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Note

Integers with a small number of minimal addition chains

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Abstract

For $n \in \mathbb{N}$ the number of minimal addition chains, $\Lambda(n)$, is examined by its representation by reduced graphs. It is shown that $\Lambda(n) = 1$ implies $n = 2^k$ or $n = 3$ answering a question of Thurber. Further those n with $\Lambda(n) = 2$ are characterized. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

An addition chain is a finite ascending ordered sequence of natural numbers starting with 1 and ending with a given $n \in \mathbb{N}$, i.e. $1 = a_0 < a_1 < \cdots < a_r = n$, such that each element except the first is the sum of two preceding elements of the chain. The length of an addition chain is by definition r. Further the minimal length of an addition chain for $n \in \mathbb{N}$ is denoted by $\ell(n)$. Formally:

Definition 1. $A(n)$ is called an addition chain for $n \in \mathbb{N}$ iff there is an $r \in \mathbb{N}_0$ such that

(i) $\forall k$ with $1 \leqslant k \leqslant r$ $\exists a_k \in \mathbb{N}$: $A(n) = (a_0, a_1, \ldots, a_r)$ with $a_0 = 1$

(ii) $\forall k$ with $1 \leqslant k \leqslant r$ $\exists i, j$ with $i < k$ and $j < k$: $a_k = a_i + a_j$

(iii) $\exists k$ with $0 \leqslant k \leqslant r$: $a_k = n$

(iv) $\forall k$ with $1 \leqslant k \leqslant r$: $a_{k-1} < a_k$

(v) $\forall i$ with $0 \leqslant i < r$ and $a_i \neq n$ $\exists k$ with $i < k \leqslant r$ $\exists j < k$: $a_k = a_i + a_j$

hold. Then r is called *the length of* $A(n)$, which is unique due to (i).

Let $\ell(n) := \min\{r \in \mathbb{N}_0 \mid \exists A(n)$ of length $r\}$. An addition chain $A(n)$ is called a *minimal addition chain* iff its length is $\ell(n)$.

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Condition (v) is a restriction as practical as (iv), to ensure that an addition chain contains no superfluous elements with respect to generating n by condition (ii). Many people replace conditions (iii) and (v) by $a_r = n$ and do not mention the deeper concept of addition chains for sets $S \subset \mathbb{N}$ or the technical difficulties which can arise from unnecessary elements of $A(n)$.

In [4], Thurber defined the number of minimal addition chains for $n \in \mathbb{N}$ as $\Lambda(n) = |\{A(n) \text{ of length } \ell(n)\}|$. He asked for those n with $\Lambda(n) = 1$ and conjectured that n must be three or a power of two to have exactly one minimal addition chain. I will present a short proof of this conjecture using the concept of reduced graphs of addition chains introduced by Knuth [3, pp. 460–462]. Hence, we need

Definition 2. A *reduced graph* $G = (V, E)$ is a finite, directed, acyclic graph with a unique source and a unique sink, such that each vertex except the source has in-degree at least two and each vertex except the sink has out-degree at least two.

Throughout this paper, we will identify $V = \{v_0, v_1, \ldots, v_k\}$ with a subset of $\mathbb{N}$, i.e. label the vertices of V with different numbers, such that, as in [3], the label of each vertex v_i of G is the number of paths from the source to v_i. This label also represents a member of any one of the addition chains that can be formed from G. Thus, v_i can be thought of as vertex $v_i \in V$ or as the number $v_i \in \mathbb{N}$ in which case it represents a member of an addition chain as well as the number of paths in G from the source to v_i. Lastly, we fix the order of the vertices by demanding $i < j \Rightarrow v_i < v_j$. This immediately implies that for the source of G $v_0 \hat{=} 1$ and for its sink $v_k \hat{=} n$ for every $A(n)$. For the ambiguity of the natural surjection between addition chains and reduced graphs due to the deficiency in the definition of addition chain see [1].

2. The case $\Lambda(n) = 1$

Lemma 1. *A reduced graph G corresponds to exactly one addition chain A iff its edges are only between successive vertices.*

Recall that by defintion a reduced graph on $k+1$ vertices must have out-degree$(v_i) \geqslant 2$ for all $i < k$.

First, we prove the $\Rightarrow$-direction of the lemma by induction on i, the index of a vertex v_i, starting with i set to k, the index of the sink, and proceeding by decreasing i by 1 down to $i = 0$, the index of the source, which will generate all edges of this acyclic graph.

