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A Lotka–Volterra type food chain model with stage structure and time delays

Rui Xu [a,*,1], M.A.J. Chaplain [b], F.A. Davidson [b]

[a] *Department of Mathematics, Institute of Shijiazhuang Mechanical Engineering, 97 Helping West Road, Shijiazhuang 050003, Hebei Province, PR China*
[b] *Department of Mathematics, University of Dundee, Dundee DD1 4HN, UK*

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Abstract

A three-species Lotka–Volterra type food chain model with stage structure and time delays is investigated. It is assumed in the model that the individuals in each species may belong to one of two classes: the immatures and the matures, the age to maturity is presented by a time delay, and that the immature predators (immature top predators) do not have the ability to feed on prey (predator). By using some comparison arguments, we first discuss the permanence of the model. By means of an iterative technique, a set of easily verifiable sufficient conditions are established for the global attractivity of the nonnegative equilibria of the model.

Keywords: Stage structure; Time delay; Permanence; Global attractivity; Iteration

1. Introduction

An important and ubiquitous problem in predator–prey theory and related topics in mathematical ecology, concerns the long term coexistence of species. Lotka–Volterra type predator–prey systems are very important in the models of multi-species populations interactions and have been studied by many authors (see, for example, [5–8]). It is assumed in the classical predator–prey

* Corresponding author.
E-mail address: rxu88@yahoo.com.cn (R. Xu).
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model that each individual predator admits the same ability to attack prey and each individual prey admits the same risk to be attacked by predator. This assumption seems not to be realistic for many animals. In the natural world, there are many species whose individuals have a life history that takes them through two stages, immature and mature, where immature predators are raised by their parents, and the rate they attacking at prey and the reproductive rate can be ignored; on the other hand, it may be reasonable for a number of animals to assume that immature prey population concealed in the mountain cave and are raised by their parents; the rate of mature predators attacking at immature prey can be ignored.

Stage-structured models have received great attention in recent years. The pioneering work of Aiello and Freedman [1] on a single species growth model with stage structure represents a mathematically more careful and biologically meaningful formulation approach. In [1], a model of single species population growth incorporating stage structure as a reasonable generalization of the classical logistic model was formulated and discussed. This model assumes an average age to maturity which appears as a constant time delay reflecting a delayed birth of immatures and a reduced survival of immatures to their maturity. Recently, many authors studied different kinds of stage-structured models and some significant work was carried out (see, for example, [2–4,9–16]).

Motivated by the recent work of Aiello and Freedman [1], in the present paper we are concerned with the effect of stage structure for each species on three species Lotka–Volterra type food chain model. To do so, we study the following delayed differential system:

$$
\begin{aligned}
\dot{x}_1(t) &= \alpha_1 e^{-\gamma_1\tau_1} x_1(t-\tau_1) - a_{11}x_1^2(t) - a_{12}x_1(t)x_2(t),\\
\dot{y}_1(t) &= \alpha_1 x_1(t) - \gamma_1 y_1(t) - \alpha_1 e^{-\gamma_1\tau_1} x_1(t-\tau_1),\\
\dot{x}_2(t) &= \alpha_2 e^{-\gamma_2\tau_2} x_1(t-\tau_2)x_2(t-\tau_2) - r_2x_2(t) - a_{22}x_2^2(t) - a_{23}x_2(t)x_3(t),\\
\dot{y}_2(t) &= \alpha_2 x_1(t)x_2(t) - \gamma_2 y_2(t) - \alpha_2 e^{-\gamma_2\tau_2} x_1(t-\tau_2)x_2(t-\tau_2),\\
\dot{x}_3(t) &= \alpha_3 e^{-\gamma_3\tau_3} x_2(t-\tau_3)x_3(t-\tau_3) - r_3x_3(t) - a_{33}x_3^2(t),\\
\dot{y}_3(t) &= \alpha_3 x_2(t)x_3(t) - \gamma_3 y_3(t) - \alpha_3 e^{-\gamma_3\tau_3} x_2(t-\tau_3)x_3(t-\tau_3),
\end{aligned}
\tag{1.1}
$$

where $x_1(t)$ and $y_1(t)$ denote the densities of the mature and immature prey population at time t, respectively; $x_2(t)$ and $y_2(t)$ represent the densities of the mature and immature predator population at time t, respectively; $x_3(t)$ and $y_3(t)$ denote the densities of the mature and immature top predator population at time t, respectively. a_{11}, a_{12}, a_{22}, a_{23}, a_{33}, r_2, r_3, α_1, α_2, α_3, γ_1, γ_2, γ_3, τ_1, τ_2 and τ_3 are positive constants. The model is derived under the following assumptions:

(A1) *The prey population*: the birth rate of the population is proportional to the existing mature population with a proportionality constant $\alpha_1 > 0$; the death rate of the immature population is proportional to the existing immature population with a proportionality constant $\gamma_1 > 0$; a_{11} is the death and intra-specific competition rate of the mature population. The term $\alpha_1 e^{-\gamma_1\tau_1} x_1(t-\tau_1)$ represents the immature prey individuals who were born at time $t-\tau_1$ and survive at time t, and therefore represents the transformation of immature prey population to mature prey population.

(A2) *The predator population*: a_{12} is the capturing rate of the mature predator, α_2/a_{12} is the conversion rate of nutrients into the reproduction of the mature predator, r_2 and a_{22} are the death rate and the intra-specific competition rate of the mature predators, respectively; the death rate of the immature population is proportional to the existing immature population with a proportionality constant $\gamma_2 > 0$. The term $\alpha_2 e^{-\gamma_2\tau_2} x_1(t-\tau_2)x_2(t-\tau_2)$ represents the number of immature predators that were born at time $t-\tau_2$ which still survive at time t and are transferred from the immature stage to the mature stage at time t. It is assumed in (1.1) that immature individual predators do not feed on prey and do not have the ability to reproduce.

(A3) *The top predator population*: a_{23} is the capturing rate of the mature top predator, α_3/a_{23} is the conversion rate of nutrients into the reproduction of the mature top predator, r_3 and a_{33} are the death rate and the intra-specific competition rate of the mature top predators, the death rate of the immature population is proportional to the existing immature population with a proportionality constant $\gamma_3>0$. The term $\alpha_3 e^{-\gamma_3\tau_3}x_2(t-\tau_3)x_3(t-\tau_3)$ denotes the number of immature top predators that were born at time $t-\tau_3$ which still survive at time t and are transferred from the immature stage to the mature stage at time t. In (1.1) we also assume that the immature top predator do not feed on predator and do not have the ability to reproduce.

The initial conditions for system (1.1) take the form

$$\begin{aligned} &x_i(\theta)=\phi_i(\theta), \qquad y_i(\theta)=\psi_i(\theta),\\ &\phi_i(0)>0, \qquad \psi_i(0)>0, \quad i=1,2,3, \end{aligned} \tag{1.2}$$

where $(\phi_1(\theta),\psi_1(\theta),\phi_2(\theta),\psi_2(\theta),\phi_3(\theta),\psi_3(\theta))\in C([-\tau,0],R^6_{+0})$, the Banach space of continuous functions mapping the interval $[-\tau,0]$ into R^6_{+0}, where $\tau=\max\{\tau_1,\tau_2,\tau_3\}$, $R^6_{+0}=\{(x_1,x_2,x_3,x_4,x_5,x_6)\mid x_i\geqslant 0,\ i=1,2,\dots,6\}$.

