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ON THE SUBWORD COMPLEXITY OF SQUARE-FREE D0L LANGUAGES

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Abstract. The subword complexity of a language K is the function which to every positive integer n assigns the number of different subwords of length n occurring in words of K. A language K is square-free if no word in it contains a subword of the form xx where x is a nonempty word. The (best) upper and lower bounds on the subword complexity of infinite square-free D0L languages are established.

1. Introduction

The problems of repetitions of subwords in words (and in infinite words) were first studied by Thue [11, 12]. Since then those problems were investigated (and rediscovered) by quite a number of authors with quite different motivations. In particular results of Thue were also used in various constructions in formal language theory (see, e.g., [3]). Recently one notices a revival of interest in Thue problems among formal language theorists (see, e.g., [1, 2, 7, 8, 10]). In particular [1, 10] it was discovered that the theory of nonrepetitive sequences of Thue is very strongly related to the theory of D0L sequences. For example, Thue's original examples of square-free sequences were constructed using D0L systems and indeed, as pointed out in [1], most (if not all) examples of nonrepetitive sequences known in the literature are either D0L sequences or codings of D0L sequences. In this way a quite significant connection is established between the theory of nonrepetitive sequences and the theory of D0L systems. The theory of nonrepetitive sequences originates a new and very interesting research area within the theory of D0L systems while the theory of D0L systems provides a better insight into the theory of nonrepetitive sequences (see, e.g., [1, 10]).

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In this paper we investigate D0L systems which generate nonrepetitive words only. In particular we investigate the upper and the lower bounds on the subword complexity of languages generated by such systems and we establish that those languages are quite 'poor' as far as number of subwords is concerned. (For a language K its subword complexity is a function assigning to each positive integer n the number of different subwords of length n occurring in words of K.) In a sense this result is quite counter intuitive: one is inclined to think that to construct an infinite language consisting of nonrepetitive words one needs a lot of different subwords to avoid repetitions. (This aspect of the problem was pointed to us by Berstel who suggested to investigate the subword complexity of D0L systems generating square-free words only. Actually Berstel conjectured that the subword complexity of such languages is bounded by a linear function; we prove that the number of subwords of length n in such languages is of order $n \log_2 n$). We believe that this paper sheds a new light on the theory of square-free languages (sequences) and that it demonstrates how known results and techniques of the theory of D0L systems contribute to the theory of nonrepetitive languages (sequences).

We assume the reader to be familiar with basic aspects of D0L systems (see, e.g., [9]).

2. Preliminaries

We will use standard notation and terminology concerning D0L systems (see, e.g., [9]). Thus a D0L system G is specified in the form $G = (\Sigma, h, \omega)$ where Σ is its alphabet, h its homomorphism and ω its axiom; $L(G)$ denotes the language of G while $E(G)$ denotes its sequence. A letter a is *erasing* if, for some $m \geqslant 1$, $h^m(a) = \Lambda$ (where Λ is the empty word), otherwise a is *nonerasing*; $\operatorname{maxr} G$ denotes $\max\{|x|: x = h(a)$ for some $a \in \Sigma\}$. Since the problems considered become trivial otherwise, we consider only D0L systems which generate infinite languages.

It turns out that the notion of the rank of a letter in a D0L system [5] will be quite useful in our investigation.

Definition. Let $G = (\Sigma, h, \omega)$ be a D0L system and let, for a letter $a \in \Sigma$, $G_a = (\Sigma, h, a)$. We say that a letter $a \in \Sigma$ is of *rank* 0 (*in* G) if $L(G_a)$ is finite. Let, for $i \geqslant 0$, Σ_i denote the set of all letters of rank i and let, for $j \geqslant 1$, $G_{(j)} = (\Sigma_{(j)}, h_{(j)}, \omega_{(j)})$ where

$$\Sigma_{(j)} = \Sigma \setminus \bigcup_{i=0}^{j-1} \Sigma_i, \qquad \omega_{(j)} = g_j(\omega)$$

and, for $b \in \Sigma_{(j)}$, $h_{(j)}(b) = g_{(j)}h(b)$ where $g_{(j)}$ is the homomorphism on Σ^* defined by $g_{(j)}(a) = a$ for $a \in \Sigma_{(j)}$, and $g_{(j)}(a) = \Lambda$ for $a \in \bigcup_{i=0}^{j-1} \Sigma_i$. If a letter $a \in \Sigma_{(j)}$ is of rank 0 in $G_{(j)}$, then we say that it is of *rank* j (*in* G). If $a \in \Sigma$ is of rank j for some $j \geqslant 0$, then we say that a has *rank in* G; otherwise we say that a is *without a rank*.