Proof. By the inductive hypothesis the predecessor of v_i, the vertex just considered, is still unused in the construction of the addition chain corresponding to G. Therefore there must be at least two out-going edges from v_{i-1} which point to v_i. If there is a further edge to v_i from a vertex v_j with $j < i$, the corresponding addition chain must stay the same when the order in which these edges are used changes, i.e. the order of the corresponding addition steps building up the addition chain must not affect the calculation of v_i. So each further edge into v_i must come from v_{i-1}, too. The same argument works now if we consider $i-1$ in place of i. □

Secondly, we prove the ⇐-direction, which is intuitively easy, but technically difficult to make precise. The reader is referred to [1] for exactness.

Proof. Suppose the edges of G are only between successive vertices. Then out-degree(v_i) = in-degree $(v_{i+1}) = t(i) \geqslant 2$ for each $i < k$, and v_{i+1} is the sum of $t(i)$ of the v_i's. There is only one possible path from v_i to v_{i+1} that reduces to a node v_i and a node v_{i+1} with $t(i)$ edges between them since each intermediate node of the 'nonreduced' graph of the addition chain A must have out-degree 1. The corresponding 'piece of A' is $v_i, v_i+v_i, v_i+v_i+v_i, \ldots, v_{i+1}$, containing $t(i)$ numbers. Thus G corresponds to exactly one addition chain. □

Lemma 2. *In a reduced graph which corresponds to a minimal addition chain there can be at most three parallel edges.*

Proof. If there are four or more parallel edges from say v_i to v_j we can generate a shorter addition chain for the same n by inserting a further vertex with value $2v_i$ after v_i and replacing each pair of edges from v_i to v_j by one edge from the new vertex to v_j. This contradicts the minimality of the given addition chain since the length of an addition chain represented by a reduced graph G is $|E|-|V|+1$ [3]. □

Theorem 1. *For any $n \in \mathbb{N}$ we have $\Lambda(n)=1$, iff either $n=3$ or $n=2^k$ with $k \in \mathbb{N}_0$.*

Proof. The simple direction: if $n=3$ then $\ell(3)=2$ and there is only one (minimal) chain $(1,2,3)$. If $n=2^k$ then $\ell(2^k)=k$ and each step in the addition chain must be a doubling to reach n in k steps starting from 1. Therefore in that case the minimal addition chain is unique, too.

Now the proof in the other direction: for each addition chain there exists at least one reduced graph, which is unique if the elements of the chain are uniquely formed from previous elements of the chain [1]. If $\Lambda(n)=1$ the addition chain $A(n)$ must be unique and the corresponding reduced graph is unique, too, as we see because of the applicability of Lemma 1 and its constructive ⇐-proof. So, we can talk about *the* reduced graph G of the addition chain instead of *any* reduced graph G of $A(n)$, which nevertheless would be also sufficient for the purpose of the proof.

Next, we use Lemma 1 and conclude that all edges $\in E$ of G can only be between successive vertices of (any) G. Because the corresponding addition chain to G should be minimal, we can use Lemma 2 and get: there are only two or three edges between successive vertices.

Hence, $n \hat{=} v_k$, which is the number of all paths from 1 to v_k in any reduced graph, equals $\prod_{i=1}^{k} |\{e \text{ is an edge from } v_{i-1} \text{ to } v_i\}| = 2^j 3^{k-j}$ for a fixed $j \in \mathbb{N}_0$. If $k > j > 0$, then 2 and 3 appear as factors in the product above. This means, a pair of edges connecting adjacent vertices (that are connected by no further edges) could be interchanged with three edges connecting another pair of adjacent vertices. Doing this, we would obtain another reduced graph for the same n from which an additional minimal addition chain for n could be produced contradicting $\Lambda(n) = 1$. Consequently, n must be either a power of two or a power of three.

Finally, we consider the case $n = 3^k$: because $n = 9 = 3^2$ has two further different minimal addition chains apart from the chain considered $(1,2,3,6,9)$ — i.e. $(1,2,4,8,9)$ and $(1,2,4,5,9)$ — we could replace the first four steps of the addition chain generated for 3^k by one of the two others, if $k > 1$. Thus, the addition chain would not be unique or the original reduced graph does not correspond to a minimal chain for n. Therefore, only $k = 1$ is possible if n is a power of three. □

3. The case $\Lambda(n) = 2$

Now let us consider those n which have two minimal addition chains. What is the structure of such n? If their reduced graph is not unique, then each of the two graphs must correspond to exactly one addition chain. But reduced graphs with a unique corresponding minimal addition chain must have $n = 2^j 3^{k-j}$ by Theorem 1. But if $\Lambda(n) = 2$ then we must have $\binom{k}{j} = 2$, too. The only possibility is $k = 2$ and $j = 1$ which means $n = 6$. And indeed $\Lambda(6) = 2$.