For continuity of the initial conditions, we further require

$$\begin{aligned} y_1(0)&=\int_{-\tau_1}^{0}\alpha_1\phi_1(s)e^{\gamma_1 s}\,ds,\\ y_2(0)&=\int_{-\tau_2}^{0}\alpha_2\phi_1(s)\phi_2(s)e^{\gamma_2 s}\,ds,\\ y_3(0)&=\int_{-\tau_3}^{0}\alpha_3\phi_2(s)\phi_3(s)e^{\gamma_3 s}\,ds. \end{aligned} \tag{1.3}$$

The paper is organized as follows. In the next section, we will discuss the positivity of solutions and the permanence of system (1.1). In Section 3, a set of easily verifiable sufficient conditions are derived for the global attractivity of the nonnegative equilibria of system (1.1) by using an iterative technique. A brief discussion is given in Section 4 to conclude this work.

2. Permanence

In this section, we are concerned with the permanence of system (1.1) with initial conditions (1.2) and (1.3).

Definition. System (1.1) is said to be permanent if there exists a compact region $D\subset \operatorname{Int} R^6_+$ such that every solution $(x_1(t),y_1(t),x_2(t),y_2(t),x_3(t),y_3(t))$ of (1.1) with initial conditions (1.2) and (1.3) eventually enters and remains in the region D.

In the following we first show the positivity of solutions to system (1.1) with initial conditions (1.2) and (1.3).

Lemma 2.1. *Solutions of system* (1.1) *with initial conditions* (1.2) *and* (1.3) *are positive for all* $t \geqslant 0$.

Proof. Let $(x_1(t), y_1(t), x_2(t), y_2(t), x_3(t), y_3(t))$ be a solution of system (1.1) with initial conditions (1.2) and (1.3). Let us first consider $y_3(t)$ for $t \in [0, \tau^*]$, where $\tau^* = \min\{\tau_1, \tau_1, \tau_2\}$. Noting that $\phi_2(\theta) \geqslant 0$, $\phi_3(\theta) \geqslant 0$ for $\theta \in [-\tau, 0]$, we obtain from the fifth equation of system (1.1) that

$$\dot{x}_3(t) = \alpha_3 e^{-\gamma_3\tau_3}\phi_2(t-\tau_3)\phi_3(t-\tau_3) - r_3x_3(t) - a_{33}x_3^2(t) \geqslant -r_3x_3(t) - a_{33}x_3^2(t).$$

By comparison, it follows that for $t \in [0, \tau^*]$,

$$x_3(t) \geqslant \frac{r_3x_3(0)}{a_{33}x_3(0)(e^{r_3t}-1)+r_3} > 0.$$

We derive from the third equation of system (1.1) that for $t \in [0, \tau^*]$,

$$\begin{aligned}\dot{x}_2(t) &= \alpha_2 e^{-\gamma_2\tau_2}\phi_1(t-\tau_2)\phi_2(t-\tau_2) - r_2x_2(t) - a_{22}x_2^2(t) - a_{23}x_2(t)x_3(t)\\ &\geqslant x_2(t)\big(-r_2 - a_{22}x_2(t) - a_{23}x_3(t)\big)\end{aligned}$$

since $\phi_1(\theta) \geqslant 0$, $\phi_2(\theta) \geqslant 0$, $\theta \in [-\tau, 0]$. A standard comparison argument shows that for $t \in [0, \tau^*]$,

$$x_2(t) \geqslant \frac{x_2(0)\exp[-\int_0^t(r_2 + a_{23}x_3(s))\,ds]}{1 + a_{22}x_2(0)\int_0^t \exp[-\int_0^s(r_2+a_{23}x_3(u))\,du]\,ds} > 0.$$

Similarly, it follows from the first equation of system (1.1) that for $t \in [0, \tau^*]$,

$$\begin{aligned}\dot{x}_1(t) &= \alpha_1 e^{-\gamma_1\tau_1}\phi_1(t-\tau_1) - a_{11}x_1^2(t) - a_{12}x_1(t)x_2(t)\\ &\geqslant x_1(t)\big[-a_{11}x_1(t) - a_{12}x_2(t)\big]\end{aligned}$$

since $\phi_1(\theta) \geqslant 0$, $\theta \in [-\tau, 0]$. By comparison, we derive that for $t \in [0, \tau^*]$,

$$x_1(t) \geqslant \frac{x_1(0)\exp[-a_{12}\int_0^t x_2(s)\,ds]}{1 + a_{11}x_1(0)\int_0^t \exp[-a_{12}\int_0^s x_2(u)\,du]\,ds} > 0.$$

In a similar way, we treat the intervals $[\tau^*, 2\tau^*], \ldots, [n\tau^*, (n+1)\tau^*], n \in N$. Thus, $x_i(t) > 0$ for all $t \geqslant 0$, $i = 1, 2, 3$.

It follows from (1.1) and (1.3) that

$$y_1(t) = \int_{t-\tau_1}^{t} \alpha_1 e^{-\gamma_1(t-s)}x_1(s)\,ds,$$

$$y_2(t) = \int_{t-\tau_2}^{t} \alpha_2 e^{-\gamma_2(t-s)}x_1(s)x_2(s)\,ds,$$

$$y_3(t) = \int_{t-\tau_3}^{t} \alpha_3 e^{-\gamma_3(t-s)} x_2(s) x_3(s)\, ds. \tag{2.1}$$

Therefore, the positivity of $y_i(t)$ $(i = 1, 2, 3)$ follows. This completes the proof. □

In order to discuss the permanence of system (1.1), we need the following result from [13].

Lemma 2.2. *Consider the following equation*:

$$\dot{x}(t) = ax(t-\tau) - bx(t) - cx^2(t),$$

where a, b, c *and* τ *are positive constants,* $x(t) > 0$ *for* $t \in [-\tau, 0]$. *We have*

(i) *if* $a > b$, *then* $\lim_{t\to+\infty} x(t) = (a-b)/c$;
(ii) *if* $a < b$, *then* $\lim_{t\to+\infty} x(t) = 0$.