For a word x, $|x|$ denotes its length while (if x is nonempty) first$x$ denotes the first letter of x. For a finite set A, $\# A$ denotes its cardinality. For a language K and a positive integer n, $\text{sub}_n K$ denotes the set of subwords of length n of K while sub$K$ denotes the set of all subwords of K. Given an alphabet Σ and $\Delta \subseteq \Sigma$, pres_Δ denotes the homomorphism on Σ^* defined by $\text{pres}_\Delta(a) = \Lambda$ if $a \in \Sigma \backslash \Delta$ and $\text{pres}_\Delta(a) = a$ if $a \in \Delta$.

We need the following notions concerning repetitions of subwords in a word.

Definition. A word is called *square-free* if it does not contain a subword of the form x^2 where x is a nonempty word. A word is called *strongly cube-free* if it does not contain a subword of the form x^2 first$x$ where x is a nonempty word. A language is called *square-free* (resp. *strongly cube-free*) if it does not contain a square-free (resp. strongly cube-free) word.

Clearly, every square-free word (language) is also strongly cube-free. Actually strongly cube-free words (languages) can be viewed also differently.

Definition. A word y is said to have an *overlap* if there exist words y_1, y_2, x_1, x_2, x_3 and x such that $y = y_1x_1x_2x_3y_2$, $x = x_1x_2 = x_2x_3$ where x_1, x_2, x_3 are nonempty words. Otherwise we say that y is *overlap-free*. A language is called *overlap-free* if each word in it is overlap-free.

Theorem 1. *A word is overlap-free if and only if it is strongly cube-free.*

Proof. (i). Let u be a word containing two overlapping occurrences of the same word. Hence $u = u_1x_1x_2x_3u_2$ where for some word x, $x_1x_2 = x_2x_3 = x$ where x_1, x_2, x_3 are all nonempty words; thus u has two different occurrences of x 'overlapping on' x_2. But then x_1x_1 firstx_1 is a subword of u and so u is not strongly cube-free.

(ii). Let u be a word which can be written in the form $u = u_1xx(\text{first } x)u_2$ where x is a nonempty word; hence u is not strongly cube-free. Then $u = u_1x(\text{first}x)y(\text{first}x)u_2$ where $x = (\text{first}x)y$. But then u can be written in the form $u = u_1z_1z_2z_3u_2$ where $z_1 = x$, $z_2 = \text{first}x$ and $z_3 = y\text{first}x$. Consequently u has two different occurrences of $z = z_1z_2 = z_2z_3$ 'overlapping on' z_2. But then u is not overlap-free.

3. Results

In this section the subword complexity of square-free D0L languages is investigated. We begin by establishing an upper bound for this complexity.

Theorem 2. *If K is a square-free D0L language then, for every positive integer n, $\# \text{sub}_n K \leqslant Cn \log_2 n$ for some positive integer constant C.*

Proof. Let $G = (\Sigma, h, \omega)$ be a DOL system generating K.

(i). If $a \in \Sigma$, then either a is of rank 0 or a does not have a rank.

This is established as follows. If a has a rank greater than 0, then G must contain a letter b of rank 1 such that, for some $m \geq 1$, $h^m(b) = ubv$ where $u, v \in \Sigma^*$, $uv \neq \Lambda$ and u and v consist of letters of rank 0 only. Since both u, $h^m(u)$, $h^{2m}(u), \ldots$ and v, $h^m(v)$, $h^{2m}(v), \ldots$ are infinite ultimately periodic sequences, $L(G)$ cannot be square-free; a contradiction.

(ii). There exists a positive integer constant q such that, if u is a subword of K consisting of letters of rank 0 only, then $|u| < q$.

