iff \( (S, C) \) satisfies the following conditions:

1. \( C \) is closed under arbitrary intersections;
2. if \( \{X_i : i \in I\} \) is totally ordered with respect to inclusion, then \( \bigcup \{X_i : i \in I\} \in C \).

We call \( C \) the convexity of \( S \) and an element of \( C \) a convex set in \( S \). The complement of a convex set in \( S \) is called a concave set in \( S \).

Note that \( \emptyset, S \in C \) by letting the index sets be empty in the above conditions.

A convexity space \( (S, C) \) is often denoted by its underlying set \( S \).

Let us denote by \( 2 \) the two-element distributive lattice \( \{0, 1\} \) equipped with the Sierpiński convexity \( \{\emptyset, \{1\}, \{0, 1\}\} \).

**Example 2.2.** Consider a vector space \( V \) over the real number field \( \mathbb{R} \). We can equip \( V \) with a natural convexity determined by the condition that \( X \subseteq V \) is convex iff, for any \( x, y \in X \) and any \( t \in [0, 1], tx + (1-t)y \in X \). In particular, the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \) for an integer \( n \geq 1 \) is naturally equipped with a convexity in this way.

Consider the \( n \)-sphere \( S^n \) for an integer \( n \geq 1 \). We can equip \( S^n \) with a natural convexity determined by the condition that \( X \subseteq S^n \) is convex iff the following hold: (i) for any \( x, y \in X \), the antipodal point of \( x \) is not \( y \); (ii) for any \( x, y \in X \), the shortest path (i.e., geodesic) between \( x \) and \( y \) on \( S^n \) is a subset of \( X \).

We can also equip the \( n \)-dimensional real projective space with a convexity (see [6, Example 6.2.7] or [28]).

By the condition 1 in **Definition 2.1**, we can define the convex hull of a subset of a convexity space as follows.

**Definition 2.3** ([6,30]). Let \( (S, C) \) be a convexity space. For \( A \subseteq S \), define

\[
\text{ch}(A) = \bigcap \{C \in C : A \subseteq C\}.
\]

Then, \( \text{ch}(A) \) is called the convex hull of \( A \).

As usual, we define a morphism of convexity spaces as follows.

**Definition 2.4** ([30,22]). Let \( (S, C), (S', D) \) be convexity spaces. A map \( f : S \to S' \) is a convexity-preserving map iff, for any \( D \in D \), we have \( f^{-1}(D) \in C \).

This definition of morphism of convexity spaces seems to be most popular, though other definitions may be possible. Even if a stronger definition of morphism of convexity spaces is employed, our duality results still work by restricting the morphism parts of the related categories.

Note that the inverse map of a bijective convexity-preserving map is not necessarily convexity preserving.

By the following proposition, we can consider the set of convex sets in a convexity space as the hom-set from the convexity space to \( 2 \).

**Proposition 2.5.** Let \( (S, C) \) be a convexity space. Then, there is a natural bijection between the set \( C \) of all convex sets in \( S \) and the set of all convexity-preserving maps from \( S \) to \( 2 \).

**Proof.** For a convex set \( C \) in \( S \), define \( f_C : S \to 2 \) by

\[
f_C(x) = \begin{cases} 
1 & \text{if } x \in C \\
0 & \text{otherwise.} 
\end{cases}
\]

Then, it is clear that \( f_C \) is a convexity-preserving map and that if \( C \neq D \) for convex sets \( C \) and \( D \) in \( S \) then we have \( f_C \neq f_D \). Thus, the map \( C \mapsto f_C \) is injective. To show the surjectivity, let \( g \) be a convexity-preserving map from \( S \) to \( 2 \). Define \( C = g^{-1}(\{1\}) \). Then, it is clear that \( C \) is a convex set in \( S \) and that \( f_C = g \). This completes the proof. \( \square \)

The notion of a polytope is defined as follows.
Definition 2.9. A poset $L$ is a convexity algebra iff it satisfies the following properties:

1. $L$ has arbitrary meets;
2. If $\{x_i \in L : i \in I\}$ is totally ordered in $L$, then $\{x_i : i \in I\}$ has a join in $L$;
3. for any doubly indexed family $\{x_{i,j} \in L : i \in I$ and $j \in J_i\}$, if $\{x_{i,j} : j \in J_i\}$ is totally ordered for every $i \in I$ and if $\bigwedge_{i \in I} x_{i,f(i)} : f \in F\}$ is totally ordered, then

$$\bigwedge_{i \in I} \bigvee_{j \in J_i} x_{i,j} = \bigvee_{f \in F} \bigwedge_{i \in I} x_{i,f(i)},$$

where $F = \prod_{i \in I} J_i = \{f : I \to \bigcup_{i \in I} J_i : \forall i \in I f(i) \in J_i\}$.

Note that a convexity algebra has the least element 0 and the greatest element 1 by letting the index sets be empty in conditions 1 and 2 above.

We call condition 3 in the above definition the chain-completely distributive law.

Definition 2.10. Let $L_1$ and $L_2$ be convexity algebras. A function $f : L_1 \to L_2$ is a homomorphism from $L_1$ to $L_2$ iff it satisfies the following properties:

1. $f(\bigwedge_{i \in I} a_i) = \bigwedge_{i \in I} f(a_i)$ for any $\{a_i : i \in I\} \subset L_1$;
2. if $\{a_i \in L_1 : i \in I\} \subset F$, then $f(\bigvee_{i \in I} a_i) = \bigvee_{i \in I} f(a_i)$.

Note that, for a homomorphism $f$ of convexity algebras, we have $f(0) = 0$ and $f(1) = 1$ by letting the index sets be empty in the above conditions.

We can easily verify the following proposition.

Proposition 2.11. Let $(S, C)$ be a convexity space. Then, $C$ forms a convexity algebra (when equipped with set-theoretical operations).

Next we define the concepts of a meet-complete filter and a chain-prime meet-complete filter (cp-mc filter for short), which correspond to a filter and a completely prime filter respectively in locale theory.

Definition 2.12. Let $L$ be a convexity algebra. A subset $F$ of $L$ is called a meet-complete filter of $L$ iff the following hold:

1. if $a \in F$ and $a \leq x$, then $x \in F$;
2. if $\{a_i : i \in I\} \subset F$, then $\bigwedge_{i \in I} a_i \in F$.

Note that a meet-complete filter $F$ is non-empty, since we have $1 \in F$ by condition 2 applied to the empty family.

A meet-complete filter may also be called Moore filter, since the related notion of a Moore family is well known.

Definition 2.13. Let $L$ be a convexity algebra. A subset $P$ of $L$ is called a chain-prime meet-complete filter (cp-mc filter for short) of $L$ iff the following hold:

1. $P$ is a meet-complete filter of $L$;
2. if $\{a_i : i \in I\}$ is totally ordered and $\bigvee_{i \in I} a_i \in P$, then there is $i \in I$ with $a_i \in P$. 
Let \( f \) be a convexity algebra.

For an arrow \( f : L_1 \to L_2 \) in \( \mathbf{CA} \), \( \text{Spec}(f) : \text{Spec}(L_2) \to \text{Spec}(L_1) \) is defined by

\[
\text{Spec}(f)(v) = v \circ f
\]

for \( v \in \text{Spec}(L_2) \).