In the other case the reduced graph G of an addition chain of n with $\Lambda(n) = 2$ is unique, but corresponds to two different addition chains. In particular, the dual graph of G [3, p. 462] must be identical to G. This means that G is edge-symmetric. Furthermore, because we know that G generates exactly two different addition chains, there must be exactly one vertex v_i of G with in-degree$(v_i) = 3$ and of the three corresponding edges one comes from v_h and two from v_j with $j \neq h$. All other vertices $\neq v_i$ must have either in-degree two or, if in-degree$\geqslant 3$, all edges come from the same predecessor vertex. In the last case only in-degree three is possible, because of the minimality of the corresponding addition chain. All these statements hold also for out-degree(v) for every $v \in V$ in place of in-degree(v) because of the edge-symmetry of G. Moreover, for every path which would go via the special vertex v_i and the corresponding dual vertex v_{k-i}, we could factor out v_{k-i} from n and create another addition chain $(1,\ldots,n/v_{k-i},\ldots,n)$ where the special vertex appears earlier. In detail: consider any path that goes through the special vertex v_i and the corresponding dual vertex v_{k-i}. If we suppose that v_{k-i} occurs at the same place or before v_i in the graph, respectively resultant addition chain, then as a consequence of the restrictions placed on the edges of the reduced graph, v_{k-i} will divide every vertex, resp. number, that occurs after it in the reduced graph, resp. addition chain. This allows the formation of an addition chain $(1,\ldots,v_i/v_{k-i},\ldots,n/v_{k-i},\ldots,n)$ with special vertex v_i/v_{k-i}. The part

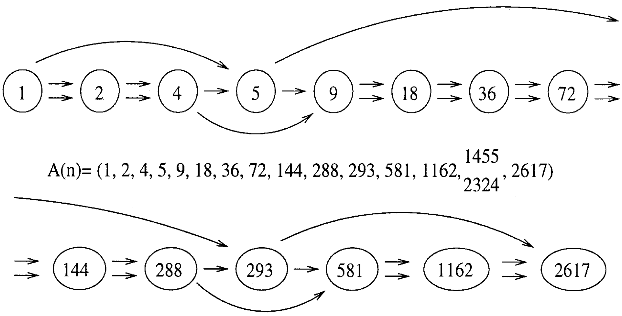

Fig. 1. The reduced graph for $n = 2617$ together with its addition chains.

of the chain from 1 to n/v_{k-i} is formed by dividing each number of the original chain from v_{k-i} to n by v_{k-i}. And the remainder of the new chain is formed by multiplying each number of the original chain from 2 to v_{k-i} by n/v_{k-i}. This new minimal addition chain must be identical to the original since $\Lambda(n) = 2$. Thus, i must be k, resulting in $v_{k-i} = v_0 = 1$. Therefore, we can formulate

Proposition 1. *The reduced graph of a minimal addition chain for $n \neq 6$ with $\Lambda(n)=2$ has its special vertex with* in-degree *three as the last vertex $v_k \mathrel{\hat{=}} n$ and the vertex $v_0 \mathrel{\hat{=}} 1$ must be its dual vertex with* out-degree *three.*

Now, define *the list of special vertices* as follows: the first special out-vertex is $v_0 \mathrel{\hat{=}} 1$. Exactly one of its edges, which does not go to its successor $v_1 \mathrel{\hat{=}} 2$ is called special. This special edge points to the first special in-vertex. By definition the next special out-vertex is the predecessor of the last special in-vertex. From this next special out-vertex there is exactly one special edge which does not point to its successor vertex. It points to the next special in-vertex and so forth. This scheme works until the special in-vertex becomes the last vertex v_k. As a result of the structure imposed upon G, a vertex of G must appear in such a list as in-vertex or out-vertex uniquely. Nodes of G which are not in this list must be strictly between the predecessor of a special in-vertex and the successor of the out-vertex that points to this in-vertex — except at the start of G where such nodes lie between the first two out vertices and at the end of G where such vertices lie between the last two in-vertices. These missing vertices can have only parallel edges to their successors. Furthermore, if there are at least two successive vertices then they must all have out-degree two because of the uniqueness of the addition chain in this part. Finally, we note that, conversely, if all minimal addition chains for n have the same reduced graph G and G has the structure as just described, then $\Lambda(n) = 2$.

