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Geometric construction of quintic parametric B-splines

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Abstract

The aim of this paper is to present a new class of B-spline-like functions with tension properties. The main feature of these basis functions consists in possessing C^3 or even C^4 continuity and, at the same time, being endowed by shape parameters that can be easily handled. Therefore they constitute a useful tool for the construction of curves satisfying some prescribed shape constraints. The construction is based on a geometric approach which uses parametric curves with piecewise quintic components.

MSC: 41A25; 41A29; 65D05

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1. Introduction

In several applications it is required to construct smooth functions or parametric curves, interpolating or approximating a given set of data, and reproducing their salient geometric properties. Classical methods based on piecewise polynomials, often do not produce interpolants or approximants satisfying the required constraints. Thus a great deal of research has focused on the study of new function spaces for building constrained curves, see for instance [5].

The methods proposed so far rarely produce curves with (analytic) smoothness order greater than two, [4], even if curves with at least C^3 continuity are often preferable in some industrial applications as in the design of robot trajectories.

Therefore, our recent studies have concentrated attention on the construction of a B-spline-like basis such that any element of the basis is of class C^r with $r \geq 3$ and possesses tension parameters to control its shape. The first results are presented in [9]. The basis functions have been obtained by a simple geometric construction and have shape parameters with a clear geometric interpretation, which is crucial for their automatic selection. The main tool for the proposed construction is given by the parametric techniques which consist in regarding the basis functions

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as special parametric curves having piecewise quintic components. The resulting basis in [9] is of class C^3 and has minimal (four intervals) support; this last property which entails a good localization, has the drawback of being able to model properly only planar curves. Indeed, by construction any curve obtained as the linear combination of such basis elements, with coefficients in $\mathbb{R}^3$, is of course of class C^3 but has zero torsion at the knots. This problem can be solved by considering basis functions with larger support.

The aim of this paper is to present a new one-parameter family of C^4 B-spline-like functions with 6 intervals support possessing shape parameters: as far as we know function spaces with such a high order of smoothness have never been proposed before in the context of constrained interpolation/approximation. These new functions are achieved by a two-step procedure.

First, we construct a two-parameter family of C^3 B-spline-like functions with 6 intervals support, obtained again by the parametric approach considering planar curves with piecewise quintic components and geometric continuity of order three. The goal is reached by coupling the parametric approach with an extension of the geometric construction presented in [3] where $C^2 \cap FC^3$ piecewise quintic curves were obtained. The resulting basis functions overcome the drawbacks of those in [9] and can be profitably used, in practical applications.

The second step consists in extending our construction to the fourth degree of smoothness by means of the geometric continuity of order four in the parametric setting. This final result is based on the detailed description of geometric B-splines given in [2].

We remark that the basis functions in [4] have a three interval support, are of class C^3, allow construction of space curves with non-necessarily zero torsion (see also [3]), and closely imitate the structure of quintic B-splines with double knots. In particular, any three consecutive intervals have a pair of basis functions associated to it. As a consequence, the dimension of the spanned space doubles the number of the intervals (plus a given number of "boundary conditions"). On the other hand, in some practical problems (mainly interpolation) the data extension matches the number of knots: so it is preferable to deal with spaces of the same dimension of the knot sequence (but for a given number of "boundary conditions"). This is exactly achieved by the basis elements proposed in [9] (in such a sense the support of these basis functions is *minimal*) and by the spaces of functions that we construct in the present paper.

The remainder of this paper is organized into 4 sections. In the next one we review the basic ideas of the parametric techniques showing how analytic continuity can be obtained through geometric continuity. In Section 3, we present the construction of B-spline-like functions of class C^3 and analyse their behavior as the shape parameters tend to limit values, while in Section 4 we extend the geometric construction in order to obtain parametric curves with geometric continuity of order four and therefore functions of class C^4. Section 5 is devoted to some applications and concluding remarks.

To prevent some possible ambiguities, we emphasize that in the paper, parametric curves are addressed from two different points of view. First the B-spline-like basis *functions* are constructed as special *parametric curves*. Then, as a possible application, these functions are used in the context of constrained approximation of parametric space curves (see Section 5).

2. Parametric approach

We construct our basis functions as particular planar parametric curves, according to the so-called parametric techniques, introduced some years ago in the context of tension methods for shape preserving interpolation, see for instance [7,8]. To help in the comprehension of the, sometimes heavy, notation, we mention that throughout the paper bold characters denote vectors, while italic characters and Greek letters are used for scalar quantities.

In the parametric approach, the graph of the function $x \to s(x)$ is seen as the support of a particular planar parametric curve. Let us consider the curve:

$$\mathbf{C}(t) := (X(t), Y(t)), \quad t \in [t_0, t_1]. \tag{1}$$

If we assume that

$$X_t(t) := \frac{\mathrm{d}}{\mathrm{d}t} X(t) > 0, \tag{2}$$

then the first component is invertible and it is well-defined $t = t(x)$, where $x = X(t)$, therefore the image of the curve (1) can be seen as the graph of a function

$$s(x) := Y(t(x)), \quad x \in [x_0, x_1],\ x_0 := X(t_0),\ x_1 := X(t_1). \tag{3}$$

If the curve $\mathbf{C}(t)$ has C^l components, then the function s is of class C^l as well and its derivatives with respect to the x variable can be expressed in terms of the derivatives of the two components of $\mathbf{C}$ with respect to t.

Dealing with piecewise functions, let us consider, in the parametric setting, two adjacent planar parametric curves of class C^l, $\mathbf{C}^+(t) := (X^+(t), Y^+(t))$ with $t \in [t_0, t_1]$, and $\mathbf{C}^-(u) := (X^-(u), Y^-(u))$, with $u \in [u_0, u_1]$, where $\mathbf{C}^+(t_0) = \mathbf{C}^-(u_1) =: \mathbf{P}$.