Theorem 2.1. *System* (1.1) *with initial conditions* (1.2) *and* (1.3) *is permanent provided that*

(H1) $A_i > 0$, $i = 1, 2$, *where*

$$\begin{aligned} A_1 &= a_{11}a_{22}a_{33} - a_{11}a_{23}\alpha_3 e^{-\gamma_3\tau_3} - a_{33}a_{12}\alpha_2 e^{-\gamma_2\tau_2} > 0, \\ A_2 &= \left(\alpha_1\alpha_2\alpha_3 e^{-\gamma_1\tau_1-\gamma_2\tau_2-\gamma_3\tau_3} - r_2 a_{11}\alpha_3 e^{-\gamma_3\tau_3} - r_3 a_{11}a_{22} - r_3 a_{12}\alpha_2 e^{-\gamma_2\tau_2}\right) \\ &\quad \times \left(1 - \frac{a_{12}\alpha_2 e^{-\gamma_2\tau_2}}{a_{11}a_{22}} - \frac{a_{23}\alpha_3 e^{-\gamma_3\tau_3}}{a_{22}a_{33}}\right) \\ &\quad - r_3 a_{12}\alpha_2 e^{-\gamma_2\tau_2}\left(\frac{a_{12}\alpha_2 e^{-\gamma_2\tau_2}}{a_{11}a_{22}} + \frac{a_{23}\alpha_3 e^{-\gamma_3\tau_3}}{a_{22}a_{33}}\right). \end{aligned} \tag{2.2}$$

Proof. Suppose $(x_1(t), y_1(t), x_2(t), y_2(t), x_3(t), y_3(t))$ is a positive solution of system (1.1) with initial conditions (1.2) and (1.3). It follows from the first equation of system (1.1) that

$$\dot{x}_1(t) \leqslant \alpha_1 e^{-\gamma_1\tau_1} x_1(t-\tau_1) - a_{11}x_1^2(t).$$

Consider the following auxiliary equation:

$$\dot{u}(t) = \alpha_1 e^{-\gamma_1\tau_1} u(t-\tau_1) - a_{11}u^2(t).$$

By Lemma 2.2, we derive that

$$\lim_{t\to+\infty} u(t) = \frac{\alpha_1 e^{-\gamma_1\tau_1}}{a_{11}}.$$

By comparison, it follows that

$$\limsup_{t\to+\infty} x_1(t) \leqslant \frac{\alpha_1 e^{-\gamma_1\tau_1}}{a_{11}}.$$

Therefore, for $\varepsilon > 0$ sufficiently small, there is a $T_{11} > 0$ such that if $t > T_{11}$,

$$x_1(t) \leqslant \frac{\alpha_1 e^{-\gamma_1\tau_1}}{a_{11}} + \varepsilon := M_1. \tag{2.3}$$

We derive from the third equation of system (1.1) for $t > T_{11} + \tau$ that

$$\dot{x}_2(t) \leqslant \alpha_2 e^{-\gamma_2\tau_2} M_1 x_2(t-\tau_2) - r_2 x_2(t) - a_{22}x_2^2(t).$$

A comparison argument shows that

$$\limsup_{t\to+\infty} x_2(t) \leqslant \frac{\alpha_2 e^{-\gamma_2\tau_2}\left(\frac{\alpha_1 e^{-\gamma_1\tau_1}}{a_{11}} + \varepsilon\right) - r_2}{a_{22}}.$$

Since $\varepsilon > 0$ is arbitrary and sufficiently small, we can conclude that

$$\limsup_{t\to+\infty} x_2(t) \leqslant \frac{\alpha_1\alpha_2 e^{-\gamma_1\tau_1-\gamma_2\tau_2} - r_2 a_{11}}{a_{11}a_{22}}.$$

Therefore, for $\varepsilon > 0$ sufficiently small there exists $T_{12} > T_{11} + \tau$ such that if $t > T_{12}$,

$$x_2(t) \leqslant \frac{\alpha_1\alpha_2 e^{-\gamma_1\tau_1-\gamma_2\tau_2} - r_2 a_{11}}{a_{11}a_{22}} + \varepsilon := M_2. \tag{2.4}$$

Similarly, we derive from the fifth equation of system (1.1) and (2.4) that

$$\limsup_{t\to+\infty} x_3(t) \leqslant \frac{\alpha_3 e^{-\gamma_3\tau_3}(\alpha_1\alpha_2 e^{-\gamma_1\tau_1-\gamma_2\tau_2} - r_2 a_{11}) - r_3 a_{11}a_{22}}{a_{11}a_{22}a_{33}}.$$

Hence, for $\varepsilon > 0$ sufficiently small there is $T_{13} > T_{12} + \tau$ such that if $t > T_{13}$,

$$x_3(t) \leqslant \frac{\alpha_3 e^{-\gamma_3\tau_3}(\alpha_1\alpha_2 e^{-\gamma_1\tau_1-\gamma_2\tau_2} - r_2 a_{11}) - r_3 a_{11}a_{22}}{a_{11}a_{22}a_{33}} + \varepsilon := M_3. \tag{2.5}$$

Set $T_1 = T_{13} + \tau$. It follows from (2.1), (2.3)–(2.5) that for $t > T_1$,

$$\begin{aligned}
y_1(t) &\leqslant \frac{\alpha_1 M_1}{\gamma_1}\left(1 - e^{-\gamma_1\tau_1}\right) := N_1,\\
y_2(t) &\leqslant \frac{\alpha_2 M_1 M_2}{\gamma_2}\left(1 - e^{-\gamma_2\tau_2}\right) := N_2,\\
y_3(t) &\leqslant \frac{\alpha_3 M_2 M_3}{\gamma_3}\left(1 - e^{-\gamma_3\tau_3}\right) := N_3.
\end{aligned} \tag{2.6}$$

Again, we derive from the first equation of system (1.1) and (2.4) that for $t > T_1$,

$$\dot{x}_1(t) \geqslant \alpha_1 e^{-\gamma_1\tau_1} x_1(t-\tau_1) - a_{11}x_1^2(t) - a_{12}M_2 x_1(t). \tag{2.7}$$

By comparison, it follows from (2.4) and (2.7) that

$$\liminf_{t\to+\infty} x_1(t) \geqslant \frac{\alpha_1 e^{-\gamma_1\tau_1} - a_{12}\left(\frac{\alpha_1\alpha_2 e^{-\gamma_1\tau_1-\gamma_2\tau_2} - r_2 a_{11}}{a_{11}a_{22}} + \varepsilon\right)}{a_{11}}.$$

Since $\varepsilon > 0$ is arbitrary and sufficiently small, we conclude that

$$\liminf_{t\to+\infty} x_1(t) \geqslant \frac{\alpha_1 e^{-\gamma_1\tau_1} - a_{12}\frac{\alpha_1\alpha_2 e^{-\gamma_1\tau_1-\gamma_2\tau_2} - r_2 a_{11}}{a_{11}a_{22}}}{a_{11}}.$$

Therefore, for $\varepsilon > 0$ sufficiently small there is $T_2 > T_1$ such that if $t > T_2$,

$$x_1(t) > \frac{\alpha_1 e^{-\gamma_1\tau_1} - a_{12}\frac{\alpha_1\alpha_2 e^{-\gamma_1\tau_1-\gamma_2\tau_2} - r_2 a_{11}}{a_{11}a_{22}}}{a_{11}} - \varepsilon := m_1. \tag{2.8}$$

It follows from the third equation of system (1.1), (2.5) and (2.8) that for $t > T_2 + \tau$,

$$\dot{x}_2(t) \geqslant \alpha_2 e^{-\gamma_2\tau_2} m_1 x_2(t-\tau_2) - r_2 x_2(t) - a_{22}x_2^2(t) - a_{23}M_3 x_2(t). \tag{2.9}$$