By Proposition 2.17, we can consider \( \text{Spec}(L) \) as the set of all cp-mc filters of \( L \) equipped with the convexity generated by the \( \{P \in \text{Spec}(L) : a \in P\}'s for \( a \in L \).

The well-definedness of the functor \( \text{Spec} \) is proven by the following lemma.

**Lemma 3.3.** Let \( f : L_1 \to L_2 \) be an arrow in \( \mathbf{CA} \). Then, \( \text{Spec}(f) : \text{Spec}(L_2) \to \text{Spec}(L_1) \) is a convexity-preserving map.
We define a contravariant functor \( \text{Conv} \) from \( \mathcal{C} \) to \( \mathbf{CA} \) as follows.

1. For an object \( S \) in \( \mathcal{C} \), \( \text{Conv}(S) \) is defined as the set of all convexity-preserving maps from \( S \) to 2 equipped with the pointwise operations. For instance, given \( f_i \in \text{Conv}(S) \) for \( i \in I \), \( \bigwedge_{i \in I} f_i \in \text{Conv}(S) \) is defined by
   \[
   \left( \bigwedge_{i \in I} f_i \right)(x) = \bigwedge_{i \in I} f_i(x).
   \]

2. For an arrow \( f : S \to S' \) in \( \mathcal{C} \), \( \text{Conv}(f) : \text{Conv}(S') \to \text{Conv}(S) \) is defined by
   \[
   \text{Conv}(f)(g) = g \circ f
   \]
   for \( g \in \text{Conv}(S') \).

For a convexity space \( S \), we can consider \( \text{Conv}(S) \) as the set of all convex sets equipped with set-theoretical operations by Proposition 2.5. Note that, in this case, the arrow part of the functor \( \text{Conv} \) can be defined by

\[
\text{Conv}(f)(C) = f^{-1}(C)
\]
for \( C \in \text{Conv}(S) \). The two definitions of \( \text{Conv} \) are essentially equivalent, and we do not have to distinguish between them.

The well-definedness of the functor \( \text{Conv} \) is proven as follows: first, \( \text{Conv}(S) \) forms a convexity algebra by Propositions 2.11 and 2.5; second, \( \text{Conv}(f) \) is a homomorphism, since all the operations of \( \text{Conv}(S) \) are defined pointwise.

For a category \( \mathcal{C} \), let \( 1_{\mathcal{C}} \) denote the identity functor from \( \mathcal{C} \) to \( \mathcal{C} \).

We define a natural transformation \( \Phi : 1_{\mathbf{CA}} \to \text{Conv} \circ \text{Spec} \) as follows. For a convexity algebra \( L \), define \( \Phi_L : L \to \text{Conv} \circ \text{Spec}(L) \) by

\[
\Phi_L(a)(v) = v(a)
\]
for \( a \in L \) and \( v \in \text{Spec}(L) \).
Then, \( \Phi_1 \) is well defined, since \( \Phi_1(a) \) is a convexity-preserving map by the following fact:

\[
\Phi_1(a)^{-1}(\{1\}) = \{v \in \text{Spec}(L) : v(a) = 1\} = \langle a \rangle.
\]

It is straightforward to verify that \( \Phi_1 \) is a homomorphism. It is proven by direct computation that \( \Phi \) is a natural transformation.

**Definition 3.8.** We define a natural transformation \( \Psi : 1_{\text{CS}} \rightarrow \text{Spec} \circ \text{Conv} \) as follows. For a convexity space \( S \), define \( \Psi_S : S \rightarrow \text{Spec} \circ \text{Conv}(S) \) by

\[
\Psi_S(x)(f) = f(x)
\]

for \( x \in S \) and \( f \in \text{Conv}(S) \).

Then, \( \Psi_S \) is well defined, since \( \Psi_S(x) \) is a homomorphism by the pointwiseness of the operations of \( \text{Conv}(S) \). Moreover, \( \Psi_S \) is a convexity-preserving map by the following fact:

\[
\Psi_S^{-1}(\{f\}) = \{x \in S : \Psi_S(x) \in \langle f \rangle\} = f^{-1}(\{1\})
\]

for \( f \in \text{Conv}(S) \). It is proven by direct computation that \( \Psi \) is a natural transformation.

Now, we show that \( \text{Spec} \) and \( \text{Conv} \) give a dual adjunction between \( \text{CA} \) and \( \text{CS} \).

**Theorem 3.9.** \( \text{Spec} \) is left adjoint to \( \text{Conv}^{\text{op}} \).

**Proof.** Let \( L \) be a convexity algebra and \( S \) a convexity space. Assume that \( f \) is a homomorphism from \( L \) to \( \text{Conv}(S) \). It suffices to show that there is a unique arrow \( g : S \rightarrow \text{Spec}(L) \) in \( \text{CS} \) such that \( \text{Conv}(g) \circ \Phi_L = f \). Now, define a map \( g : S \rightarrow \text{Spec}(L) \) by

\[
g(x)(a) = \Psi_S(x)(f(a))
\]

for \( x \in S \) and \( a \in L \). Then, since \( f \) and \( \Psi_S(x) \) are homomorphisms, we have \( g(x) \in \text{Spec}(L) \). Moreover, \( g \) is a convexity-preserving map by \( f(a) \in \text{Conv}(S) \) and the following fact:

\[
g^{-1}(\langle a \rangle) = \{x \in S : g(x) \in \langle a \rangle\}
\]

\[
= \{x \in S : g(x)(a) = 1\}
\]

\[
= \{x \in S : f(a)(x) = 1\}
\]

\[
= f(a)^{-1}(\{1\}).
\]

For \( a \in L \) and \( x \in S \), we have

\[
(\text{Conv}(g) \circ \Phi_L)(a)(x) = \Phi_L(a) \circ g(x)
\]

\[
= g(x)(a)
\]

\[
= f(a)(x).
\]

Hence, we have \( \text{Conv}(g) \circ \Phi_L = f \).

To show the uniqueness, suppose that \( h \) is a convexity-preserving map from \( S \) to \( \text{Spec}(L) \) and that \( \text{Conv}(h) \circ \Phi_L = f \). Then, it follows that \( \text{Conv}(h) \circ \Phi_L = f = \text{Conv}(g) \circ \Phi_L \). Here, for \( a \in L \) and \( x \in S \), we have

\[
(\text{Conv}(h) \circ \Phi_L)(a)(x) = \Phi_L(a) \circ h(x) = h(x)(a).
\]

Similarly, we have

\[
(\text{Conv}(g) \circ \Phi_L)(a)(x) = g(x)(a).
\]

Thus, we conclude that \( h = g \). \( \square \)

Recall that a left adjoint functor preserves colimits and a right adjoint functor preserves limits (see [2]). Thus, some categorical constructions in one category can be transferred into the other category via the above adjunction.