Following [6], we say that $\mathbf{C}^+$ and $\mathbf{C}^-$ have *Geometric Continuity* of order $r \leq l$ (GC^r for short) at $\mathbf{P}$ if there exists an algebraic curve which meets both curves at $\mathbf{P}$ with contact of order r, that is if there exists a reparameterization $u = u(t)$, of $\mathbf{C}^-$, with $u(t_0) = u_1$, such that

$$\frac{\mathrm{d}^p \mathbf{C}^+(t)}{\mathrm{d}^p t}_{|t=t_0} = \frac{\mathrm{d}^p \mathbf{C}^-(u(t))}{\mathrm{d}^p t}_{|u(t_0)}, \quad p = 0, \ldots, r. \tag{4}$$

Hence, setting

$$w_p := \frac{\mathrm{d}^p u(t)}{\mathrm{d}^p t}_{|t=t_0}, \quad w_1 > 0,\ p \geq 1,$$

$\mathbf{C}^+$, $\mathbf{C}^-$ are GC^4 continuous at $\mathbf{P}$ if and only if

$$\begin{pmatrix} \mathbf{C}_t^+(t_0^+) \\ \mathbf{C}_{tt}^+(t_0^+) \\ \mathbf{C}_{ttt}^+(t_0^+) \\ \mathbf{C}_{tttt}^+(t_0^+) \end{pmatrix} = \begin{bmatrix} w_1 & 0 & 0 & 0 \\ w_2 & w_1^2 & 0 & 0 \\ w_3 & 3w_1w_2 & w_1^3 & 0 \\ w_4 & 3w_2^2 + 4w_1w_3 & 6w_1^2w_2 & w_1^4 \end{bmatrix} \begin{pmatrix} \mathbf{C}_u^-(u_1^-) \\ \mathbf{C}_{uu}^-(u_1^-) \\ \mathbf{C}_{uuu}^-(u_1^-) \\ \mathbf{C}_{uuuu}^-(u_1^-) \end{pmatrix}. \tag{5}$$

We refer to [6], Chapter 5, for further comments on the geometric meaning of GC^r continuity and on its connections with the maybe better known Frénet continuity (FC).

Let us assume now $X_t^+(t) > 0$ and $X_u^-(u) > 0$. According to (3) we may define

$$s^+(x) := Y^+(t(x)), \qquad s^-(x) := Y^-(u(x)).$$

From (4), by elementary computations, it turns out that $s^+(x)$ and $s^-(x)$ join with continuous derivatives up to order r at $x_0 := X^+(t_0) = X^-(u_1)$ if and only if $\mathbf{C}^+$, $\mathbf{C}^-$ are GC^r continuous at $\mathbf{P}$. Summarizing, we can obtain a function of class C^r considering a collection of adjacent segments of GC^r planar curves satisfying (2).

3. C^3 B-spline-like basis

Our aim is to build up a B-spline-like basis with C^3 continuity. Each basis function is regarded as a particular parametric planar curve whose components are piecewise quintic polynomials with GC^3 continuity. If we express the curve segments in the Bézier form we may derive conditions on the Bézier control points which assure the fulfillment of GC^3 continuity. Let us put, for $t \in [t_i, t_{i+1}]$,

$$\mathbf{C}_i(t) := \sum_{j=0}^{n} \mathbf{b}_{i,j} b_j^{(n)}(\tau), \quad \tau := \frac{t - t_i}{h_i},\ h_i := t_{i+1} - t_i,$$

where $\mathbf{b}_{i,j} \in \mathbb{R}^2$ are the Bézier control points of $\mathbf{C}_i$ and $b_j^{(n)}$, $j = 0, \ldots, n$, denote the Bernstein polynomials

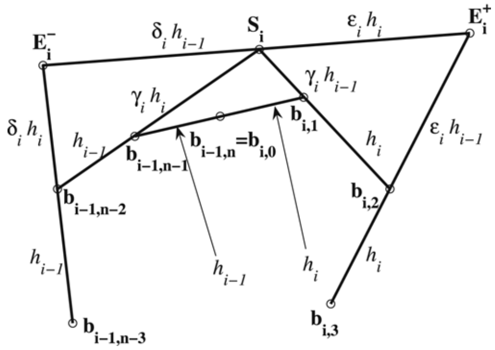

Fig. 1. Geometric construction of a C^1, FC^3 join for Bézier curves of degree n.

of degree n. It is well-known (see [2], [6]) that $\mathbf{C}_{i-1}$ and $\mathbf{C}_i$ meet with C^1 and FC^3 continuities if and only if the last 4 Bézier control points of $\mathbf{C}_{i-1}$ and the first 4 Bézier control points of $\mathbf{C}_i$ satisfy the relations determined by the geometric construction shown in Fig. 1, where $\gamma_i, \delta_i, \epsilon_i$, are free design parameters. If $\gamma_i = \delta_i = \epsilon_i = 1$ this construction reduces to the geometric construction of classical C^3 analytic continuity. FC^3 continuity is less restrictive than GC^3, [6], and indeed in the last case the design parameters are no longer independent. For symmetry reasons we may set $\delta_i = \epsilon_i$, and considering quintic curves (i.e. $n = 5$), with some computation we obtain, from (5), that $\mathbf{C}_{i-1}$ and $\mathbf{C}_i$ meet with GC^3 continuity at $\mathbf{b}_{i-1,5} = \mathbf{b}_{i,0}$ if and only if

$$\delta_i = \epsilon_i = \frac{\gamma_i(\gamma_i + 1)}{2(2 - \gamma_i)}, \quad \gamma_i \neq 2. \tag{6}$$

If $\gamma_i = 1$ then (6) gives $\delta_i = \epsilon_i = 1$, that is we recover C^3 continuity.

Given now the values $h_i > 0,\ i = -5, \ldots, N + 4$, and the increasing vector

$$\xi_{-5} < \xi_{-4} < \cdots < \xi_i < \cdots < \xi_{N+5}, \tag{7}$$

we put $t_{-5} := 0, t_{i+1} := t_i + h_i,\ i = -5, \cdots, N + 4$ and

$$\mathbf{D}_i^{(i)} := (\xi_i, 1)\,, \qquad \mathbf{D}_i^{(k)} := (\xi_i, 0)\,, \quad \text{if } i \neq k. \tag{8}$$

Supposing also to have assigned the two following sets of parameters

$$\Gamma := \{0 < \gamma_i < 2, i = -5, \ldots, N + 5\}, \qquad \Lambda := \{0 < \lambda_i, i = -5, \ldots, N + 5\}, \tag{9}$$

for $k = -2, \ldots, N + 2$, we can consider the piecewise quintic Bézier curve

$$\mathbf{C}_i^{(k)}(t; \Gamma, \Lambda) := (X_i(t; \Gamma, \Lambda), \quad Y_i^{(k)}(t; \Gamma, \Lambda)), \quad t \in [t_i, t_{i+1}],\ i = -3, \ldots, N + 2,$$

whose Bézier control points, $\mathbf{b}_{i,j}^{(k)},\ j = 0, \ldots, 5$, are determined according to the geometric construction given in Fig. 2, with ϵ_i, δ_i as in (6), see also [3]. In more detail