By comparison, we obtain from (2.5), (2.8) and (2.9) that

$$\liminf_{t\to+\infty} x_2(t) \geqslant \frac{1}{a_{22}}\left\{\alpha_2 e^{-\gamma_2\tau_2}\left(\frac{\alpha_1 e^{-\gamma_1\tau_1} - a_{12}\frac{\alpha_1\alpha_2 e^{-\gamma_1\tau_1-\gamma_2\tau_2}-r_2a_{11}}{a_{11}a_{22}}}{a_{11}} - \varepsilon\right) - r_2 \right.$$
$$\left. - a_{23}\left(\frac{\alpha_3 e^{-\gamma_3\tau_3}(\alpha_1\alpha_2 e^{-\gamma_1\tau_1-\gamma_2\tau_2} - r_2a_{11}) - r_3a_{11}a_{22}}{a_{11}a_{22}a_{33}} + \varepsilon\right)\right\}.$$

Since $\varepsilon > 0$ is arbitrary small, we can conclude that

$$\liminf_{t\to+\infty} x_2(t) \geqslant \frac{\left(\frac{\alpha_1\alpha_2 e^{-\gamma_1\tau_1-\gamma_2\tau_2}}{a_{11}} - r_2\right)\left(1 - \frac{a_{12}\alpha_2 e^{-\gamma_2\tau_2}}{a_{11}a_{22}} - \frac{a_{23}\alpha_3 e^{-\gamma_3\tau_3}}{a_{22}a_{33}}\right) + \frac{r_3a_{23}}{a_{33}}}{a_{22}}.$$

Hence, for $\varepsilon > 0$ sufficiently small there is a $T_3 > T_2 + \tau$ such that if $t > T_3$,

$$x_2(t) > \frac{\left(\frac{\alpha_1\alpha_2 e^{-\gamma_1\tau_1-\gamma_2\tau_2}}{a_{11}} - r_2\right)\left(1 - \frac{a_{12}\alpha_2 e^{-\gamma_2\tau_2}}{a_{11}a_{22}} - \frac{a_{23}\alpha_3 e^{-\gamma_3\tau_3}}{a_{22}a_{33}}\right) + \frac{r_3a_{23}}{a_{33}}}{a_{22}} - \varepsilon := m_2. \tag{2.10}$$

Similarly, we derive from the fifth equation of system (1.1) that

$$\liminf_{t\to+\infty} x_3(t) \geqslant \frac{A_2}{a_{11}a_{22}a_{33}},$$

where A_2 is defined in (2.2). Therefore, for $\varepsilon > 0$ sufficiently small there exists a $T_4 > T_3 + \tau$ such that if $t > T_4$,

$$x_3(t) > \frac{A_2}{a_{11}a_{22}a_{33}} - \varepsilon := m_3. \tag{2.11}$$

We note that if (H1) holds and $\varepsilon > 0$ is chosen sufficiently small, $m_i > 0$.

It follows from (2.1), (2.8), (2.10) and (2.11) that there is $T > T_4 + \tau$ such that if $t > T$,

$$y_1(t) \geqslant \frac{\alpha_1 m_1}{\gamma_1}\left(1 - e^{-\gamma_1\tau_1}\right) > 0,$$
$$y_2(t) \geqslant \frac{\alpha_2 m_1 m_2}{\gamma_2}\left(1 - e^{-\gamma_2\tau_2}\right) > 0,$$
$$y_3(t) \geqslant \frac{\alpha_3 m_2 m_3}{\gamma_3}\left(1 - e^{-\gamma_3\tau_3}\right) > 0.$$

This completes the proof. □

3. Global attractivity of nonnegative equilibria

In this section, we discuss the global attractivity of the nonnegative equilibria of system (1.1) by using an iterative technique developed by some authors (see, for example, [3,13,14,16]).

It is easy to show that system (1.1) has at least two nonnegative equilibria: $E_0(0,0,0,0,0,0)$, $E_1(\alpha_1 e^{-\gamma_1\tau_1}/a_{11}, \alpha_1^2 e^{-\gamma_1\tau_1}(1 - e^{-\gamma_1\tau_1})/(a_{11}\gamma_1), 0, 0, 0, 0)$. By analyzing the corresponding characteristic equations, we know that E_0 is always unstable; if $\alpha_1\alpha_2 e^{-\gamma_1\tau_1-\gamma_2\tau_2} > r_2a_{11}$, E_1 is locally unstable, if $\alpha_1\alpha_2 e^{-\gamma_1\tau_1-\gamma_2\tau_2} < r_2a_{11}$, E_1 is locally stable. If $\alpha_1\alpha_2 e^{-\gamma_1\tau_1-\gamma_2\tau_2} > r_2a_{11}$, system (1.1) has another nonnegative equilibrium $E_2(x_1^0, y_1^0, x_2^0, y_2^0, 0, 0)$, where

$$x_1^0 = \frac{a_{22}\alpha_1 e^{-\gamma_1\tau_1} + r_2a_{12}}{a_{11}a_{22} + a_{12}\alpha_2 e^{-\gamma_2\tau_2}}, \qquad x_2^0 = \frac{\alpha_1\alpha_2 e^{-\gamma_1\tau_1-\gamma_2\tau_2} - r_2a_{11}}{a_{11}a_{22} + a_{12}\alpha_2 e^{-\gamma_2\tau_2}},$$
$$y_1^0 = \frac{\alpha_1 x_1^0}{\gamma_1}\left(1 - e^{-\gamma_1\tau_1}\right), \qquad y_2^0 = \frac{\alpha_2 x_1^0 x_2^0}{\gamma_2}\left(1 - e^{-\gamma_2\tau_2}\right). \tag{3.1}$$

Furthermore, system (1.1) admits a unique positive equilibrium $E^*(x_1^*, y_1^*, x_2^*, y_2^*, x_3^*, y_3^*)$ if the following holds:

(H2) $\Delta_3 > 0$, where

$$x_i^* = \frac{\Delta_i}{\Delta} \quad (i = 1, 2, 3), \qquad y_1^* = \frac{\alpha_1 x_1^*}{\gamma_1}\left(1 - e^{-\gamma_1\tau_1}\right),$$
$$y_2^* = \frac{\alpha_2 x_1^* x_2^*}{\gamma_2}\left(1 - e^{-\gamma_2\tau_2}\right), \qquad y_3^* = \frac{\alpha_3 x_2^* x_3^*}{\gamma_3}\left(1 - e^{-\gamma_3\tau_3}\right), \tag{3.2}$$

in which

$$\Delta_1 = a_{22}a_{33}\alpha_1 e^{-\gamma_1\tau_1} - r_3 a_{12}a_{23} + a_{23}\alpha_1\alpha_3 e^{-\gamma_1\tau_1-\gamma_3\tau_3} + r_2 a_{12}a_{33},$$
$$\Delta_2 = a_{33}\alpha_1\alpha_2 e^{-\gamma_1\tau_1-\gamma_2\tau_2} - r_2 a_{11}a_{33} + r_3 a_{11}a_{23},$$
$$\Delta_3 = \alpha_1\alpha_2\alpha_3 e^{-\gamma_1\tau_1-\gamma_2\tau_2-\gamma_3\tau_3} - r_2 a_{11}\alpha_3 e^{-\gamma_3\tau_3} - r_3 a_{11}a_{22} - r_3 a_{12}\alpha_2 e^{-\gamma_2\tau_2},$$
$$\Delta = a_{11}a_{22}a_{33} + a_{11}a_{23}\alpha_3 e^{-\gamma_3\tau_3} + a_{12}a_{33}\alpha_2 e^{-\gamma_2\tau_2}. \tag{3.3}$$

We first give a result on the global attractivity of the positive equilibrium E^* of system (1.1).