### 4. Duality between SpCA and SobCS

In this section, we introduce the notions of spatial convexity algebra and sober convexity space. Then, by restricting the dual adjunction between \( \text{CA} \) and \( \text{CS} \), we shall show a duality between the category \( \text{SpCA} \) of spatial convexity algebras and the category \( \text{SobCS} \) of sober convexity spaces.
4.1. Spatiality

We define the notion of spatiality as the existence of “enough” cp-mc filters.

**Definition 4.1.** For a convexity algebra \( L \), \( L \) is spatial iff, for any \( a, b \in L \) with \( a \nleq b \), there is a cp-mc filter \( P \) of \( L \) such that \( a \in P \) and \( b \notin P \).

We can characterize the spatiality of a convexity algebra \( L \) as the injectivity of \( \Phi_L \).

**Lemma 4.2.** Let \( L \) be a convexity algebra. The following are equivalent:

1. \( L \) is spatial;
2. \( \Phi_L \) is injective; i.e., for any \( a, b \in L \) with \( a \neq b \) there is \( v \in \text{Spec}(L) \) with \( v(a) \neq v(b) \);
3. if \( (a) \subset (b) \) for \( a, b \in L \), then \( a \leq b \).

**Proof.** By Proposition 2.17, it is straightforward to show that 1 implies 2 and that 3 implies 1. We show that 2 implies 3. Assume 2. To show the contrapositive of 3, assume that \( a \nleq b \). Then, since \( \Phi_L \) is an injective homomorphism, we have \( \Phi_L(a) \nleq \Phi_L(b) \). Therefore, there is \( v \in \text{Spec}(L) \) such that

\[
v(a) = \Phi_L(a)(v) > \Phi_L(b)(v) = v(b).
\]

Hence, we have \( v \in (a) \) and \( v \notin (b) \). This completes the proof. \( \square \)

The following proposition provides many natural examples of spatial convexity algebras.

**Proposition 4.3.** Let \( S \) be a convexity space. Then, \( \text{Conv}(S) \) is a spatial convexity algebra.

**Proof.** Let \( f, g \in \text{Conv}(S) \) with \( f \neq g \). Then, we have \( f(x) \neq g(x) \) for some \( x \in S \). Let \( v = \Psi_S(x) \). Then, we have

\[
\Phi_{\text{Conv}(S)}(f)(v) = v(f) = f(x).
\]

We also have

\[
\Phi_{\text{Conv}(S)}(g)(v) = v(g) = g(x).
\]

Thus, \( \Phi_{\text{Conv}(S)} \) is injective, and so \( \text{Conv}(S) \) is spatial by Lemma 4.2. \( \square \)

In fact, \( \Phi_L \) is always surjective as follows.

**Lemma 4.4.** Let \( L \) be a convexity algebra. Then, \( \Phi_L \) is surjective.

**Proof.** Based on Proposition 2.5, we can consider \( \text{Conv} \circ \text{Spec}(L) \) as the set of all convex sets in \( \text{Spec}(L) \). Thus, since \( \Phi_L(a)^{-1}((1)) = (a) \), we can consider \( \Phi_L(a) = (a) \). Then, it follows from Lemma 3.5 that

\[
\{(a) : a \in L\} = \text{Conv} \circ \text{Spec}(L).
\]

Hence, \( \Phi_L \) is surjective by \( \Phi_L(a) = (a) \). \( \square \)

By Lemma 4.2, Proposition 4.3 and Lemma 4.4, we obtain the following proposition (recall that \( \Phi_L \) is a homomorphism for any convexity algebra \( L \)).

**Proposition 4.5.** For a convexity algebra \( L \), \( L \) is spatial iff \( \Phi_L : L \to \text{Conv} \circ \text{Spec}(L) \) is an isomorphism.

This proposition implies that any spatial convexity algebra can be represented as the convexity algebra of convex sets in a convexity space.

We can provide an algebraic characterization of spatiality as follows.

**Proposition 4.6.** Let \( L \) be a convexity algebra. Then, \( L \) is spatial iff \( L \) is chain-algebraic.

**Proof.** Assume that \( L \) is spatial. By Proposition 4.5, \( L \) is isomorphic to the convexity algebra of all convex sets in a convexity space, which is shown to be chain-algebraic by combining Propositions 2.15 and 2.8. Assume that \( L \) is chain-algebraic. Let \( a, b \in L \) with \( a \nleq b \). Let \( A \) be the set of chain-compact elements that are less than or equal to \( a \) and \( B \) the set of chain-compact elements that are less than or equal to \( b \). Since \( L \) is chain-algebraic, we have \( a = \bigvee A \) and \( b = \bigvee B \). Therefore, it follows from \( a \nleq b \) that there is \( c \in A \) such that \( c \nleq b \). Define \( P = \{x \in L : c \leq x\} \). Then, since \( c \) is a chain-compact element, \( P \) is a cp-mc filter by Proposition 2.16, and also we have both \( a \in P \) and \( b \notin P \). Thus, \( L \) is spatial. \( \square \)

**Definition 4.7.** Let \( L \) be a convexity algebra and \( \text{MCF}(L) \) the set of all meet-complete filters of \( L \). Then, \( L \) is filter-closed iff for any non-empty totally ordered subset \( \{X_i : i \in I\} \) of \( \text{MCF}(L) \), \( \bigcup_{i \in I} X_i \) is a meet-complete filter.

For instance, every successor ordinal is a filter-closed convexity algebra and also the finite product of successor ordinals is a filter-closed convexity algebra. More generally, we have the following characterization of filter-closed convexity algebra.

**Proposition 4.8.** For a convexity algebra \( L \), \( L \) is filter-closed iff there is no infinite descending chain in \( L \).
**Definition.** A convexity space is sober if for every polytope $x$ in it, there exists a unique point $y$ such that $x = \text{ch}(y)$.

Let $L$ be a filter-closed convexity algebra. Then, $L$ is spatial.

Proof. We first show that filter-closedness implies the non-existence of an infinite descending chain. In order to prove the contrapositive, assume that there exists an infinite descending chain $\{a_i : i \in I\}$ in $L$. Since, if not, there exists $k \in I$ such that $a_k \leq a_i$ for any $i \in I$, i.e., $[a_k \in L : i \in I]$ is not an infinite descending chain. Define $A_i = \{x \in L : a_i \leq x\}$, which is a meet-complete filter. Clearly, $\bigcup_{i \in I} A_i$ is totally ordered. Moreover, $\bigcup_{i \in I} A_i$ is not a meet-complete filter, since we have both $a_i \in \bigcup_{i \in I} A_i$ for any $i \in I$ and $\bigcup_{i \in I} A_i \notin \bigcup_{i \in I} A_i$ by the fact that $\bigcup_{i \in I} a_i \not< a_k$ for any $k \in I$. Therefore, $L$ is not filter-closed.

To show the converse, assume that there is no infinite descending chain in $L$. Let $\{X_i : i \in I\}$ be a non-empty totally ordered subset of $\text{MCF}(L)$. Since any meet-complete filter $X$ is generated by $\bigwedge X$, $\{\bigwedge X_i : i \in I\}$ is totally ordered in $L$. However, it follows from assumption that $\{\bigwedge X_i : i \in I\}$ is not an infinite descending chain. Thus, $\bigcap_{i \in I} X_i$ is not an infinite ascending chain. Then, there is $j \in I$ such that $X_j = \bigcup_{i \in I} X_i$. Hence, $\bigcup_{i \in I} X_i$ is a meet-complete filter. Thus, $L$ is filter-closed. □

An analogue of the prime filter theorem for distributive lattices holds for filter-closed convexity algebras.