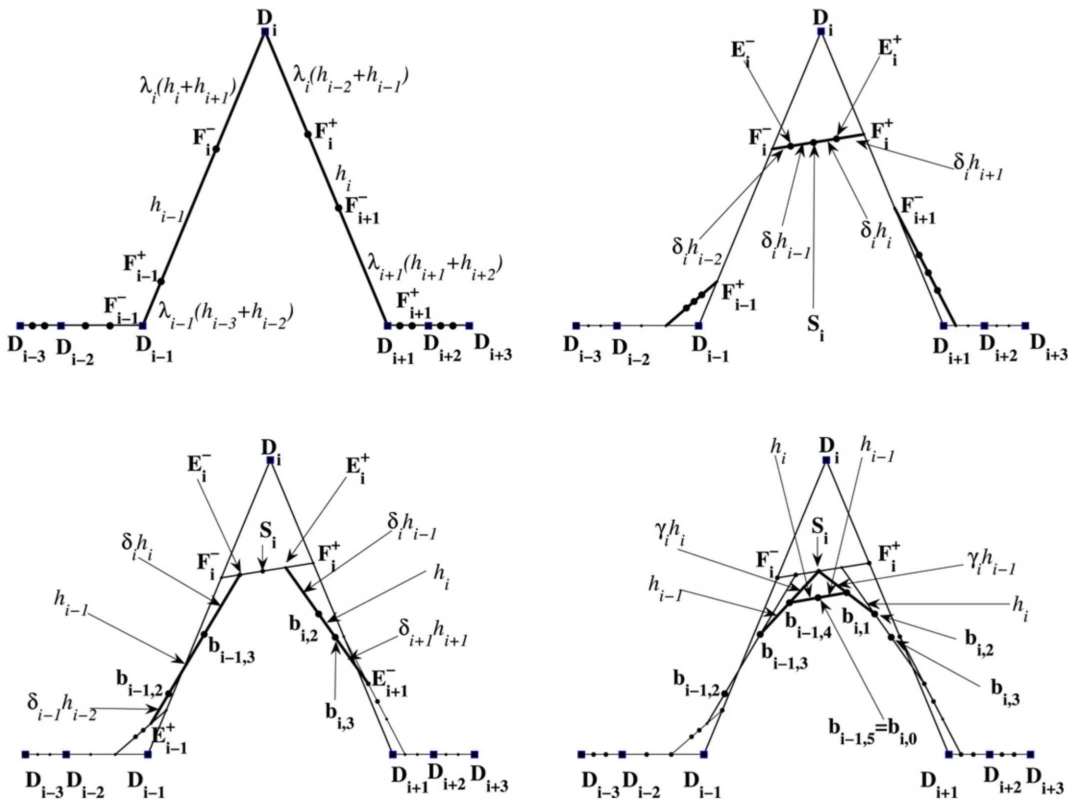

Fig. 2. Geometric construction of the Bézier coefficients of $\mathbf{C}_i^{(k)}$ for $k = i$, see (10)–(18). For graphical convenience the superscript k is omitted.

$$\mathbf{F}_i^{(-,k)} := \frac{(\lambda_{i-1}(h_{i-2}+h_{i-3})+h_{i-1})\mathbf{D}_i^{(k)} + \lambda_i(h_i+h_{i+1})\mathbf{D}_{i-1}^{(k)}}{h_{i-1}+\lambda_{i-1}(h_{i-2}+h_{i-3})+\lambda_i(h_i+h_{i+1})}, \tag{10}$$

$$\mathbf{F}_i^{(+,k)} := \frac{(\lambda_{i+1}(h_{i+1}+h_{i+2})+h_i)\mathbf{D}_i^{(k)} + \lambda_i(h_{i-2}+h_{i-1})\mathbf{D}_{i+1}^{(k)}}{h_i+\lambda_{i+1}(h_{i+1}+h_{i+2})+\lambda_i(h_{i-2}+h_{i-1})}, \tag{11}$$

$$\mathbf{E}_i^{(-,k)} := \frac{h_{i-2}\mathbf{F}_i^{(+,k)} + (h_{i-1}+h_i+h_{i+1})\mathbf{F}_i^{(-,k)}}{h_{i-2}+h_{i-1}+h_i+h_{i+1}}, \tag{12}$$

$$\mathbf{E}_i^{(+,k)} := \frac{(h_{i-2}+h_{i-1}+h_i)\mathbf{F}_i^{(+,k)} + h_{i+1}\mathbf{F}_i^{(-,k)}}{h_{i-2}+h_{i-1}+h_i+h_{i+1}}, \tag{13}$$

$$\mathbf{S}_i^{(k)} := \frac{h_{i-1}\mathbf{E}_i^{(+,k)} + h_i\mathbf{E}_i^{(-,k)}}{h_{i-1}+h_i}, \tag{14}$$

$$\mathbf{b}_{i-1,3}^{(k)} := \frac{\delta_i h_i \mathbf{E}_{i-1}^{(+,k)} + (\delta_{i-1}h_{i-2}+h_{i-1})\mathbf{E}_i^{(-,k)}}{\delta_i h_i + \delta_{i-1}h_{i-2} + h_{i-1}}, \tag{15}$$

$$\mathbf{b}_{i,2}^{(k)} := \frac{(\delta_{i+1}h_{i+1}+h_i)\mathbf{E}_i^{(+,k)} + \delta_i h_{i-1}\mathbf{E}_{i+1}^{(-,k)}}{\delta_{i+1}h_{i+1}+h_i+\delta_i h_{i-1}}, \tag{16}$$

$$\mathbf{b}_{i-1,4}^{(k)} := \frac{\gamma_i h_i \mathbf{b}_{i-1,3}^{(k)} + h_{i-1}\mathbf{S}_i^{(k)}}{\gamma_i h_i + h_{i-1}}, \qquad \mathbf{b}_{i,1}^{(k)} := \frac{\gamma_i h_{i-1}\mathbf{b}_{i,2}^{(k)} + h_i\mathbf{S}_i^{(k)}}{\gamma_i h_{i-1} + h_i}, \tag{17}$$

$$\mathbf{b}_{i-1,5}^{(k)} := \mathbf{b}_{i,0}^{(k)} := \frac{h_i \mathbf{b}_{i-1,4}^{(k)} + h_{i-1}\mathbf{b}_{i,1}^{(k)}}{h_i + h_{i-1}}. \tag{18}$$

Since $h_i > 0$, from (9) and (6) the points $\mathbf{b}_{i,j}^{(k)}$ are convex combinations of $\mathbf{D}_i^{(k)}$. Then from (7), the x components of $\mathbf{b}_{i,j}^{(k)}$ form a strictly increasing sequence and $X_i(t;\Gamma,\Lambda)$ is strictly increasing as well, and therefore invertible. Setting

$$\begin{aligned} &x_{-3} := X_{-3}(t_{-3};\Gamma,\Lambda), \qquad x_{N+3} := X_{N+2}(t_{N+3};\Gamma,\Lambda)\\ &x_i := X_i(t_i;\Gamma,\Lambda) = X_{i-1}(t_i;\Gamma,\Lambda), \quad i=-2,\ldots,N+2, \end{aligned} \tag{19}$$

for $k=-2,\ldots,N+2$ we can then define, according to (3),

$$B^{(k)}(x;\Gamma,\Lambda) := Y_i^{(k)}(t(x);\Gamma,\Lambda), \quad x\in[x_i,x_{i+1}],\ i=-3,\ldots,N+2. \tag{20}$$

By construction, for every fixed k, the curve segments $\mathbf{C}_i^{(k)},\ i=-2,\ldots,N+2$ define a C^1 piecewise quintic GC^3 curve. From the above considerations and from the properties of Bézier–Bernstein representation, observing also that the ordinates of $\mathbf{b}_{i,j}^{(k)}$ are convex combinations of those of $\mathbf{D}_i^{(k)}$, we have the following proposition (see Fig. 3).