This is proved by contradiction as follows. Let u be 'an arbitrarily long' subword of K consisting of letters of rank 0 only. Since it is well known (see, e.g., [9]) that subwords consisting of erasing letters only are shorter than certain constant, u must contain 'arbitrarily many' nonerasing letters. Let $E(G) = \omega_0, \omega_1, \omega_2, \ldots$ where for some $i \geq 1$, $\omega_i = xuy$. Notice that in words $\omega_0, \omega_1, \ldots, \omega_{i-1}$ we can distinguish (occurrences of) subwords $u_0, u_1, \ldots, u_{i-1}$ respectively which are the shortest subwords which are ancestors of u. Let j be the minimal integer such that $|u_j| \geq 2$. So let $u_j = av_jb$ where $a, b \in \Sigma$, $v_j \in \Sigma^*$. Clearly $|u_j| \leq \max\{|\omega|, \operatorname{maxr} G\}$ and v_j, if nonempty, consists of letters of rank 0 only (because its contribution to ω_i is either empty or it consists of letters of rank 0 only). Let $u = cvd$ where $c, d \in \Sigma$ and $v \in \Sigma^*$. The situation can be best illustrated as in Fig. 1.

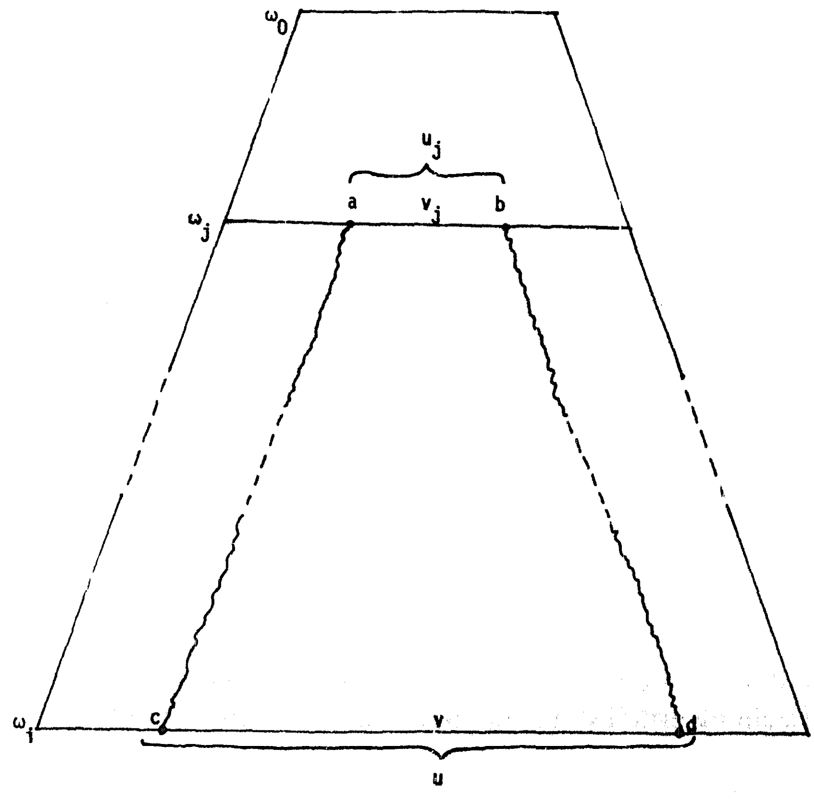

Fig. 1.

Since the length of v_i is limited and u is arbitrarily long either on the path from a to c or on the path from b to d there must be a symbol, say e, repeating at least twice which contributes to v a subword which contains a nonerasing letter; since both cases are symmetric assume that e occurs on the path from a to c. Hence for some $m \geq 1$ $h^m(e) = z_1ez_2$ where z_2 is nonempty and consists of type 0 letters only with at least one of them being nonerasing. Since, clearly, z_2, $h^m(z_2)$, $h^{2m}(z_2), \ldots$ is an infinite ultimately periodic sequence of nonempty words, $L(G)$ must contain a word which is not square-free; a contradiction. Hence there exists a positive integer constant q such that each subword of $L(G)$ consisting of letters of rank 0 only must be shorter than q.

(iii). Now let $\bar{G} = (\bar{\Sigma}, \bar{h}, \bar{\omega})$ be the D0L system defined as follows:

$$\bar{\Sigma} = \{[u, a, v]: u, v \in \Sigma_0^*, |u| < q, |v| < q \text{ and } a \in \Sigma \backslash \Sigma_0\},$$

$$\bar{\omega} = [u_1, a_1, \Lambda][u_2, a_2, \Lambda] \cdots [u_l, a_l, u_{l+1}],$$

where

$$u_1, u_2, u_3, \ldots, u_{l+1} \in \Sigma_0^*, \quad a_1, \ldots, a_l \in \Sigma \backslash \Sigma_0, \qquad l \geq 1$$

and

$$\omega = u_1a_1u_2a_2 \cdots u_la_lu_{l+1},$$

for $[u, a, v] \in \bar{\Sigma}$,

$$\bar{h}([u, a, v]) = [z_0, b_1, \Lambda] \cdots [z_{k-1}, b_k, z_k],$$

where

$$k \geq 1, \quad h(a) = x_0b_1x_1b_2 \cdots b_kx_k, \qquad x_0, \ldots, x_k \in \Sigma_0^*, \quad b_1, \ldots, b_k \in \Sigma \backslash \Sigma_0,$$

$$z_0 = h(u)x_0, \quad z_1 = x_1, \quad z_2 = x_2, \ldots, z_{k-1} = x_{k-1} \quad \text{and } z_k = x_kh(v).$$