**Proposition 4.9.** Let $L$ be a filter-closed convexity algebra. Then, $L$ is spatial.

Proof. Let $a, b \in L$, with $a \not< b$. Let $\mathcal{H}$ be the set of meet-complete filters $F$ of $L$ such that $a \in F$ and $b \notin F$. Since $\{x \in L : a \leq x\} \in \mathcal{H}$, $\mathcal{H}$ is not empty. Since $L$ is filter-closed, every totally ordered subset $\{F_i : i \in I\}$ of $\mathcal{H}$ has an upper bound $\bigcup_{i \in I} F_i$ in $\mathcal{H}$. Thus, by Zorn’s lemma, we have a maximal element $P$ in $\mathcal{H}$. Clearly, $a \in P$ and $b \notin P$.

In order to complete the proof, we show that $P$ is a cp-mc filter of $L$. Let $\{a_i : i \in I\}$ be a totally ordered subset of $L$ and $\bigvee_{i \in I} a_i \in P$. Suppose for contradiction that $a_i \notin P$ for any $i \in I$. Then, it follows from the maximality of $M$ that for every $i \in I$ there exists $p_i \in P$ such that $a_i \wedge p_i \leq b$. Let $p = \bigwedge_{i \in I} p_i$.

Since $P$ is a meet-complete filter, we have $p \in P$. Clearly, $a_i \wedge p \leq b$. Hence, we have $\bigvee_{i \in I} (a_i \wedge p) \geq b$.

It follows from the chain-completely distributive law (i.e., item 3 in Definition 2.9) that $\bigvee_{i \in I} (a_i \wedge p) = \left(\bigvee_{i \in I} a_i\right) \wedge p$.

Since $\bigvee_{i \in I} a_i \in P$ and $p \in P$, we have $(\bigvee_{i \in I} a_i) \wedge p \in P$ and so $\bigvee_{i \in I} (a_i \wedge p) \in P$. By $\bigvee_{i \in I} (a_i \wedge p) \leq b$, we have $b \in P$, which is a contradiction. Thus, $P$ is a cp-mc filter. Hence, $L$ is spatial. □

### 4.2. Sobriety

In order to define sober convexity space, we first define a chain-irreducible convex set, whose role in our duality theory is analogous to that of an irreducible closed set in Isbell duality.

**Definition 4.10.** Let $(S, \mathcal{C})$ be a convexity space. A convex set $C$ in $S$ is said to be chain-irreducible if, for any totally ordered subset $\{C_i : i \in I\}$ of $\mathcal{C}$, there exists $i \in I$ such that $C = C_i$.

Note that a convex set in a convexity space $(S, \mathcal{C})$ is chain-irreducible if it is a chain-compact element in the convexity algebra $\mathcal{C}$. By Proposition 2.7, we have the following lemma.

**Lemma 4.11.** Let $S$ be a convexity space. Then, a convex subset of $S$ is chain-irreducible if and only if it is a polytope in the convexity space.

Now, we introduce the notion of sober convexity space.

**Definition 4.12.** A convexity space $S$ is said to be sober if, for every chain-irreducible convex set $C$ in $S$, there is a unique point $x \in S$ such that $C = \text{ch}(x)$.

By Lemma 4.11, we obtain the following alternative definition of sobriety, which clarifies the convexity theoretical meaning of sobriety.

**Proposition 4.13.** A convexity space is sober if and only if every polytope in it is the convex hull of a unique point.
We remark that not all natural examples of convexity spaces are sober. For example, by the above proposition, \( \mathbb{R}^n \) with the usual convexity (see Example 2.2) is not a sober convexity space, though it is a sober topological space.

**Example 4.14.** Consider \( 2^n \), i.e., the set of all functions from the set \( \omega \) of all non-negative integers to \( 2 \) (= \{0, 1\}). Let \( C_0 = 2^n \). For \( k \in \omega \) with \( k \geq 1 \) and \( n_1, \ldots, n_k \in \omega \), let

\[
C_k(n_1, \ldots, n_k) = \{ f \in 2^n : f(n_1) = f(n_2) = \cdots = f(n_k) = 1 \}.
\]

Equip \( 2^n \) with the convexity generated by

\[
\{ C_k(n_1, \ldots, n_k) : k \in \omega \text{ and } n_1, \ldots, n_k \in \omega \}.
\]

Then, \( 2^n \) forms a sober convexity space.

The next proposition provides many natural examples of sober convexity spaces.

**Proposition 4.15.** Let \( L \) be a convexity algebra. Then, \( \text{Spec}(L) \) is a sober convexity space.

**Proof.** Assume that \( C \) is a chain-irreducible convex set in \( \text{Spec}(L) \). Define

\[
a = \bigcap \{ x \in L : C = \langle x \rangle \},
\]

where \( \{ x \in L : C = \langle x \rangle \} \) is non-empty, since any convex set in \( \text{Spec}(L) \) is of the form \( \langle x \rangle \) for \( x \in L \) by Lemma 3.5. Then, we have \( C = \langle a \rangle \) by Lemma 3.4. We claim that \( a \) is a chain-compact element in \( L \). Suppose that \( a \leq \bigvee_{i \in I} a_i \) for a totally ordered subset \( \{ a_i : i \in I \} \) of \( L \). Then, by Lemma 3.4, we have

\[
C = \langle a \rangle = \left( \bigvee_{i \in I} a_i \right) \cap \langle a \rangle = \bigcup_{i \in I} \{ a_i \wedge a \}.
\]

Since \( C \) is chain-irreducible and since \( \{ (a_i \wedge a) : i \in I \} \) is totally ordered, there exists \( i \in I \) such that

\[
C = \langle a \rangle = \langle a \wedge a_i \rangle.
\]

Thus, it follows from the definition of \( a \) that \( a \leq a \wedge a_i \), whence we have \( a \leq a_i \). Therefore, \( a \) is a chain-compact element in \( L \). Let \( P_a = \{ x \in L : a \leq x \} \). Then, \( P_a \) is a cp-mc filter. In the following, we do not distinguish between cp-mc filters and homomorphisms into \( 2 \), based on Proposition 2.17. Then, we have

\[
C = \langle a \rangle = \bigcap \{ \langle x \rangle : x \in P_a \} = \bigcap \{ \langle x \rangle : P_a \subseteq \langle x \rangle \} = \text{ch}(P_a).
\]

To show the uniqueness, assume that, for \( P, Q \in \text{Spec}(L) \), \( \text{ch}(P) = C = \text{ch}(Q) \). Suppose for contradiction that \( P \neq Q \). Then, we may assume that there is \( b \in L \) such that \( b \in P \) and \( b \notin Q \). Therefore, we have \( \text{ch}(P) \subseteq \langle b \rangle \) and \( \lnot (\text{ch}(Q) \subseteq \langle b \rangle) \), which is a contradiction. \( \square \)

By letting \( L \) be the convexity algebra of convex sets in a convexity space, Spec(L) can be considered as the space of polytopes in the convexity space. Spaces of polytopes in convexity spaces seem to be natural examples of sober convexity spaces.