Proposition 1. *The functions* $B^{(k)}(.;\Gamma,\Lambda),\ k=-2,\ldots,N+2$, *defined in* (20) *are non-negative and of class* C^3. *Moreover,*

$$\begin{aligned} &B^{(k)}(x;\Gamma,\Lambda)=0, \quad x\notin(x_{k-3},x_{k+3}),\\ &\sum_{k=-2}^{N+2} B^{(k)}(x;\Gamma,\Lambda)=1, \quad x\in[x_0,x_N]. \quad \square \end{aligned}$$

In the case $\lambda_i=\gamma_i=1,\ \forall i$, setting

$$\xi_i := \frac{1}{5}(t_{i-2}+t_{i-1}+t_i+t_{i+1}+t_{i+2}), \tag{21}$$

by using a computer algebra system, we have that ξ_i plays the role of the classical Greville abscissa, see [1],[1] that is $X_i(t)=t, t\in[t_i,t_{i+1}]$ so that the $B^{(k)}$ in (20) are C^3 piecewise quintic functions.

The free parameters λ_i,γ_i act as *tension parameters* and allow us to control the shape of $B^{(k)}$. Indeed, from the construction (see Fig. 2), it is not difficult to see that, as λ_i,γ_i approach 0 then the Bézier control points

$$\mathbf{b}_{i-1,3}^{(k)},\ \mathbf{b}_{i-1,4}^{(k)},\ \mathbf{b}_{i-1,5}^{(k)}, \mathbf{b}_{i,0}^{(k)},\ \mathbf{b}_{i,1}^{(k)}, \mathbf{b}_{i,2}^{(k)} \ \to \mathbf{D}_i^{(k)}.$$

Thus if λ_i,γ_i tend to zero, $i=k-3,\ldots,k+3$, the function $B^{(k)}(.;\Gamma,\Lambda)$ approaches the polygonal line, $\mathcal{P}^{(k)}$, with vertices $\mathbf{D}_j^{(k)}$, $j=-5,\ldots,N+5$. Summarizing, the parameters in Λ and Γ can be seen as local shape parameters stretching the graph of the function $B^{(k)}(.;\Gamma,\Lambda)$ from a quintic C^3 spline function (in the case $\lambda_i=\gamma_i=1,\ \forall i$) to the polygonal line $\mathcal{P}^{(k)}$ and then they can be used to control the shape of the functions $B^{(k)}(.;\Gamma,\Lambda)$. Moreover, such parameters inherit a clear geometric interpretation from the geometric construction depicted in Fig. 2. These facts make the functions in (20) a useful tool in constrained interpolation/approximation, both in the functional and in the parametric setting, whenever a high smoothness is required.

We will refer to the functions $B^{(k)}(.;\Gamma,\Lambda),\ k=-2,\ldots,N+2$, as *six interval supported*, C^3 *tensioned quintic parametric B-splines*, with knots (19), associated to the sets of shape parameters Γ and Λ. We remark that, in contrast with the basis functions provided in [9], if used to construct spatial curves, the functions $B^{(k)}$ produce curves whose torsion does not necessarily vanish at the knots.

4. C^4 B-spline-like basis

In this Section we briefly outline how the above procedure can be extended to obtain a B-spline-like basis with C^4 continuity. Using the parametric approach, C^4 continuity can be achieved defining each basis function as a parametric planar piecewise polynomial curve with GC^4 continuity, see (5).

[1] (21) is the classical expression for Greville abscissas for quintic B-splines but the result is not obvious in this context because we are dealing with a proper subspace.