Theorem 3.1. *Let* (H2) *hold. Then the positive equilibrium* $E^*(x_1^*, y_1^*, x_2^*, y_2^*, x_3^*, y_3^*)$ *of system* (1.1) *is globally attractive provided that*

(H3) $a_{11}a_{22}a_{33} > a_{11}a_{23}\alpha_3 e^{-\gamma_3\tau_3} + a_{33}a_{12}\alpha_2 e^{-\gamma_2\tau_2}$.

Proof. Let $(x_1(t), y_1(t), x_2(t), y_2(t), x_3(t), y_3(t))$ be a positive solution to system (1.1) with initial conditions (1.2) and (1.3).

Denote

$$U_i = \limsup_{t\to+\infty} x_i(t), \qquad V_i = \liminf_{t\to+\infty} x_i(t) \quad (i = 1, 2, 3).$$

We now claim that $U_i = V_i = x_i^*$ $(i = 1, 2, 3)$.

It follows from the first equation of system (1.1) that

$$\dot{x}_1(t) \leqslant \alpha_1 e^{-\gamma_1\tau_1} x_1(t-\tau_1) - a_{11}x_1^2(t).$$

By comparison, we derive that

$$U_1 = \limsup_{t\to+\infty} x_1(t) \leqslant \frac{\alpha_1 e^{-\gamma_1\tau_1}}{a_{11}} := N_1^{x_1}.$$

Hence, for $\varepsilon > 0$ sufficiently small there is $T_{11} > 0$ such that if $t > T_{11}$, $x_1(t) \leqslant N_1^{x_1} + \varepsilon$.

We derive from the third equation of system (1.1) that for $t > T_{11} + \tau$,

$$\dot{x}_2(t) \leqslant \alpha_2 e^{-\gamma_2\tau_2}\left(N_1^{x_1} + \varepsilon\right)x_2(t-\tau_2) - r_2 x_2(t) - a_{22}x_2^2(t).$$

A standard comparison argument shows that

$$U_2 = \limsup_{t\to+\infty} x_2(t) \leqslant \frac{\alpha_2 e^{-\gamma_2\tau_2}(N_1^{x_1} + \varepsilon) - r_2}{a_{22}}.$$

Since this is true for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that $U_2 \leqslant N_1^{x_2}$, where

$$N_1^{x_2} = \frac{\alpha_2 e^{-\gamma_2\tau_2} N_1^{x_1} - r_2}{a_{22}}.$$

Hence, for $\varepsilon > 0$ sufficiently small, there is $T_{21} \geqslant T_{11} + \tau$ such that if $t > T_{21}$, $x_2(t) \leqslant N_1^{x_2} + \varepsilon$.

We derive from the fifth equation of system (1.1) that for $t > T_{21} + \tau$,

$$\dot{x}_3(t) \leqslant \alpha_3 e^{-\gamma_3\tau_3}\big(N_1^{x_2} + \varepsilon\big)x_3(t-\tau_3) - r_3 x_3(t) - a_{33}x_3^2(t).$$

By comparison, it follows that

$$U_3 = \limsup_{t\to+\infty} x_3(t) \leqslant \frac{\alpha_3 e^{-\gamma_3\tau_3}(N_1^{x_2} + \varepsilon) - r_3}{a_{33}}.$$

Since this is true for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that $U_3 \leqslant N_1^{x_3}$, where

$$N_1^{x_3} = \frac{\alpha_3 e^{-\gamma_3\tau_3} N_1^{x_2} - r_3}{a_{33}}.$$

Therefore, for $\varepsilon > 0$ sufficiently small, there is $T_{31} > T_{21} + \tau$ such that if $t > T_{31}$, $x_3(t) \leqslant N_1^{x_3} + \varepsilon$.

Again, we derive from the first equation of system (1.1) that for $t > T_{31}$,

$$\dot{x}_1(t) \geqslant \alpha_1 e^{-\gamma_1\tau_1} x_1(t-\tau_1) - a_{11}x_1^2(t) - a_{12}x_1(t)\big(N_1^{x_2} + \varepsilon\big).$$

Thus, if for $t > T_{31}$ we denote by $v(t)$ the solution of

$$\dot{v}(t) = \alpha_1 e^{-\gamma_1\tau_1} v(t-\tau_1) - a_{11}v^2(t) - a_{12}v(t)\big(N_1^{x_2} + \varepsilon\big)$$

with suitable initial condition, then $x_1(t) \geqslant v(t)$ and hence

$$V_1 = \liminf_{t\to+\infty} x_1(t) \geqslant \lim_{t\to+\infty} v(t) = \frac{\alpha_1 e^{-\gamma_1\tau_1} - a_{12}(N_1^{x_2} + \varepsilon)}{a_{11}}.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$V_1 \geqslant \frac{\alpha_1 e^{-\gamma_1\tau_1} - a_{12}N_1^{x_2}}{a_{11}} := M_1^{x_1}.$$

Therefore, for any $\varepsilon > 0$ sufficiently small, there exists $T_{12} > T_{31} + \tau$ such that if $t > T_{12}$, $x_1(t) \geqslant M_1^{x_1} - \varepsilon$.

It follows from the third equation of system (1.1) that for $t > T_{12} + \tau$,

$$\dot{x}_2(t) \geqslant \alpha_2 e^{-\gamma_2\tau_2}\big(M_1^{x_1} - \varepsilon\big)x_2(t-\tau_2) - r_2x_2(t) - a_{22}x_2^2(t) - a_{23}\big(N_1^{x_3} + \varepsilon\big)x_2(t).$$

By comparison, we obtain that

$$V_2 = \liminf_{t\to+\infty} x_2(t) \geqslant \frac{\alpha_2 e^{-\gamma_2\tau_2}(M_1^{x_1} - \varepsilon) - r_2 - a_{23}(N_1^{x_3} + \varepsilon)}{a_{22}}.$$

Since this is true for arbitrary $\varepsilon > 0$ sufficiently small, we have

$$V_2 \geqslant \frac{\alpha_2 e^{-\gamma_2\tau_2} M_1^{x_1} - r_2 - a_{23}N_1^{x_3}}{a_{22}} := M_1^{x_2}.$$

Hence for $\varepsilon > 0$ sufficiently small, there is $T_{22} > T_{12} + \tau$ such that if $t > T_{22}$, $x_2(t) \geqslant M_1^{x_2} - \varepsilon$.

Similarly, it follows from the fifth equation of system (1.1) that for $t > T_{22} + \tau$,

$$\dot{x}_3(t) \geqslant \alpha_3 e^{-\gamma_3\tau_3}\big(M_1^{x_2} - \varepsilon\big) - r_3 x_3(t) - a_{33} x_3^2(t),$$

which yields

$$V_3 = \liminf_{t\to+\infty} x_3(t) \geqslant \frac{\alpha_3 e^{-\gamma_3\tau_3} M_1^{x_2} - r_3}{a_{33}} := M_1^{x_3}.$$

Thus, for $\varepsilon > 0$ sufficiently small, there is $T_{32} > T_{22} + \tau$ such that if $t > T_{32}$, $x_3(t) \geqslant M_1^{x_3} - \varepsilon$.