**Proposition 4.16.** Let \( S \) be a sober convexity space and \( C \) its convexity. Then, \( \Psi_S : S \to \text{Spec} \circ \text{Conv}(S) \) is an isomorphism in \( \text{CS} \).

**Proof.** We first show that \( \Psi_S \) is injective. Assume that \( \Psi_S(x) = \Psi_S(y) \) for \( x, y \in S \). Then, we have \( \Psi_S(x)^{-1}(\{1\}) = \Psi_S(y)^{-1}(\{1\}) \); i.e.,

\[
\{ f \in \text{Conv}(S) : f(x) = 1 \} = \{ f \in \text{Conv}(S) : f(y) = 1 \}.
\]

By Proposition 2.5, we have \( \{ C \in C : x \in C \} = \{ C \in C : y \in C \} \). By taking the intersections, it follows from the definition of convex hull that

\[
\text{ch}(\{x\}) = \bigcap \{ C \in C : x \in C \} = \bigcap \{ C \in C : y \in C \} = \text{ch}(\{y\}).
\]

Since \( S \) is sober and since \( \text{ch}(\{x\}) \) is a chain-irreducible convex set, we have \( x = y \). Thus, \( \Psi_S \) is injective.

We next show that \( \Psi_S \) is surjective. Let \( v \in \text{Spec} \circ \text{Conv}(S) \). By Proposition 2.17, \( v^{-1}(\{1\}) \) is a cp-mc filter of \( \text{Conv}(S) \). By Proposition 2.16, \( \bigwedge v^{-1}(\{1\}) \) is a chain-compact element in \( \text{Conv}(S) \). Since \( \text{Conv}(S) \) is isomorphic to the convexity algebra \( C \) via the map \( f \mapsto f^{-1}(\{1\}) \), it follows that

\[
\bigcap \{ f^{-1}(\{1\}) : f \in v^{-1}(\{1\}) \}
\]

is a chain-compact element in \( C \) and is thus a chain-irreducible convex set in \( S \). Since \( S \) is sober, there is \( x \in S \) such that

\[
\bigcap \{ f^{-1}(\{1\}) : f \in v^{-1}(\{1\}) \} = \text{ch}(\{x\}).
\]
We claim that $\Psi_S(x) = v$. Let $g \in \text{Conv}(S)$. We first assume that $v(g) = 1$. Then, we have $g \in v^{-1}([1])$. By the choice of $x$, we have $x \in \text{ch}(\{x\}) \subset g^{-1}([1])$. Thus, it follows that $\Psi_S(x)(g) = 1 = v(g)$. We next assume that $v(g) = 0$. Suppose for contradiction that $\Psi_S(x)(g) = 1$; i.e., $g(x) = 1$. Since $g^{-1}([1])$ is a convex set in $S$ and $x \in g^{-1}([1])$, we have
\[
\bigcap_{g^{-1}([1])} f \in v^{-1}([1])) = \text{ch}(\{x\}) \subset g^{-1}([1])).
\]
Thus, we have $\bigwedge g^{-1}([1]) \leq g \in \text{Conv}(S)$. Since $v^{-1}([1])$ is a cp-mc filter and $\bigwedge g^{-1}([1]) \in v^{-1}([1])$, we have $g \in v^{-1}([1])$, which contradicts $v(g) = 0$. Therefore, we have $\Psi_S(x)(g) = 0 = v(g)$. Thus, we obtain $\Psi_S(x) = v$. Hence, $\Psi_S$ is surjective.

It has already been shown that $\Psi_S$ is a convexity-preserving map. To complete the proof, we show that $\Psi_S^{-1}$ is a convexity-preserving map. Let $C$ be a convex set in $S$. Define $f_C : S \to 2$ as in the proof of Proposition 2.5. We claim that $\Psi_S(C) = \langle f_C \rangle$. Suppose that $v \in \Psi_S(C)$. Then, $v = \Psi_S(x)$ for some $x \in C$, whence we have
\[
v(f_C) = \Psi_S(x)(f_C) = f_C(x) = 1.
\]
Hence, we have $v \in \langle f_C \rangle$. Conversely, suppose that $v \in \langle f_C \rangle$. Since $\Psi_S$ is surjective, there exists $x \in S$ such that $\Psi_S(x) = v$. By $v \in \langle f_C \rangle$, we have $\Psi_S(x)(f_C) = f_C(x) = 1$; i.e., $x \in C$. Hence, we have $v \in \Psi_S(C)$. $\square$

In this way, we can recover the points of a sober convexity space from the convexity algebra of convex sets in it. The above proposition implies that any sober convexity space can be represented as $\text{Spec}(S)$ for a convexity algebra $L$.

By Propositions 4.15 and 4.16, we have the following characterization of sobriety.

**Proposition 4.17.** For a convexity space $S$, $S$ is sober iff $\Psi_S$ is an isomorphism in CS.

### 4.3. Duality between SpCA and SobCS

$\text{SpCA}$ denotes the category of spatial convexity algebras and homomorphisms. $\text{SobCS}$ denotes the category of sober convexity spaces and convexity-preserving maps. Finally we obtain the following duality between spatial convexity algebras and sober convexity spaces.

**Theorem 4.18.** $\text{SpCA}$ and $\text{SobCS}$ are dually equivalent via the functors $\text{Spec}$ and $\text{Conv}$.

**Proof.** By Proposition 4.15, (the restriction of) $\text{Spec}$ is well defined. By Proposition 4.3, (the restriction of) $\text{Conv}$ is well defined. By Proposition 4.5, $\phi : 1_{\text{SpCA}} \to \text{Conv} \circ \text{Spec}$ is a natural isomorphism. By Proposition 4.16, $\psi : 1_{\text{SobCS}} \to \text{Spec} \circ \text{Conv}$ is a natural isomorphism. $\square$

This is a convexity-theoretical analogue of Isbell duality between spatial frames and sober topological spaces. However, there is a big difference between the above duality and Isbell duality, especially between the notion of sobriety for convexity spaces and the notion of sobriety for topological spaces. That is, most of ordinary topological spaces such as $\mathbb{R}^n$ are sober and so fall into Isbell duality, while most of ordinary convexity spaces such as $\mathbb{R}^n$ are not sober and so do not fall into the above duality. In the next section, we consider another duality into which most of ordinary convexity spaces do fall.

## 5. Duality between mSpCA and mSobCS

In this section, by introducing the notions of m-sparsity, m-homomorphism and m-sobriety, we shall show a duality between the category of m-spatial convexity algebras and m-homomorphisms and the category of m-sober convexity spaces and convexity-preserving maps.

### 5.1. m-sparsity and m-sobriety

For a convexity algebra $L$, we mean by an m-mc filter of $L$ a maximal meet-complete filter of $L$ where maximality means that with respect to inclusion.

Consider a convexity space $(S, C)$ such that $\{x\}$ is convex for any $x \in S$. Then, for $x \in S$, $\{C \in C : x \in C\}$ is an m-mc filter of the convexity algebra $C$.