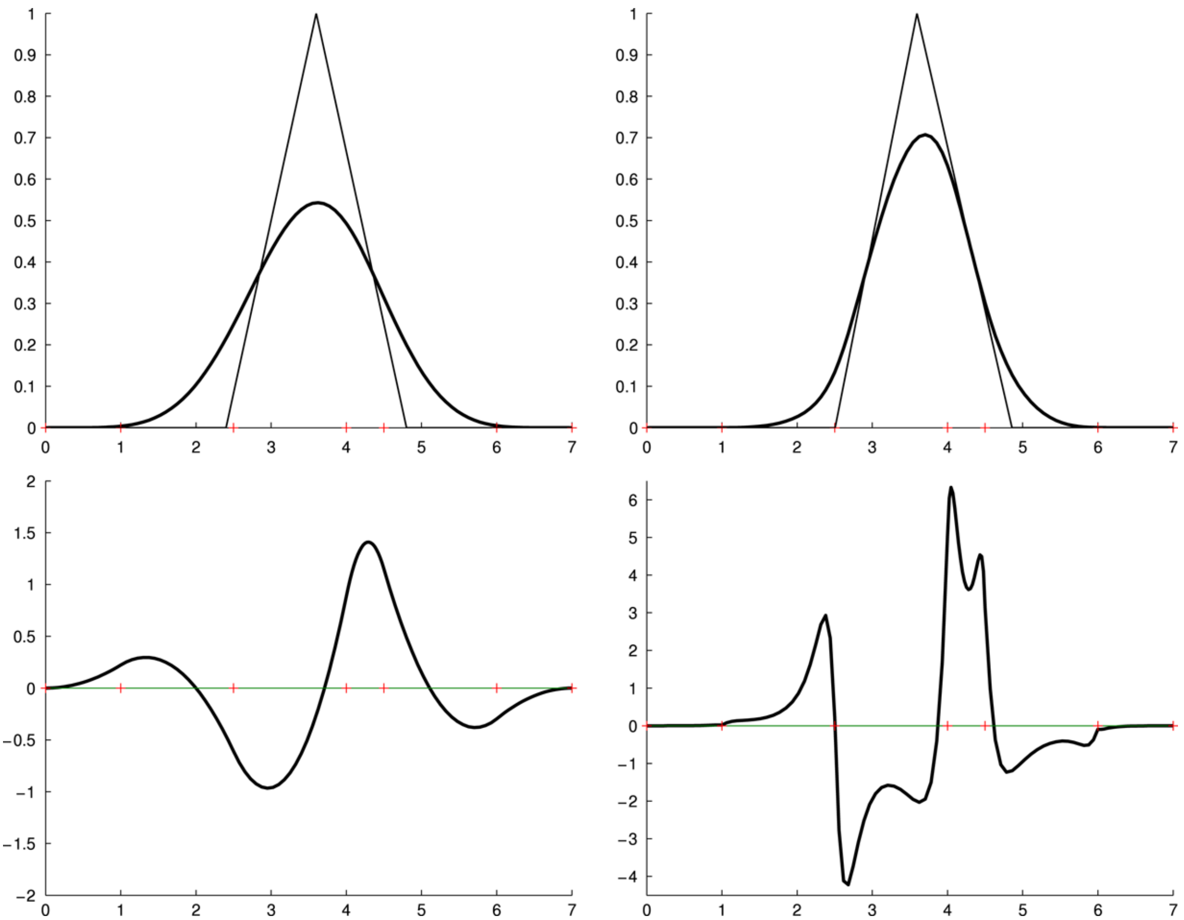

Fig. 3. Six interval supported C^3 tensioned quintic parametric B-spline with the corresponding polygon $\mathcal{P}^{(k)}$ (top) and its third derivative (bottom). Knot sequence: [0 1 2.5 4 4.5 6 7]. Left: $\lambda_i = \gamma_i = 1,\ \forall\, i$. Right: $\lambda_i = .1,\ \gamma_i = .9,\ \forall\, i$.

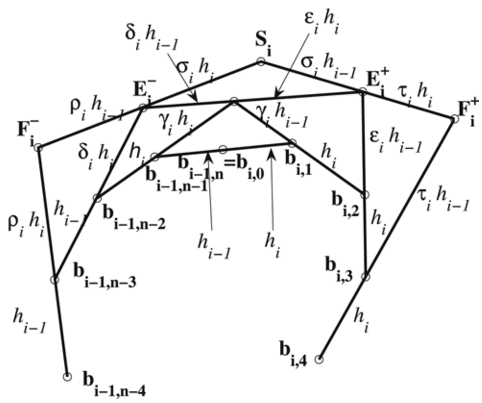

Fig. 4. Geometric construction of a C^1, FC^4 join for Bézier curves of degree n.

We recall, see [2,6], that two polynomial curve segments $\mathbf{C}_{i-1}$ and $\mathbf{C}_i$ meet with C^1 and FC^4 continuity if and only if the last 5 Bézier control points of $\mathbf{C}_{i-1}$ and the first 5 Bézier control points of $\mathbf{C}_i$ satisfy the relations determined by the geometric construction shown in Fig. 4, where $\gamma_i, \delta_i, \epsilon_i, \rho_i, \sigma_i, \tau_i$ are free design parameters and if they are all equal to 1 lead to the geometric construction of classical C^4 analytic continuity.

Dealing with the more restrictive GC^4 continuity these design parameters are no longer independent.

For the sake of simplicity we shall study a one-parameter family of basis functions, setting for instance γ_i, $i = -5, \ldots, N+5$ as the free parameter. For symmetry reason we may set $\delta_i = \epsilon_i$, $\rho_i = \tau_i$, and considering quintic curves (i.e. $n = 5$), with some computation we obtain from (5) that $\mathbf{C}_{i-1}$ and $\mathbf{C}_i$ meet with GC^4 continuity at $\mathbf{b}_{i-1,5} = \mathbf{b}_{i,0}$ if and only if ϵ_i, δ_i are as in (6) and

$$\rho_i = \tau_i = \frac{\gamma_i(\gamma_i^2 + 2\gamma_i + 4)}{7\gamma_i^2 - 40\gamma_i + 40}, \qquad \sigma_i = \frac{(\gamma_i + 1)(\gamma_i^2 + 2\gamma_i + 4)\gamma_i^2}{11\gamma_i^4 + 23\gamma_i^3 - 108\gamma_i^2 + 56\gamma_i + 32}. \tag{22}$$

If $\gamma_i = 1$ then (6) and (22) give $\delta_i = \epsilon_i = \rho_i = \tau_i = \sigma_i = 1$, that is C^4 continuity. With a construction similar to that used for building GC^3 curves, assuming to be given the values $h_i > 0$, $i = -5, \ldots, N+4$, and the strictly increasing vector (7), taking the points $\mathbf{D}_i^{(k)}$ as in (8), we can consider, for an assigned set of parameters $\overline{\Gamma} := \{0 < \gamma_i \le 1,\ i = -5, \ldots, N+5\}$, the piecewise quintic Bézier curve $\overline{\mathbf{C}}_i^{(k)}(t; \overline{\Gamma}) := (\overline{X}_i(t; \overline{\Gamma}), \overline{Y}_i^{(k)}(t; \overline{\Gamma}))$, $t \in [t_i, t_{i+1}]$, whose Bézier control points are determined, according to Section 3 in [2], by (14)–(18), with

$$\begin{aligned}
\mathbf{F}_i^{(-,k)} &:= \frac{(\psi_{i-1}h_{i-2} + \omega_{i-1}h_{i-3} + h_{i-1})\mathbf{D}_i^{(k)} + (\bar{\psi}_{i-1}h_i + \bar{\omega}_{i-1}h_{i+1})\mathbf{D}_{i-1}^{(k)}}{\psi_{i-1}h_{i-2} + \omega_i h_{i-3} + h_{i-1} + \bar{\psi}_{i-1}h_i + \bar{\omega}_{i-1}h_{i+1}}, \\
\mathbf{F}_i^{(+,k)} &:= \frac{(\bar{\psi}_i h_{i+1} + \bar{\omega}_i h_{i+2} + h_i)\mathbf{D}_i^{(k)} + (\omega_i h_{i-2} + \psi_i h_{i-1})\mathbf{D}_{i+1}^{(k)}}{h_i + \bar{\psi}_i h_{i+1} + \bar{\omega}_i h_{i+2} + \omega_i h_{i-2} + \psi_i h_{i-1}}, \\
\mathbf{E}_i^{(-,k)} &:= \frac{\theta_i h_{i-2}\mathbf{F}_i^{(+,k)} + (\delta_i(h_{i-1} + h_i) + \bar{\theta}_i h_{i+1})\mathbf{F}_i^{(-,k)}}{\theta_i h_{i-2} + \delta_i(h_{i-1} + h_i) + \bar{\theta}_i h_{i+1}}, \\
\mathbf{E}_i^{(+,k)} &:= \frac{(\theta_i h_{i-2} + \delta_i(h_{i-1} + h_i))\mathbf{F}_i^{(+,k)} + \bar{\theta}_i h_{i+1}\mathbf{F}_i^{(-,k)}}{\theta_i h_{i-2} + \delta_i(h_{i-1} + h_i) + \bar{\theta}_i h_{i+1}},
\end{aligned}$$