As an illustration the reduced graph for $n = 2617$ is given in Fig. 1, where the label of each vertex is obtained by counting the number of paths from the source to this vertex. We get its (minimal) addition chains of length 14 by inserting either 1455 or 2324 before the last value 2617. The corresponding list of special vertices is

Table 1

Restrictions on $i, m \in \mathbb{N}$	Family (n_i)	First n	$\ell(n_i)$	$s(n_i)$
	–	6	3	1
$i \geqslant 2$	$2^i + 1$	5	$i+1$	1
$i \geqslant 2,\ 2i \neq m,\ \lvert m-i \rvert > 2$	$(2^{i+1}+1)^2 \cdot 2^m + (2^i+1)^2$	2467	$2i+m+5$	3
		91913	20	4

Table 2

k	3	5	7	9	11	13, $m \geqslant 7$	15	17	19
n	9, 12	7, 27, 48	18, 192	768	3072	$359, 11 \cdot 2^m + 7, 3 \cdot 2^{12}$	$11, 3 \cdot 2^{14}$	$711, 3 \cdot 2^{16}$	$3 \cdot 2^{18}$

$(1, 5, 4, 9, 5, 293, 288, 581, 293, 2617)$. We are left with the question: which $n \in \mathbb{N}$ have only such a reduced graph for their corresponding minimal addition chains? The known $n \in \mathbb{N}$ with $\Lambda(n) = 2$ including all these n up to 2^{17} are as shown in Table 1.

The entries in this tables were found by examining all minimal addition chains for n up to 2^{17} computationally and then the families were recognized by a bit of perspicacity. The last column, indicating the number of small steps in a minimal addition chain for n can be calculated as $s(n) = \ell(n) - \lfloor \log_2(n) \rfloor$. The family with $s(n_i) = 3$ could be proved by the complete 3-smallstep-analysis of Flammenkamp [2], which would characterize all further 3-smallstep addition chains with $\Lambda(n) = 2$, if such exist.

4. The case $\Lambda(n) \geqslant 3$

For even k, there are infinitely many numbers n with $\Lambda(n) = k$ [4, p. 290]. Thurber showed also in his Theorem 2 that $\Lambda(3 \cdot 2^{k-1}) = k$ for all $k \in \mathbb{N}$. Hence, at least one number n exists for each $k \in \mathbb{N}$. But for which odd $k \in \mathbb{N}$ are there infinitely many $n \in \mathbb{N}$ with $\Lambda(n) = k$? If we have a look at numbers n of the form $2^m + 2^{m'} + 1$ for $m > m' > 0$, we notice that $\Lambda(n)$ covers certain odd values:

$$m' = 1: \quad n = 2^m + 3 \quad \text{and } m \geqslant 6, \text{ then } \Lambda(n) = 23,$$

$$m' = 2: \quad n = 2^m + 5 \quad \text{and } m \geqslant 7, \text{ then } \Lambda(n) = 21,$$

$$m' \geqslant 3: \quad n = 2^m + 2^{m'} + 1 \quad \text{and } m \geqslant 2m' + 1, \text{ then } \Lambda(n) = 4m' + 9.$$

These facts, found by computation up to some m, can be easily proved for all (larger) m by the computer program of Flammenkamp, which just counts minimal addition chains represented in exactly such algebraic representations, see [2]. Thus, if there are only finitely many n for which $\Lambda(n) = k$ for some $k \in \mathbb{N}_0$, then $k \equiv 3 \pmod 4$ or $k < 20$. Table 2 lists the known n for odd $k < 20$:

As we see, only in the case $k = 13$, are infinitely many n known with exactly that number of minimal addition chains — as mentioned above, this could also be proved automatically, as it is covered by the 3-smallstep addition chains of case 125: `125 6*`

`@(p)+@(q)+@(r)+@(s)+@(−p+q+s+1)+@(−p+r+s+1)` of Flammenkamp's diploma thesis, setting $p=m+3,\ q=m+1,\ r=m,\ s=2$.

During the first revision of this paper, one referee kindly communicated to me another family, namely $n=187\cdot 2^m+63$ for $m\geqslant 7,\ m\neq 9$, which appears to have $\Lambda(n)=13$, too.

We are left with the open problem similar to Thurber's question: for which n does a reduced graph exist presenting each minimal addition chain for n? The answer to this question would describe all such n with e.g. $\Lambda(n)=2,3$ uniquely.

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