We derive from the first equation of system (1.1) that for $t > T_{32}$,

$$\dot{x}_1(t) \leqslant \alpha_1 e^{-\gamma_1\tau_1} x_1(t-\tau_1) - a_{11} x_1^2(t) - a_{12}\big(M_1^{x_2} - \varepsilon\big)x_1(t).$$

A standard comparison argument shows that

$$U_1 = \limsup_{t\to+\infty} x_1(t) \leqslant \frac{\alpha_1 e^{-\gamma_1\tau_1} - a_{12}(M_1^{x_2} - \varepsilon)}{a_{11}}.$$

Since $\varepsilon > 0$ is arbitrary and sufficiently small, we derive

$$U_1 \leqslant \frac{\alpha_1 e^{-\gamma_1\tau_1} - a_{12} M_1^{x_2}}{a_{11}} := N_2^{x_1}.$$

Hence, for $\varepsilon > 0$ sufficiently small, there is $T_{13} > T_{32} + \tau$ such that if $t > T_{13}$, $x_1(t) \leqslant N_2^{x_1} + \varepsilon$.

It follows from the third equation of system (1.1) that for $t > T_{13} + \tau$,

$$\dot{x}_2(t) \leqslant \alpha_2 e^{-\gamma_2\tau_2}\big(N_2^{x_1} + \varepsilon\big)x_2(t-\tau_2) - r_2 x_2(t) - a_{22} x_2^2(t) - a_{23}\big(M_1^{x_3} - \varepsilon\big)x_2(t).$$

By comparison, we derive that

$$U_2 = \limsup_{t\to+\infty} x_2(t) \leqslant \frac{\alpha_2 e^{-\gamma_2\tau_2}(N_2^{x_1} + \varepsilon) - r_2 - (M_1^{x_3} - \varepsilon)}{a_{22}}.$$

Since $\varepsilon > 0$ is arbitrary and sufficiently small, we derive that

$$U_2 \leqslant \frac{\alpha_2 e^{-\gamma_2\tau_2} N_2^{x_1} - r_2 - M_1^{x_3}}{a_{22}} := N_2^{x_2}.$$

Therefore, for $\varepsilon > 0$ sufficiently small, there is $T_{23} > T_{13} + \tau$ such that if $t > T_{23}$, $x_2(t) \leqslant N_2^{x_2} + \varepsilon$.

Similarly, we derive from the fifth equation of system (1.1) that for $t > T_{23} + \tau$,

$$\dot{x}_3(t) \leqslant \alpha_3 e^{-\gamma_3\tau_3}\big(N_2^{x_2} + \varepsilon\big)x_3(t-\tau_3) - r_3 x_3(t) - a_{33} x_3^2(t).$$

By comparison, it follows that

$$U_3 = \limsup_{t\to+\infty} x_3(t) \leqslant \frac{\alpha_3 e^{-\gamma_3\tau_3}(N_2^{x_2} + \varepsilon) - r_3}{a_{33}}.$$

Since $\varepsilon > 0$ is arbitrary and sufficiently small, we get

$$U_3 \leqslant \frac{\alpha_3 e^{-\gamma_3\tau_3} N_2^{x_2} - r_3}{a_{33}} := N_2^{x_3}.$$

Continuing this process, we obtain six sequences $M_n^{x_1}$, $N_n^{x_1}$, $M_n^{x_2}$, $N_n^{x_2}$, $M_n^{x_3}$, $N_n^{x_3}$ $(n = 1, 2, \ldots)$ such that for $n \geqslant 2$,

$$N_n^{x_1} = \frac{\alpha_1 e^{-\gamma_1\tau_1} - a_{12} M_{n-1}^{x_2}}{a_{11}},$$
$$N_n^{x_2} = \frac{\alpha_2 e^{-\gamma_2\tau_2} N_n^{x_1} - r_2 - a_{23} M_{n-1}^{x_3}}{a_{22}},$$
$$N_n^{x_3} = \frac{\alpha_3 e^{-\gamma_3\tau_3} N_n^{x_2} - r_3}{a_{33}},$$
$$M_n^{x_1} = \frac{\alpha_1 e^{-\gamma_1\tau_1} - a_{12} N_n^{x_2}}{a_{11}},$$
$$M_n^{x_2} = \frac{\alpha_2 e^{-\gamma_2\tau_2} M_n^{x_1} - r_2 - a_{23} N_n^{x_3}}{a_{22}},$$
$$M_n^{x_3} = \frac{\alpha_3 e^{-\gamma_3\tau_3} M_n^{x_2} - r_3}{a_{33}}. \tag{3.4}$$

Clearly, we have

$$M_n^{x_i} \leqslant V_i \leqslant U_i \leqslant N_n^{x_i}, \quad i = 1, 2, 3. \tag{3.5}$$

It follows from (3.4) that for $n \geqslant 2$,

$$N_{n+1}^{x_3} = \frac{\Delta_3(a_{11}a_{22}a_{33} - a_{11}a_{23}\alpha_3 e^{-\gamma_3\tau_3} - a_{33}a_{12}\alpha_2 e^{-\gamma_2\tau_2})}{(a_{11}a_{22}a_{33})^2} + \frac{(a_{11}a_{23}\alpha_3 e^{-\gamma_3\tau_3} + a_{33}a_{12}\alpha_2 e^{-\gamma_2\tau_2})^2}{(a_{11}a_{22}a_{33})^2} N_n^{x_3}, \tag{3.6}$$

where Δ_3 is defined in (3.3).

We therefore rewrite (3.6) into

$$N_{n+1}^{x_3} = \frac{(a_{11}a_{22}a_{33})^2 - (a_{11}a_{23}\alpha_3 e^{-\gamma_3\tau_3} + a_{33}a_{12}\alpha_2 e^{-\gamma_2\tau_2})^2}{(a_{11}a_{22}a_{33})^2} x_3^* + \frac{(a_{11}a_{23}\alpha_3 e^{-\gamma_3\tau_3} + a_{33}a_{12}\alpha_2 e^{-\gamma_2\tau_2})^2}{(a_{11}a_{22}a_{33})^2} N_n^{x_3}. \tag{3.7}$$

Noting that $N_n^{x_3} \geqslant x_3^*$ and $a_{11}a_{22}a_{33} > a_{11}a_{23}\alpha_3 e^{-\gamma_3\tau_3} + a_{33}a_{12}\alpha_2 e^{-\gamma_2\tau_2}$, we derive from (3.7) that

$$\begin{aligned} N_{n+1}^{x_3} - N_n^{x_3} &= \frac{(a_{11}a_{22}a_{33})^2 - (a_{11}a_{23}\alpha_3 e^{-\gamma_3\tau_3} + a_{33}a_{12}\alpha_2 e^{-\gamma_2\tau_2})^2}{(a_{11}a_{22}a_{33})^2} x_3^* \\ &\quad + \left\{ \frac{(a_{11}a_{23}\alpha_3 e^{-\gamma_3\tau_3} + a_{33}a_{12}\alpha_2 e^{-\gamma_2\tau_2})^2}{(a_{11}a_{22}a_{33})^2} - 1 \right\} N_n^{x_3} \\ &\leqslant \frac{(a_{11}a_{22}a_{33})^2 - (a_{11}a_{23}\alpha_3 e^{-\gamma_3\tau_3} + a_{33}a_{12}\alpha_2 e^{-\gamma_2\tau_2})^2}{(a_{11}a_{22}a_{33})^2} x_3^* \\ &\quad + \left\{ \frac{(a_{11}a_{23}\alpha_3 e^{-\gamma_3\tau_3} + a_{33}a_{12}\alpha_2 e^{-\gamma_2\tau_2})^2}{(a_{11}a_{22}a_{33})^2} - 1 \right\} x_3^*. \end{aligned}$$