where $\delta_i, \epsilon_i, \rho_i, \sigma_i, \tau_i$ are as in (6) and (22) and the remaining auxiliary parameters, provided in [2], are

$$\theta_i := \frac{\rho_{i-1}\delta_i h_{i-1}(\delta_{i-1}h_{i-2} + h_{i-1} + \delta_i h_i)}{(\delta_{i-1}h_{i-2} + h_{i-1})(h_{i-1} + \gamma_i h_i) - \rho_{i-1}\delta_i h_{i-2}h_i}, \tag{23}$$

$$\bar{\theta}_i := \frac{\delta_i \rho_{i+1} h_i(\delta_i h_{i-1} + h_i + \delta_{i+1}h_{i+1})}{(h_{i-1}\gamma_i + h_i)(h_i\delta_{i+1}h_{i+1}) - \rho_{i+1}\delta_i h_{i-1}h_{i+1}}, \tag{24}$$

$$\bar{\lambda}_i := \frac{\rho_i \delta_{i+1} h_i(\rho_i h_{i-1} + h_i + \gamma_{i+1}h_{i+1})}{(\delta_i h_{i-1} + h_i)(h_i + \gamma_{i+1}h_{i+1}) - \rho_i \delta_{i+1} h_{i-1}h_{i+1}}, \tag{25}$$

$$\phi_i := \frac{\delta_{i-1}\rho_i h_{i-1}(\gamma_{i-1}h_{i-2} + h_{i-1} + \rho_i h_i)}{(\gamma_{i-1}h_{i-2} + h_{i-1})(h_{i-1} + \delta_i h_i) - \rho_i \delta_{i-1} h_{i-2}h_i}, \tag{26}$$

$$\lambda_i := \frac{\sigma_i \theta_i h_{i-1}(\phi_i h_{i-2} + \rho_i h_{i-1} + \sigma_i h_i)}{(\phi_i h_{i-2} + \rho_i h_{i-1})(\delta_i h_{i-1} + \delta_i h_i) - \sigma_i \theta_i h_{i-2}h_i}, \tag{27}$$

$$\bar{\phi}_i := \frac{\sigma_i \bar{\theta}_i h_i(\sigma_i h_{i-1} + \rho_i h_i + \bar{\lambda}_i h_{i+1})}{(\delta_i h_{i-1} + \delta_i h_i)(\bar{\lambda}_i h_{i+1} + h_i \rho_i) - \sigma_i \bar{\theta}_i h_{i-1}h_{i+1}}, \tag{28}$$

$$\psi_i := \frac{\sigma_i h_i(\delta_i h_{i-1} + \delta_i h_i + \bar{\theta}_i h_{i+1})}{(\delta_i h_{i-1} + \delta_i h_i)(\bar{\lambda}_i h_{i+1} + \rho_i h_i) - \sigma_i \bar{\theta}_i h_{i-1}h_{i+1}}, \tag{29}$$

$$\bar{\psi}_{i-1} := \frac{\sigma_i h_{i-1}(\theta_i h_{i-2} + \delta_i h_{i-1} + \delta_i h_i)}{(\phi_i h_{i-2} + \rho_i h_{i-1})(\delta_i h_{i-1} + \delta_i h_i) - \sigma_i \theta_i h_{i-2}h_i}, \tag{30}$$

$$\bar{\omega}_{i-1} := \frac{\bar{\phi}_i(\lambda_i h_{i-2} + \sigma_i h_{i-1} + \rho_i h_i + \bar{\lambda}_i h_{i+1})(h_{i-1} + \bar{\psi}_{i-1}h_i)}{(\phi_i h_{i-2} + \rho_i h_{i-1} + \sigma_i h_i)(\sigma_i h_{i-1} + h_i\rho_i + \bar{\lambda}_i h_{i+1}) - \lambda_i \bar{\phi}_i h_{i-2}h_{i+1}}, \tag{31}$$

$$\omega_i := \frac{\lambda_i(\phi_i h_{i-2} + \rho_i h_{i-1} + \sigma_i h_i + \bar{\phi}_i h_{i+1})(\psi_i h_{i-1} + h_i)}{(\phi_i h_{i-2} + \rho_i h_{i-1} + \sigma_i h_i)(\sigma_i h_{i-1} + \rho_i h_i + \bar{\lambda}_i h_{i+1}) - \lambda_i \bar{\phi}_i h_{i-2}h_{i+1}}. \tag{32}$$