We can clearly assume that $\bar{G}$ is an everywhere growing D0L system (i.e., for every $a \in \Sigma$, $|h(a)| \geq 2$); if $\bar{G}$ is not such a system, then we can speed it up (see, e.g., [8]) and then deal with a finite number of D0L systems $\bar{G}_1, \ldots, \bar{G}_m$ each of which is everywhere growing. From the construction of $\bar{G}$ it directly follows that $L(G) = g(L(\bar{G}))$ where g is the homomorphism on $\bar{\Sigma}^*$ defined by $g([u, a, v]) = uav$. It is proved in [6] that if H is an everywhere growing D0L system and f is a nonerasing homomorphism, then, for every positive integer n, $\# \text{sub}_n f(L(H)) \leq Dn \log_2 n$ for some positive integer D.

Thus the theorem holds.

We demonstrate now that the above established upper bound ($n \log_2 n$) is the best possible.

Theorem 3. *There exist a square-free D0L language K and a positive constant D such that for every $n \geq 1$, $\# \text{sub}_n K \geq Dn \log_2 n$.*

Proof. Consider the D0L system $G=(\Sigma, h, \omega)$ with $\Sigma=\{0,1,2\}$, $h(0)=012$, $h(1)=02$, $h(2)=1$ and $\omega=0$ from [8]. It is shown in [8] (see also [1]) that $L(G)$ is square-free. Let $G_{(3)}=(\Sigma, h_{(3)}, 0)$ where for $a\in\Sigma$, $h_{(3)}(a)=h^3(a)$; thus $G_{(3)}$ results from G by starting with the axiom 0 and then taking only each third word of G. Clearly also $L(G_{(3)})$ is square-free. Notice that, if f_G and $f_{G_{(3)}}$ denote the growth functions of G and $G_{(3)}$ respectively, then

$$f_G(n)\leqslant 3^n \quad\text{and}\quad f_{G_{(3)}}(n)>4^n \quad \text{for } n\geqslant 0. \tag{1}$$

Now let $H=(\Theta, g, 0\bar{0}\bar{\bar{0}})$ be the $D0L$ system where $\Theta=\Sigma\cup\bar{\Sigma}\cup\bar{\bar{\Sigma}}$ with $\bar{\Sigma}=\{\bar{a}: a\in\Sigma\}$ and $\bar{\bar{\Sigma}}=\{\bar{\bar{a}}: a\in\Sigma\}$, $g(a)=h_{(3)}(a)$, $g(\bar{a})=\overline{h(a)}$ and $g(\bar{\bar{a}})=\overline{\overline{h_{(3)}(a)}}$ for $a\in\Sigma$ (where for a word $\alpha\in\Sigma^+$, $\bar{\alpha}$ results from α by replacing every letter a in it by $\bar{a}$ and $\bar{\bar{\alpha}}$ results from α by replacing every letter a in it by $\bar{\bar{a}}$).

Clearly also $L(H)$ is square-free. Let $n\geqslant 1$ and let us estimate a lower bound for $\#\mathrm{sub}_{3n}L(H)$. To this aim consider the word $z=g^m(0\bar{0}\bar{\bar{0}})$ where $m=\lceil\log_4 2n\rceil$. Then $z=z_1z_2z_3$ where $z_1\in\Sigma^+$, $z_2\in\bar{\Sigma}^+$ and $z_3\in\bar{\bar{\Sigma}}^+$. Notice that it follows from (1) that $|z_3|\geqslant 2n$. Let y be the prefix of z_3 of length $2n$. Since $L(H)$ is square-free (and so by Theorem 1 also overlap-free) all subwords of y or length n are different. Let u be one fixed subword out of these n subwords. Note that $E(G)$ has the strong prefix property (that is $h^{n+1}(\omega)=h^n(\omega)\alpha_n$ for each $n\geqslant 0$ where $\alpha_n\in\Sigma^+$) hence we can talk about the 'fixed occurrence of u' in z_3 and in all suffixes of all consecutive words of $L(H)$ where we consider the longest suffixes which are over the alphabet $\bar{\bar{\Sigma}}$. Now let us estimate the lower bound for the number of all those subwords of $L(H)$ that end on this fixed occurrence of u and are of length $3n$.