**Lemma 5.1.** Let $M$ be an m-mc filter of a convexity algebra $L$. Then, $M$ is a cp-mc filter of $L$.

**Proof.** Assume that $\bigvee_{i \in I} a_i \in M$ for a totally ordered subset $\{a_i : i \in I\}$ of $L$. Suppose for contradiction that, for any $i \in I$, $a_i \notin M$. Since $M$ is an m-mc filter, we have the following. For any $i \in I$, there is $b_i \in M$ such that $a_i \wedge b_i = 0$. Then, we have $\bigwedge_{i \in I} b_i \in M$ by $b_i \in M$. We also have $a_i \wedge (\bigwedge_{i \in I} b_i) = 0$, whence it follows that
\[
\bigvee_{i \in I} \left( a_i \wedge \left( \bigwedge_{i \in I} b_i \right) \right) = 0.
\]
Since \( \{a_i : i \in I\} \) is totally ordered, it follows from the chain-completely distributive law (i.e., item 3 in Definition 2.9) that

\[
\left( \bigvee_{i \in I} a_i \right) \wedge \left( \bigwedge_{i \in I} b_i \right) = 0.
\]

Since \( \bigwedge_{i \in I} b_i \in M \) and \( \bigvee_{i \in I} a_i \in M \), we have 0 \( \in M \), which is a contradiction. Thus, there is \( i \in I \) such that \( a_i \in M \). Hence, \( M \) is a cp-mc filter of \( L \). \( \square \)

Then, m-spatiality is defined as follows.

**Definition 5.2.** A convexity algebra \( L \) is called m-spatial iff for any \( a, b \in L \) with \( a \nleq b \) there is an m-mc filter \( M \) of \( L \) such that \( a \in M \) and \( b \notin M \).

By Lemma 5.1, we obtain the following proposition.

**Proposition 5.3.** Let \( L \) be a convexity algebra. If \( L \) is m-spatial then \( L \) is spatial.

We remark that, although m-spatiality implies spatiality, m-soberness defined below does not imply sobriety.

We next introduce the notion of m-homomorphism. A similar notion is used also in the context of duality theory for distributive semilattices (see [14, 16]).

**Definition 5.4.** An m-homomorphism \( f : L_1 \to L_2 \) between convexity algebras \( L_1 \) and \( L_2 \) is defined as a homomorphism of convexity algebras such that, for any m-mc filter \( M \) of \( L_2 \), \( f^{-1}(M) \) is an m-mc filter of \( L_1 \).

It shall be shown that the dual notion of a convexity-preserving map between m-sober (defined below) convexity spaces is an m-homomorphism and is not a homomorphism.

Let us review the concept of an atomistic poset (see [10]). Recall that an atom in a poset \( P \) with a least element 0 is an element of \( P \) that is minimal in \( P \setminus \{0\} \).

**Definition 5.5.** A poset with a least element 0 is called atomistic iff any element of the poset is the join of a set of atoms of the poset.

Note that, in general, being atomistic is not equivalent to being atomic.

We can provide an algebraic characterization of m-spatiality as follows.

**Proposition 5.6.** For a convexity algebra \( L \), the following are equivalent:

1. \( L \) is m-spatial;
2. \( L \) is atomistic.

**Proof.** We first show that 1 implies 2. Let \( a \in L \). If \( a = 0 \) then \( a \) is the join of \( \emptyset \). Assume that \( a > 0 \). Let \( a' \) be the join of those atoms \( x \in L \) such that \( x \leq a \). It suffices to show that \( a = a' \). Since if there is no atom \( x \in L \) with \( x \leq a \) then we have \( a' = 0 \), it follows from the choice of \( a' \) that \( a \geq a' \). Suppose for contradiction that \( a > a' \). Since \( L \) is m-spatial, there is an m-mc filter \( M \) of \( L \) such that \( a \in M \) and \( a' \notin M \). Since \( M \) is an m-mc filter of \( L \), \( \bigwedge M \) is an atom of \( L \). By \( a \in M \), we also have \( \bigwedge M \leq a \). Then, it follows from the definition of \( a' \) that

\[
\bigcap M \leq a'.
\]

Since \( M \) is a meet-complete filter, we have \( \bigwedge M \in M \), and so \( a' \in M \), which contradicts \( a' \notin M \). Hence, \( a = a' \).

We next show that 2 implies 1. Let \( a, b \in L \) with \( a \nleq b \). Since \( L \) is atomistic, there is a set \( A \) of atoms of \( L \) such that \( \bigvee A = a \). Similarly, there is a set \( B \) of atoms of \( L \) such that \( \bigvee B = b \). Then, we may assume that \( B \) is the set of those atoms \( x \in L \) such that \( x \leq b \). By \( a \nleq b \), there is \( c \in A \) such that \( c \notin B \). Define

\[
M = \{ x \in L : c \leq x \}.
\]

Then, we have both \( a \in M \) and \( b \notin M \), since \( B \) is the set of those atoms \( x \in L \) such that \( x \leq b \). Now, it remains to show that \( M \) is an m-mc filter of \( L \), which follows from the fact that \( c \) is an atom of \( L \). \( \square \)

We next introduce the notion of m-soberness.

**Definition 5.7.** A convex space \( S \) is called m-sober iff \( \{x\} \) is convex for any \( x \in S \).

Many ordinary convexity spaces are m-sober, including those in Example 2.2.

An m-sober convexity space is not necessarily sober. For example, the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \) with the usual convexity (see Example 2.2) is not sober and is m-sober. The same thing holds also for other convexity spaces such as those in Example 2.2. Thus, we may consider that the notion of m-soberness is more natural than that of sobriety.
5.2. Duality between mSpCA and mSobCS

In this subsection we show a dual equivalence between categories mSpCA and mSobCS, which are defined as follows.

**Definition 5.8.** mSobCS denotes the category of m-sober convexity spaces and convexity-preserving maps. mSpCA denotes the category of m-spatial convexity algebras and m-homomorphisms.

We introduce a functor mSpec based on the view that a point is an m-mc filter.

**Definition 5.9.** We define a contravariant functor mSpec from mSpCA to mSobCS as follows.

1. For an object L in mSpCA, mSpec(L) is defined as the set of all m-mc filters of L equipped with the convexity generated by \{ \{a\}_m : a \in L\}, where \( \langle a \rangle_m = \{M \in m\text{Spec}(L) : a \in M\} \).

2. For an arrow \( f : L_1 \to L_2 \) in mSpCA, mSpec(f) : mSpec(L_2) \to mSpec(L_1) is defined by \( m\text{Spec}(f)(M) = f^{-1}(M) \) for \( M \in m\text{Spec}(L_2) \).

The well-definedness of the functor mSpec is shown by the following proposition.

**Proposition 5.10.** For a convexity algebra L, mSpec(L) is m-sober.

**Proof.** Let \( M \in m\text{Spec}(L) \). We claim that \( \langle \bigwedge M \rangle_m = \{ M \} \). Since \( \bigwedge M \in M \), we have \( \{ M \} \subseteq \langle \bigwedge M \rangle_m \).