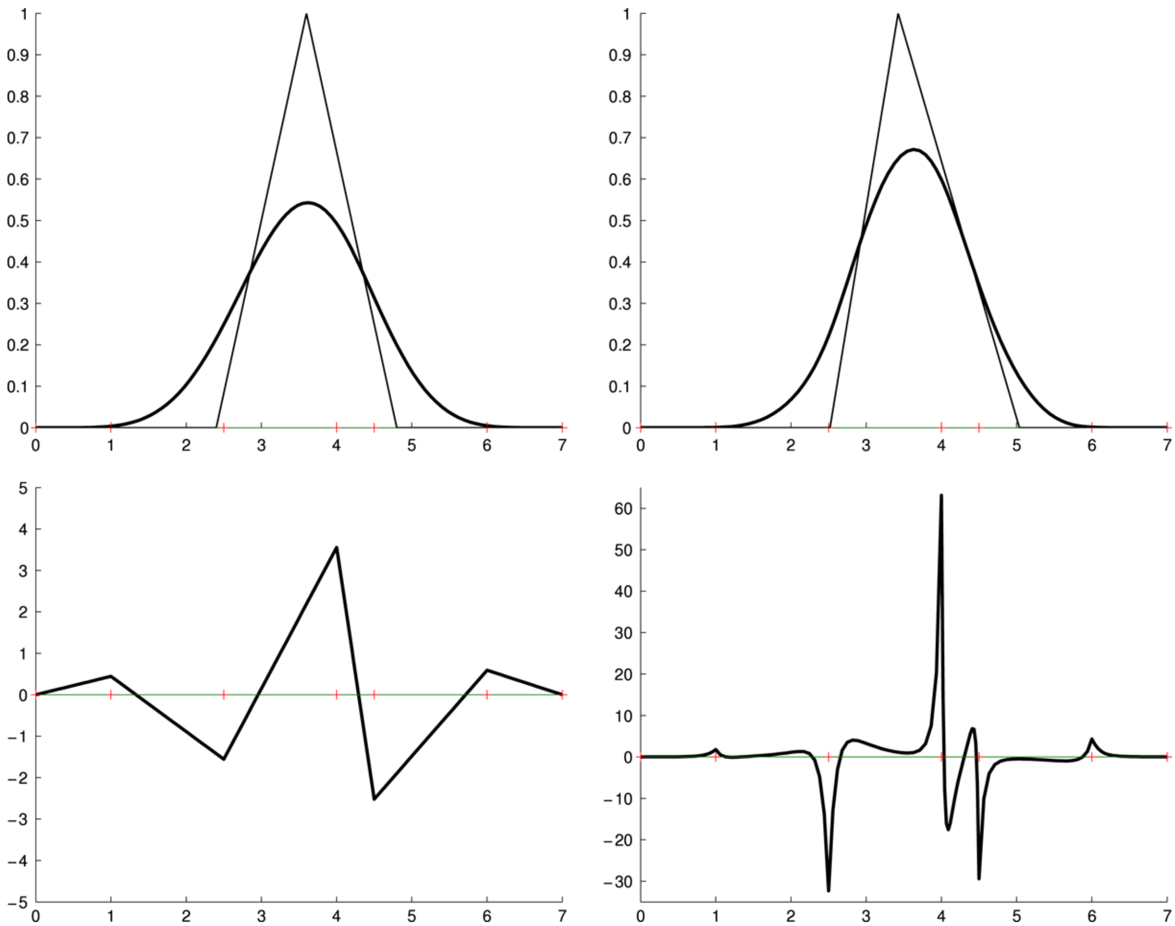

Fig. 5. C^4 tensioned quintic parametric B-spline with the corresponding polygon $\mathcal{P}^{(k)}$ (top) and its fourth derivative (bottom). Knot sequence: [0 1 2.5 4 4.5 6 7]. Left: $\gamma_i = 1, \ \forall i$. Right: $\gamma_i = .75, \ \forall i$.

The previous construction, borrowed from [2], produces in general FC^4 curves. In our case, since $\gamma_i, \delta_i, \epsilon_i, \rho_i, \sigma_i, \tau_i$ satisfy (6) and (22), the curves $\overline{\mathbf{C}}^{(k)}$ are GC^4 continuous. Thus, whenever the parameters $\gamma_i, \delta_i, \epsilon_i, \rho_i, \sigma_i, \tau_i$ and the auxiliary ones in (23)–(32) are non-negative,[2] the curves $\overline{\mathbf{C}}^{(k)}$ allow us to define, according to (3), functions, $\overline{B}^{(k)}$, of class C^4. In particular, if $\gamma_i = 1$, the previous construction reduces to classical geometric construction of C^4 quintic B-splines, hence the choice (21) produces, as for the GC^3 construction, $X_i(t) = t, t \in [t_i, t_{i+1}]$.

As for the C^3 case, due to the properties of the points $\mathbf{D}_i^{(k)}$, these functions form a non-negative partition of unity and are supported in the intervals $[\bar{x}_{k-3}, \bar{x}_{k+3}]$ (see also Proposition 1), where

$$\begin{aligned} &\bar{x}_{-3} := \overline{X}_{-3}(t_{-3}; \overline{\Gamma}), \qquad \bar{x}_{N+3} := \overline{X}_{N+2}(t_{N+3}; \overline{\Gamma}), \\ &\bar{x}_i := \overline{X}_i(t_i; \overline{\Gamma}) = \overline{X}_{i-1}(t_i; \overline{\Gamma}), \quad i = -2, \ldots, N+2. \end{aligned} \tag{33}$$

Thus the functions $\overline{B}^{(k)}$ possess all the classical properties of quintic B-splines and we will refer to them as C^4 *tensioned quintic parametric B-splines*, with knots (33), associated to the set of shape parameters $\overline{\Gamma}$. Also in this case the free parameters γ_i act as local *tension parameters*. Indeed, if they tend to zero the same occurs for $\delta_i, \epsilon_i, \rho_i, \sigma_i, \tau_i$,

[2] This is clearly the case at least for γ_i close to 1 or γ_i close to 0 (from a simple asymptotic analysis we have that all the above coefficients approach 0^+ as γ_i tend to 0). We refer to [2], Section 3.2, for more details on this point.

see (6) and (22), and for the auxiliary parameters in (23)–(32) as well. Thus the function $\overline{B}^{(k)}(.; \overline{\Gamma})$ approaches the polygonal line, $\mathcal{P}^{(k)}$, with vertices $\mathbf{D}_j^{(k)}$, $j = -5, \ldots, N+5$, see Fig. 5.

5. Numerical examples

Before presenting a graphical example some remarks concerning the knot sequences (19) and (33) are in order.

For the sake of brevity, let us discuss the case of C^4 basis functions, similar arguments apply to the C^3 case. The knot sequence (33) is determined by the sequence of "parametric knots" $\{t_i\}$, by the "abscissas of the de Boor control points", (7), and by the set of shape parameters, Γ, in a highly non-linear way. Of course, when the shape parameters are all equal to 1 (setting ξ_i as in (21)) the knots $\overline{x}_i$ agree with the t_i's, while they approach the ξ_i's as the shape parameters tend to zero (see end of Section 4), but in the general case it is almost impossible to derive precise a priori information on their location more stringently than from those arising from the convex combination. In practical application, provided that no extreme reduction of the shape parameters is necessary, we can proceed as follows. First, independently select both the sequences $\{t_i\}$, and $\{\overline{x}_k\}$ according to a given suitable parameterization (usually the two sequences coincide). Then, determine the sequence (7) so that, for the current values of the shape parameters, the relations (33) hold. Denoting by $b_{i,0}^{(k,y)}$ the y component of $\mathbf{b}_{i,0}^{(k)}$, this is equivalent to solving the linear system, with a pentadiagonal matrix[3]

$$\sum_k \xi_k b_{i,0}^{(k,y)} = \overline{x}_i. \tag{34}$$

Now, we briefly illustrate the performances of the spaces of functions spanned by the B-spline-like bases introduced in Sections 3 and 4 by means of a classical test in constrained spatial curve approximation: the so-called perturbed "bean data", [4]. Considering the chord length parameterization for the data, we assume that a sequence of 9 knots, $\mathcal{T}$, is provided in the input, [4]. We want to construct a C^3 (C^4) curve approximating the data and reproducing their shape, that is reproducing the behavior of the piecewise linear, $\mathbf{L}^*$, with knots $\mathcal{T}$, which is the best least squares approximation to the data, see the left side of Fig. 6. In particular, the sign of the torsion of the approximating curve is required to be consistent with the sign of the discrete torsion of $\mathbf{L}^*$ [4,5].