Therefore, the sequence $N_n^{x_3}$ is monotonically decreasing. Accordingly, $\lim_{n\to+\infty} N_n^{x_3}$ exists. Taking $n \to +\infty$, it follows from (3.7) that

$$\lim_{n\to+\infty} N_n^{x_3} = x_3^*. \tag{3.8}$$

We further derive from (3.4) and (3.8) that

$$\lim_{n\to+\infty} N_n^{x_2} = x_2^*, \qquad \lim_{n\to+\infty} M_n^{x_1} = x_1^*, \qquad \lim_{n\to+\infty} M_n^{x_2} = x_2^*,$$
$$\lim_{n\to+\infty} M_n^{x_3} = x_3^*, \qquad \lim_{n\to+\infty} N_n^{x_1} = x_1^*. \tag{3.9}$$

It follows from (3.5), (3.8) and (3.9) that

$$U_1 = V_1 = x_1^*, \qquad U_2 = V_2 = x_2^*, \qquad U_3 = V_3 = x_3^*. \tag{3.10}$$

As a consequence, we obtain that

$$\lim_{t\to+\infty} x_i(t) = x_i^* \quad (i = 1, 2, 3).$$

Using L'Hospital's rule, it follows from (2.1) that

$$\lim_{t\to+\infty} y_i(t) = y_i^* \quad (i = 1, 2, 3).$$

This completes the proof. □

Next, we discuss the global stability of the nonnegative equilibria E_1 of system (1.1).

Theorem 3.2. *If $\alpha_1\alpha_2 e^{-\gamma_1\tau_1-\gamma_2\tau_2} < r_2 a_{11}$, the nonnegative equilibrium E_1 of system* (1.1) *is globally asymptotically stable.*

Proof. Noting that the nonnegative equilibrium E_1 is locally stable if $\alpha_1\alpha_2 e^{-\gamma_1\tau_1-\gamma_2\tau_2} < r_2 a_{11}$, it suffices to show that E_1 is globally attractive.

Let $\varepsilon > 0$ be sufficiently small satisfying

$$\alpha_2 e^{-\gamma_2\tau_2}\left(\frac{\alpha_1 e^{-\gamma_1\tau_1}}{a_{11}} + \varepsilon\right) - r_2 < 0, \qquad \alpha_3 e^{-\gamma_3\tau_3}\varepsilon - r_3 < 0. \tag{3.11}$$

We derive from the first equation of system (1.1) that

$$\dot{x}_1(t) \leqslant \alpha_1 e^{-\gamma_1\tau_1} x_1(t-\tau_1) - a_{11} x_1^2(t),$$

by comparison which yields

$$\limsup_{t\to+\infty} x_1(t) \leqslant \frac{\alpha_1 e^{-\gamma_1\tau_1}}{a_{11}}. \tag{3.12}$$

Therefore, for $\varepsilon > 0$ sufficiently small satisfying (3.11) there is $T_1 > 0$ such that if $t > T_1$, $x_1(t) \leqslant \alpha_1 e^{-\gamma_1\tau_1}/a_{11} + \varepsilon$.

It follows from the third equation of system (1.1) that for $t > T_1 + \tau$,

$$\dot{x}_2(t) \leqslant \alpha_2 e^{-\gamma_2\tau_2}\left(\frac{\alpha_1 e^{-\gamma_1\tau_1}}{a_{11}} + \varepsilon\right) x_2(t-\tau_2) - r_2 x_2(t) - a_{22} x_2^2(t).$$

Consider the following auxiliary equation:

$$\dot{u}(t) = \alpha_2 e^{-\gamma_2\tau_2}\left(\frac{\alpha_1 e^{-\gamma_1\tau_1}}{a_{11}} + \varepsilon\right) u(t-\tau_2) - r_2 u(t) - a_{22} u^2(t). \tag{3.13}$$

By Lemma 2.2, we derive from (3.11) that

$$\lim_{t\to+\infty} u(t) = 0.$$

By comparison, it follows that

$$\lim_{t\to+\infty} x_2(t) = 0.$$

Hence, for $\varepsilon > 0$ sufficiently small satisfying (3.11), there exists $T_2 > T_1 + \tau$ such that if $t > T_2$, $0 < x_2(t) < \varepsilon$.

It follows from the fifth equation of system (1.1) that for $t > T_2 + \tau$,

$$\dot{x}_3(t) \leqslant \alpha_3 e^{-\gamma_3\tau_3}\varepsilon x_3(t-\tau_3) - r_3 x_3(t) - a_{33}x_3^2(t),$$

which, together with (3.11), yields

$$\lim_{t\to+\infty} x_3(t) = 0.$$

We derive from the first equation of system (1.1) that for $t > T_2$,

$$\dot{x}_1(t) \geqslant \alpha_1 e^{-\gamma_1\tau_1} x_1(t-\tau_1) - a_{11}x_1^2(t) - a_{12}\varepsilon x_1(t).$$

By comparison, it follows that

$$\liminf_{t\to+\infty} x_1(t) \geqslant \frac{\alpha_1 e^{-\gamma_1\tau_1} - a_{12}\varepsilon}{a_{11}}.$$

Since this is true for arbitrary $\varepsilon > 0$ sufficiently small, we can conclude that

$$\liminf_{t\to+\infty} x_1(t) \geqslant \frac{\alpha_1 e^{-\gamma_1\tau_1}}{a_{11}},$$

which, together with (3.12), leads to

$$\lim_{t\to+\infty} x_1(t) = \frac{\alpha_1 e^{-\gamma_1\tau_1}}{a_{11}}.$$

Using L'Hospital's rule, we obtain from (2.1) that

$$\lim_{t\to+\infty} y_1(t) = \frac{\alpha_1^2 e^{-\gamma_1\tau_1}}{a_{11}\gamma_1}\left(1 - e^{-\gamma_1\tau_1}\right), \qquad \lim_{t\to+\infty} y_2(t) = \lim_{t\to+\infty} y_3(t) = 0.$$

The proof is complete. □

Finally, we show the global attractivity of the nonnegative equilibrium E_2 of system (1.1).

Theorem 3.3. *The nonnegative equilibrium* $E_2(x_1^0, y_1^0, x_2^0, y_2^0, 0, 0)$ *is globally attractive provided that*

(H4) $0 < \alpha_3 e^{-\gamma_3\tau_3}(\alpha_1\alpha_2 e^{-\gamma_1\tau_1-\gamma_2\tau_2} - a_{11}r_2) < a_{11}a_{22}r_3$,
(H5) $a_{11}a_{22} > a_{12}\alpha_2 e^{-\gamma_2\tau_2}$.