Assume that \( \bigwedge M \in N \) for \( N \in m\text{Spec}(L) \). It follows from \( M, N \in m\text{Spec}(L) \) that \( M \subseteq N \), and so \( M = N \) by maximality. Thus, we have \( \langle \bigwedge M \rangle_m \subseteq \{ M \} \).

Therefore, we have \( \langle \bigwedge M \rangle_m = \{ M \} \). Hence, \( \{ M \} \) is convex. □

**Remark 5.11.** Throughout this section, based on Proposition 2.5, we consider Conv as a functor from mSobCS to mSpCA as follows. For an object \( S \) in mSobCS, Conv(S) is defined as the convexity algebra of all convex subsets of \( S \). For an arrow \( f : S_1 \to S_2 \) in mSobCS, Conv(f) : Conv(S_2) \to Conv(S_1) is defined by Conv(f)(C) = f^{-1}(C) for \( C \in Conv(S_2) \).

Then, the well-definedness of the functor Conv : mSobCS \to mSpCA is shown by the following two propositions.

**Proposition 5.12.** Let \( S \) be an object in mSobCS. Then, Conv(S) is an m-spatial convexity algebra.

**Proof.** Let \( C_1, C_2 \in Conv(S) \) such that \( C_1 \) is not a subset of \( C_2 \). Then, there is \( x \in C_1 \) with \( x \notin C_2 \). Define

\[ M = \{ C \in Conv(S) : x \in C \} \]

Then, we have both \( C_1 \in M \) and \( C_2 \notin M \). Now, it suffices to show that \( M \) is an m-mc filter of Conv(S). It is straightforward to verify that \( M \) is a meet-complete filter. Since \( S \) is m-sober, \( \{x\} \) is convex and so \( \{x\} \in M \). If \( C \notin M \) for \( C \in Conv(S) \), then \( C \cap \{x\} = \emptyset \). Thus, \( M \) is maximal. □

Since an arrow in mSpCA is an m-homomorphism, not a homomorphism, it is important to verify that the arrow part of Conv is well defined.

**Proposition 5.13.** Let \( f : S_1 \to S_2 \) be an arrow in mSobCS. Then, Conv(f) : Conv(S_2) \to Conv(S_1) is an m-homomorphism.

**Proof.** Clearly, Conv(f) is a homomorphism. Let \( M \) be an m-mc filter of Conv(S_1). Since \( \bigwedge M \subseteq M \) and \( M \neq \bigwedge M \), we have \( \bigwedge M \neq \emptyset \), and so there is \( m \in \bigwedge M \). Then, \( M \subseteq \{ C \in Conv(S_1) : m \in C \} \). Since \( \{ C \in Conv(S_1) : m \in C \} \) is a proper meet-complete filter, it follows from the maximality of \( M \) that

\[ M = \{ C \in Conv(S_1) : m \in C \} \]

Thus, it follows that

\[ Conv(f)^{-1}(M) = \{ C \in Conv(S_2) : f^{-1}(C) \in M \} = \{ C \in Conv(S_2) : m \in f^{-1}(C) \} = \{ C \in Conv(S_2) : f(m) \in C \} \]

Since \( S_2 \) is m-sober, \( \{f(m)\} \) is convex and so \( \{ C \in Conv(S_2) : f(m) \in C \} \) is an m-mc filter of Conv(S_2). This completes the proof. □
We next define two natural transformations. Let \( \text{Id}_1 \) denote the identity functor on \( \text{mSpCA} \) and \( \text{Id}_2 \) the identity functor on \( \text{mSobCS} \).

**Definition 5.14.** We define a natural transformation \( \alpha : \text{Id}_1 \to \text{Conv} \circ \text{mSpec} \) as follows. For an \( m \)-spatial convexity algebra \( L \), define \( \alpha_L : L \to \text{Conv} \circ \text{mSpec}(L) \) by
\[
\alpha_L(a) = \{ M \in \text{mSpec}(L) : a \in M \} = \langle a \rangle_m.
\]

It is straightforward to verify that \( \alpha \) is actually a natural transformation.

**Proposition 5.15.** For an \( m \)-spatial convexity algebra \( L \), \( \alpha_L : L \to \text{Conv} \circ \text{mSpec}(L) \) is an isomorphism in \( \text{mSpCA} \).

**Proof.** Since an isomorphism in \( \text{CA} \) is always an isomorphism in \( \text{mSpCA} \), it suffices to show that \( \alpha_L \) is an isomorphism in \( \text{CA} \). We first show that \( \alpha_L \) is a homomorphism. By \( \text{Lemma 5.1} \), an \( m \)-mc filter is a \( cp \)-mc filter. Thus, we have
\[
\left( \bigvee_{i \in I} a_i \right)_m = \bigcup_{i \in I} \langle a_i \rangle_m
\]
for a totally ordered subset \( \{ a_i : i \in I \} \) of \( L \). We also have
\[
\left( \bigwedge_{i \in I} a_i \right)_m = \bigcap_{i \in I} \langle a_i \rangle_m
\]
for a subset \( \{ a_i : i \in I \} \) of \( L \). Thus, \( \alpha_L \) is a homomorphism. It is straightforward to see that \( \alpha_L \) is injective by the \( m \)-spatiality of \( L \). By arguing as in the proof of \( \text{Lemma 3.5} \), it is shown that \( \{ \langle a \rangle_m : a \in L \} \) coincides with the convexity of \( \text{mSpec}(L) \). Thus, \( \alpha_L \) is surjective. This completes the proof. □

**Definition 5.16.** We define a natural transformation \( \beta : \text{Id}_2 \to \text{mSpec} \circ \text{Conv} \) as follows. For an \( m \)-sober convexity space \( S \), define \( \beta_S : S \to \text{mSpec} \circ \text{Conv}(S) \) by
\[
\beta_S(x) = \{ C \in \text{Conv}(S) : x \in C \}.
\]

It is straightforward to verify that \( \beta \) is actually a natural transformation.

**Proposition 5.17.** For an \( m \)-sober convexity space \( S \), \( \beta_S : S \to \text{mSpec} \circ \text{Conv}(S) \) is an isomorphism in \( \text{mSobCS} \).

**Proof.** Since \( S \) is \( m \)-sober, \( \beta_S(x) \) is an \( m \)-mc filter for \( x \in S \) and so \( \beta_S \) is well defined. Clearly, \( \beta_S \) is injective. Since \( S \) is \( m \)-sober, an \( m \)-mc filter of \( \text{Conv}(S) \) is of the form
\[
\{ C \in \text{Conv}(S) : x \in C \}
\]
for some \( x \in S \). Thus, \( \beta_S \) is surjective. Since \( \beta_S^{-1}(\langle C \rangle_m) = C \) for \( C \in \text{Conv}(S) \), \( \beta_S \) is convexity preserving. It is easily verified that
\[
\beta_S(C) = \langle C \rangle_m
\]
for \( C \in \text{Conv}(S) \), whence \( \beta_S^{-1} \) is convexity preserving. Hence, \( \beta_S \) is an isomorphism in \( \text{mSobCS} \). □

In this way, we can recover the points of an \( m \)-sober convexity space from the convexity algebra of convex sets in it. This proposition implies that any \( m \)-sober convexity space can be represented as \( \text{mSpec}(L) \) for a convexity algebra \( L \), where note that most of ordinary convexity spaces are \( m \)-sober.