The required curve can be obtained as a proper C^3 reparameterization of a GC^3 piecewise quintic curve considering a combination of the basis functions (20) associated to the knots $\mathcal{T}$ (the sequence (7) has been constructed according to the procedure outlined above). Determining the coefficients via best least squares approximation, if $\lambda_k = \gamma_k = 1, \ \forall\, k$, we obtain a C^3 piecewise quintic curve (thicker line in Fig. 6, top left) whose torsion does not satisfy the required constraints in the fourth and in the last interval (see the thicker line in Fig. 6, top center). The six interval supported C^3 tensioned quintic parametric B-spline curve obtained with $\lambda_k = .4, \ \gamma_k = .9, \forall\, k$, (thicker line in Fig. 6, middle left) fulfills the requirement on the sign of its torsion still retaining the C^3 continuity (thicker line in Fig. 6, middle center). The thinner lines in Fig. 6, top and middle, refer to the four interval supported C^3 tensioned quintic parametric B-spline curve, obtained with the same procedure and the same shape parameters by the construction presented in [9] whose torsion always vanishes at the knots. The results obtained by the same procedure considering a C^4 curve constructed according to Section 4 are depicted in Fig. 6 bottom. The thinner lines refer to the case $\gamma_k = 1, \ \forall k$, while the thicker ones to the choice $\gamma_k = .8, \ \forall k$. The right column in Fig. 6 depicts the various porcupine plots of the torsion, [4].

We have described the construction and salient properties of C^3 and C^4 B-spline-like functions with shape parameters. The construction is based on two main mathematical tools: the parametric techniques and the characterization of smoothness conditions between two segments through control points.

Both the C^3 and the C^4 B-spline-like functions are described in terms of a geometric construction, they have a similar structure and a similar behavior. The C^4 functions depend only on a single shape parameter (which is usually considered a positive fact because the related algorithms are simpler). On the other hand, in the construction of the

[3] The sequence (7) has to be increasing. This is surely the case if the shape parameters are not too far from 1 but, for extremely odd choices of the shape parameters and of the sequence $\{t_i\}$, (34) can produce non-increasing sequences. In such a case some heuristic modification of the initial choice of $\{\overline{x}_k\}$ has to be applied.

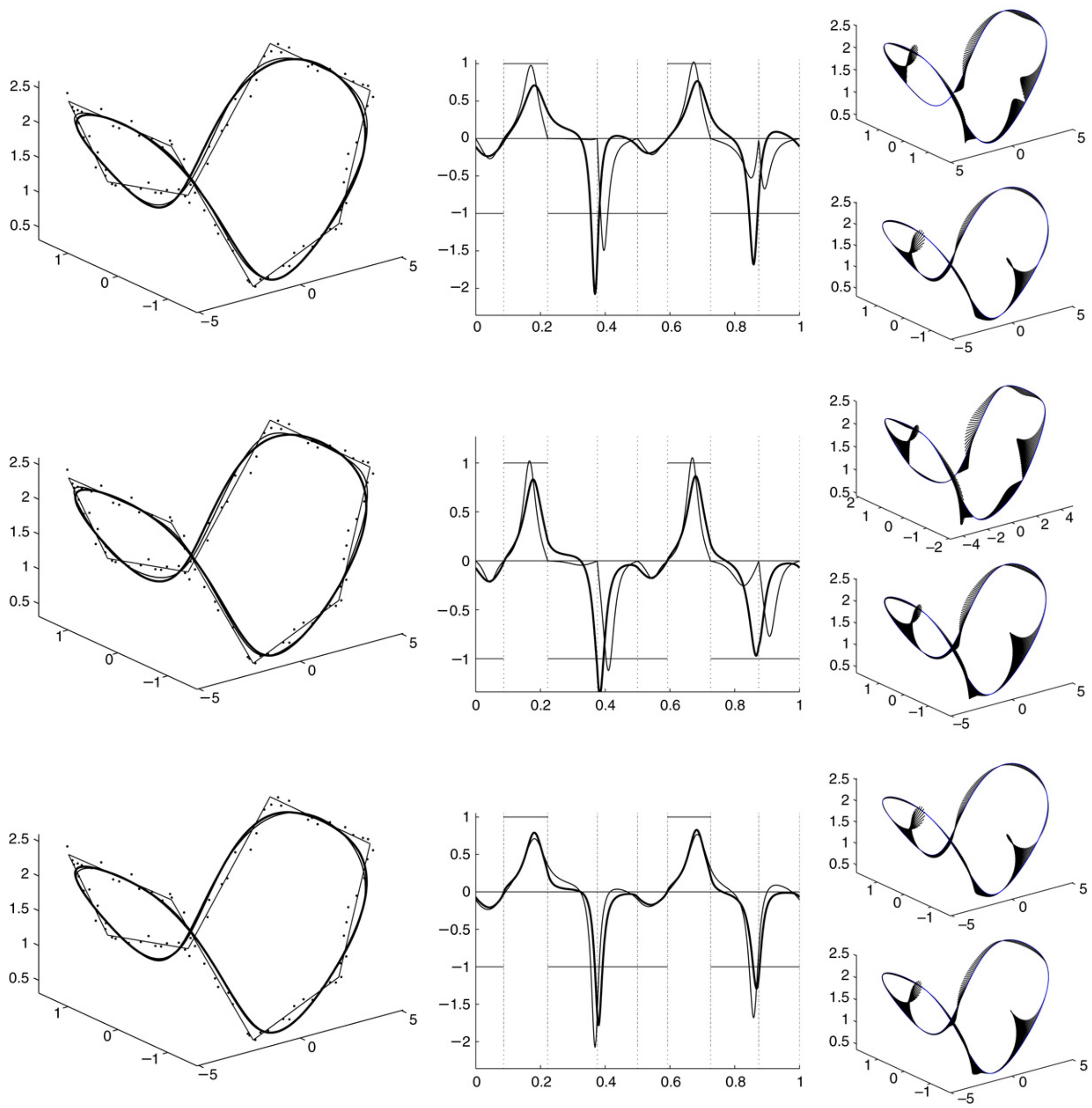

Fig. 6. Spatial curve approximation. Top and center: C^3. Bottom: C^4. The horizontal segments in the middle column indicate the sign of the discrete torsion of $\mathbf{L}^*$.

knot sequence (33) as described at the beginning of this Section, the difficulties discussed in the footnote [3] can (rarely) occur, while this was not the case in all the (numerous) numerical experiments that we performed for the C^3 case.

Due to page limitation, we cannot discuss in detail the practical choice of the shape parameters, but we emphasize that the geometric construction and the Bézier representation greatly simplify this task. In fact constraints on the shape of the obtained functions or curves can be easily translated into (sufficient) constraints on the (possibly subdivided) related Bézier control polygon, which can be manipulated in a much easier way, see also [9] for further comments.

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