Note that, if t and t' are such two different subwords where $|\mathrm{pres}_{\bar{\Sigma}}t|\leqslant n$ and $|\mathrm{pres}_{\bar{\Sigma}}t'|\leqslant n$, then $t\neq t'$ (because f_G is a monotonically growing function). Hence, let us estimate a bound on a positive integer p having the property that, if $x=g^{m+p}(0\bar{0}\bar{\bar{0}})$, then $|\mathrm{pres}_{\bar{\Sigma}}x|\leqslant n$. First of all, as long as $3^{m+p}\leqslant n$, then (by (1)) p has the desired property. Thus $(m+p)\log_4 3\leqslant\log_4 n$ and consequently $p\leqslant C\log_4 n-0.5$, where $C=(1-\log_4 3)/\log_4 3$. Since we have n possible choices for u we get that

$$\#\mathrm{sub}_{3n}L(H)\geqslant n(C\log_4 n-0.5).$$

Consequently there exists a positive constant C_1 such that for all $n\geqslant 4$

$$\#\mathrm{sub}_{3n}L(H)\geqslant C_1 n\log_4 n$$

(any C_1 such that $C_1\leqslant C-0.5$ will do).

Then it is rather easy to see that there exists a positive constant D such that $\#\mathrm{sub}_nL(H)\geqslant Dn\log_2 n$ for every $n\geqslant 1$.

Hence the theorem holds.

We turn now to the lower bound on the subword complexity of square-free D0L languages.

Theorem 4. *If K is an infinite square-free language, then $\#\mathrm{sub}_n K \geqslant n$ for every positive integer n.*

Proof. Let n be a positive integer. If $n = 1$, then clearly $\#\mathrm{sub}_n K \geqslant n$. So let $n \geqslant 2$ and let z in K be such that $|z| \geqslant 2n-1$. Let $z_1, z_2, \ldots, z_{n-1}$ be words resulting from z by erasing from it the first, the two first, ..., and the $(n-1)$ first letters respectively. Now let $y, y_1, \ldots, y_{n-1}$ be prefixes of length n of words $z, z_1, \ldots, z_{n-1}$ respectively. Note that all those words $y, y_1, \ldots, y_{n-1}$ appear as subwords of z in such a way that any two of them overlap in z. Since K is square-free, Theorem 1 implies that K is overlap-free and consequently $y, y_1, \ldots, y_{n-1}$ are all different subwords of z. Thus $\#\mathrm{sub}_n K \geqslant n$.

Finally we demonstrate that the linear bound on the subword complexity of square-free D0L languages is the best possible.

Theorem 5. *There exist a square-free D0L language K and a positive integer constant C such that for every positive integer n, $\#\mathrm{sub}_n K \leqslant Cn$.*

Proof. It is well known (see, e.g., [1]) that there exists a square-free D0L language defined by a uniformly growing D0L system. (A D0L system $G = (\Sigma, h, \omega)$ is called uniformly growing if there exists a positive integer constant t such that, for every $a \in \Sigma$, $|h(a)| = t$.) However, if G is a uniformly growing D0L system, then [4] there exists a positive integer constant C such that, for all $n \geqslant 0$, $\#\mathrm{sub}_n L(G) \leqslant Cn$.

We conclude this paper with the following two remarks:

(1). In this paper we have established lower and upper bounds on the subword complexity of square-free D0L languages. Thue's original interest (as well as the interest of the most of his followers) was in square-free infinite words. For this reason [1] and [9] consider D0L systems (Σ, h, ω) with the property that ω is a prefix of $h(\omega)$; each D0L system of this kind defines a unique infinite word. It is easy to see that all results we have presented in this paper are also valid for D0L systems of this particular kind.

(2). Analogously to the notion of a square-free word (language), for every $k \geqslant 2$ we can consider the notion of a k-repetitions-free word (language); Thue considered 3-repetitions-free words which he called *cube-free*. It is easy to see that our lower and upper bounds for the subword complexity remain valid also in the general case of k-repetitions-free D0L languages.

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