Proof. Let $(x_1(t), y_1(t), x_2(t), y_2(t), x_3(t), y_3(t))$ be a solution of system (1.1) with initial conditions (1.2) and (1.3).

Let $\varepsilon > 0$ be sufficiently small satisfying

$$\alpha_3 e^{-\gamma_3\tau_3}\left(\frac{\alpha_1\alpha_2 e^{-\gamma_1\tau_1-\gamma_2\tau_2} - r_2 a_{11}}{a_{11}a_{22}} + \varepsilon\right) - r_3 < 0. \tag{3.14}$$

We derive from the fifth equation of system (1.1) and (2.4) that there is $T_1 > 0$ such that if $t > T_1$,

$$\dot{x}_3(t) \leqslant \alpha_3 e^{-\gamma_3\tau_3}\left(\frac{\alpha_1\alpha_2 e^{-\gamma_1\tau_1-\gamma_2\tau_2} - r_2 a_{11}}{a_{11}a_{22}} + \varepsilon\right)x_3(t-\tau_3) - r_3 x_3(t) - a_{33}x_3^2(t).$$

Consider the following auxiliary equation:

$$\dot{u}(t) = \alpha_3 e^{-\gamma_3\tau_3}\left(\frac{\alpha_1\alpha_2 e^{-\gamma_1\tau_1-\gamma_2\tau_2} - r_2 a_{11}}{a_{11}a_{22}} + \varepsilon\right)u(t-\tau_3) - r_3 u(t) - a_{33}u^2(t). \tag{3.15}$$

By Lemma 2.2, it follows from (3.14) and (3.15) that

$$\lim_{t\to+\infty} u(t) = 0.$$

By comparison, we derive

$$\lim_{t\to+\infty} x_3(t) = 0.$$

Therefore, for $\varepsilon > 0$ sufficiently small there is $T_2 > T_1$ such that if $t > T_2$, $0 < x_3(t) < \varepsilon$.

It therefore follows from the first and the third equation of system (1.1) that for $t > T_2$,

$$\begin{aligned}
&\dot{x}_1(t) = \alpha_1 e^{-\gamma_1\tau_1}x_1(t-\tau_1) - a_{11}x_1^2(t) - a_{12}x_1(t)x_2(t),\\
&\dot{x}_2(t) \geqslant \alpha_2 e^{-\gamma_2\tau_2}x_1(t-\tau_2)x_2(t-\tau_2) - (r_2 + a_{23}\varepsilon)x_2(t) - a_{22}x_2^2(t),
\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}
&\dot{x}_1(t) = \alpha_1 e^{-\gamma_1\tau_1}x_1(t-\tau_1) - a_{11}x_1^2(t) - a_{12}x_1(t)x_2(t),\\
&\dot{x}_2(t) \leqslant \alpha_2 e^{-\gamma_2\tau_2}x_1(t-\tau_2)x_2(t-\tau_2) - r_2 x_2(t) - a_{22}x_2^2(t).
\end{aligned} \tag{3.17}$$

We consider the following auxiliary system:

$$\begin{aligned}
&\dot{u}_1(t) = \alpha_1 e^{-\gamma_1\tau_1}u_1(t-\tau_1) - a_{11}u_1^2(t) - a_{12}u_1(t)u_2(t),\\
&\dot{u}_2(t) = \alpha_2 e^{-\gamma_2\tau_2}u_1(t-\tau_2)u_2(t-\tau_2) - r u_2(t) - a_{22}u_2^2(t).
\end{aligned} \tag{3.18}$$

It is easy to see that if $\alpha_1\alpha_2 e^{-\gamma_1\tau_1-\gamma_2\tau_2} > a_{11}r$, system (3.18) has a unique positive equilibrium $E_1^*(u_1^0, u_2^0)$, where

$$u_1^0 = \frac{a_{22}\alpha_1 e^{-\gamma_1\tau_1} + r a_{12}}{a_{11}a_{22} + a_{12}\alpha_2 e^{-\gamma_2\tau_2}}, \qquad u_2^0 = \frac{\alpha_1\alpha_2 e^{-\gamma_1\tau_1-\gamma_2\tau_2} - r a_{11}}{a_{11}a_{22} + a_{12}\alpha_2 e^{-\gamma_2\tau_2}}.$$

If (H5) holds, using an iterative technique similar to that in the proof of Theorem 3.1, we can derive that

$$\lim_{t\to+\infty} u_1(t) = u_1^0, \qquad \lim_{t\to+\infty} u_2(t) = u_2^0.$$

By comparison, it follows from (3.16) that

$$\begin{aligned}
&\liminf_{t\to+\infty} x_1(t) \geqslant x_1^0 + \frac{a_{23}\varepsilon}{a_{11}a_{22} + a_{12}\alpha_2 e^{-\gamma_2\tau_2}},\\
&\liminf_{t\to+\infty} x_2(t) \geqslant x_2^0 - \frac{a_{23}\varepsilon}{a_{11}a_{22} + a_{12}\alpha_2 e^{-\gamma_2\tau_2}}.
\end{aligned}$$

Since this true for arbitrary $\varepsilon > 0$ sufficiently small, we can conclude that

$$\liminf_{t\to+\infty} x_1(t) \geqslant x_1^0, \qquad \liminf_{t\to+\infty} x_2(t) \geqslant x_2^0. \tag{3.19}$$

Similarly, by comparison we derive from (3.17) that

$$\limsup_{t\to+\infty} x_1(t) \leqslant x_1^0, \qquad \limsup_{t\to+\infty} x_2(t) \leqslant x_2^0,$$

which, together with (3.19), yields

$$\lim_{t\to+\infty} x_1(t) = x_1^0, \qquad \lim_{t\to+\infty} x_2(t) = x_2^0.$$

Using L'Hospital's rule, we can easily show from (2.1) that

$$\limsup_{t\to+\infty} y_1(t) = y_1^0, \qquad \limsup_{t\to+\infty} y_2(t) = y_2^0, \qquad \lim_{t\to+\infty} y_3(t) = 0.$$

This completes the proof. □

4. Discussion

In this paper, motivated by the work of Aiello and Freedman [1], we incorporated stage structures into a three-species Lotka–Volterra type simple food chain model. By using some comparison arguments we first established sufficient conditions for the permanence of system (1.1). By using an iterative technique, we discussed the global attractivity of the feasible equilibria of system (1.1). By Theorem 3.1, we see that if the intra-specific competition rates dominate the capturing rates of the mature predator and the mature top predator and the transformation rates of the immature predator and the immature top predator, the positive equilibrium of system (1.1) is globally attractive. By Theorem 3.2, we see that if the transformation rate of immature prey population to mature prey population and the transformation rate of the immature predator population to mature predator population are low, and the death rate of the mature predator and the intra-specific competition rate of the mature prey are high, the prey population will be persistent, but the predator and the top predator populations will go to extinction. By Theorem 3.3 we see that if the death rate of the mature top predator is high enough satisfying (H4)–(H5), the top predator population will go to extinction, but the prey and the predator populations will be permanent.

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