By \( \text{Propositions 5.15 and 5.17} \), \( \alpha \) and \( \beta \) are natural isomorphisms, and thus we obtain the following duality between \( m \)-spatial convexity algebras and \( m \)-sober convexity spaces.

**Theorem 5.18.** \( \text{mSpCA} \) and \( \text{mSobCS} \) are dually equivalent via the functors \( \text{mSpec} \) and \( \text{Conv} \).

Most of ordinary convexity spaces are \( m \)-sober (recall that a singleton is usually convex), and thus fall into the above duality.

We remark that convexity-preserving maps between \( m \)-sober convexity spaces correspond to \( m \)-homomorphisms between \( m \)-spatial convexity algebras and do not correspond to homomorphisms.
6. Concluding remarks

In this work, we have obtained the following main results with other modest ones: (1) Spec and Conv give a dual adjunction between CA and CS; (2) SpCA and SobCS are dually equivalent via Spec and Conv; (3) mSpCA and mSobCS are dually equivalent via mSpec and Conv. Note that many ordinary convexity spaces are not sober but m-sober, while most of ordinary topological spaces are sober, which is a striking difference between topology and convex geometry. Now, (1) and (2) are based on the view that a point is a cp-mc filter, while (3) is based on the view that a point is an m-mc filter. Then, natural questions arise. Which view is better? Which notion of point is the proper one? Our answers are as follows.

It seems difficult to obtain a dual adjunction between the category of all convexity algebras and the category of all convexity spaces based on the view that a point is an m-mc filter. Some of the reasons are as follows: (1) For a convexity space S and x ∈ S, βS(x) (see Definition 5.16) is not always an m-mc filter of Conv(S); (2) the left adjoint functor of Conv is uniquely determined up to isomorphism (see [2]) and it is Spec, not mSpec. Thus, we may consider that the view that a point is a cp-mc filter is superior to the view that a point is an m-mc filter from a category-theoretic standpoint.

However, the proper notion of a point seems to be an m-mc filter. Consider the n-dimensional Euclidean space \( \mathbb{R}^n \) equipped with the usual convexity. Then, Spec \( \circ \text{Conv}(\mathbb{R}^n) \) does not coincide with \( \mathbb{R}^n \) (i.e., \( \mathbb{R}^n \) is not a sober convexity space, though it is a sober topological space) but coincides with the set of polytopes in \( \mathbb{R}^n \) by Proposition 2.15. The same thing holds true not only for \( \mathbb{R}^n \) but also for many other ordinary convexity spaces such as vector spaces over \( \mathbb{R} \) and manifolds including the n-sphere and the n-dimensional real projective space (for their convexities, see Example 2.2). Actually, for any convexity space S, the space of polytopes in S coincides with Spec \( \circ \text{Conv}(S) \) by Proposition 2.15 and is thus sober by Proposition 4.15, whence we can notice that the space of polytopes in a convexity space is the sobriﬁcation of the convexity space. Note that conversely any sober convexity space can be represented as the space of polytopes in a convexity space by Proposition 4.16.

Therefore, we conclude that a cp-mc filter (or a homomorphism into \( 2 \)) actually represents a polytope, not a point, and an m-mc filter properly represents a point in many ordinary cases. In a nutshell, an m-mc filter is a point and a cp-mc filter is a “general” point, which makes it possible to generate a polytope as the convex hull of a unique point, as in algebraic geometry a prime ideal is considered as a generic point, which makes it possible to generate an irreducible algebraic variety as the closure of a unique point (see [17]). In this sense, the notion of a polytope in convex geometry corresponds to that of an irreducible algebraic variety in algebraic geometry.

We remark that pointless convex geometry is closely related to domain theory (for domain theory, see [13]). In this paper, the notions of convexity space and convexity algebra are defined in terms of chains. However, it is possible to define them in terms of directed sets instead of chains, and most of arguments in this paper work well even if we replace chains with directed sets. Interestingly, the notion of a continuous lattice, which is well known in domain theory, coincides with the notion of convexity algebra deﬁned in terms of directed sets, which follows from [13, Theorem 1-2.7]. Moreover, under this reformulation of related notions, the duality between spatial convexity algebras and sober convexity spaces turns out to reveal the convexity-theoretical nature of the Hoffman–Mislove–Stralka duality between algebraic (continuous) lattices and join-semilattices with the least elements (for this duality, see [13, Theorem IV-1.16] and [18]), though there is a minor difference between the morphism parts of the two dualities. Here note that algebraic lattices coincide with spatial convexity algebras. By combining the two dualities, we notice that sober convexity spaces are equivalent to join-semilattices with the least elements. A consequence of this observation is that the set of polytopes in any convexity space forms a join-semilattice with the least element and conversely any join-semilattice with the least element can be represented as the join-semilattice of polytopes in a convexity space. Another consequence of it is that the set of ideals of any join-semilattice forms a sober convexity on the join-semilattice and conversely any sober convexity space can be represented as a join-semilattice equipped with the convexity consisting of ideals of the join-semilattice. In some sense, pointless convex geometry formulated in terms of directed sets is nothing but the theory of continuous lattices.

We next discuss the relationships between pointless convex geometry and Hilbert’s philosophy. In the introduction of [8], Coquand states that Hilbert’s program may be reformulated using pointless topology. According to the results in this paper and the idea in [8], we notice that pointless convex geometry may also be considered to be in harmony with Hilbert’s philosophy, especially his instrumentalism (see [11]). We remark that this view seems to hold true of locale theory, but it is not clear whether or not the view on locale theory is the same as Coquand’s one. This harmony may be explained as follows. Convexity algebras correspond to real objects in Hilbert’s sense, which actually exist. On the other hand, the points of convexity algebras correspond to ideal objects in his sense, which do not actually exist, and are mere instruments for the study of the real objects. We may work in the category of convexity spaces by using the functors Spec and mSpec, which correspond to the introduction of ideal objects in his sense. However, results obtained by using the ideal objects can (sometimes) be pulled back to the category of convexity algebras via the functor Conv, which corresponds to the elimination of ideal objects in his sense.

Finally, we discuss the relationships between our view on pointless mathematics and Husserl’s phenomenology (for Husserl’s phenomenology, see [19,29]; for the relationships between Husserl’s phenomenology and Brouwer’s notion of the continuum, which may be thought of as an origin of pointless topology, see [29]). In [25], Husserl’s phenomenology of space is summarized as follows: “Epistemology of space before ontology of space”. According to our view in the first paragraph of Section 1, this may be paraphrased as follows: “Region before point” in terms of pointless geometry, “Theory before model” in logical terms and “Algebra before spectrum” in terms of duality theory and algebraic geometry. We conclude the paper by emphasizing that a region is the epistemological ingredient of the notion of space and a point is the ontological
or metaphysical ingredient of it, whence duality theory clarifies the relationships between epistemological and ontological aspects of the notion of space.

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