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Inverse Scattering Theory for Wave Equations in Stratified Media

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takes different constant values on $\{y > 0\}$ and $\{y < 0\}$. The exponentially decreasing perturbation $c(x, y) - c_0(y)$ is uniquely reconstructed from the scattering matrix of an arbitrarily fixed energy.

1. INTRODUCTION

In many wave propagation problems of classical physics, one encounters layered media. The seismic wave propagates inside the earth through the layered structure and in acoustic problems in fluids such as atmosphere or oceans, the media often have a stratified form. Many works have been devoted to the spectral and forward scattering problems for the stratified media (see, e.g., Wilcox [22], Dermenjian and Guillot [2], Weder [21], Kikuchi and Tamura [14], and Shimizu [19]). A complete set of generalized eigenfunctions are constructed, the wave operators are defined and are shown to be asymptotically complete, and the scattering matrices are proved to be unitary. Little is known, however, about the inverse problem, in spite of its great importance. The construction of the perturbation from the scattering matrix is one of the main themes of the study of the wave propagation problem and it certainly has considerable significance in practical applications.

In the present paper we shall pick up the simplest and the most fundamental case of the wave equation in a stratified medium. Namely, we study the equations

$$(\partial/\partial t)^2 u = c_0(y)^2 \Delta_{x,y} u, \qquad (\partial/\partial t)^2 u = c(x, y)^2 \Delta_{x,y} u, \tag{1.1}$$

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in $\mathbf{R}^{n+1}$, $n \geqslant 2$, where $\Delta_{x,y} = \sum_{i=1}^{n} (\partial/\partial x_i)^2 + (\partial/\partial y)^2$. Here and henceforth, $x = (x_1, ..., x_n) \in \mathbf{R}^n$, $y \in \mathbf{R}^1$. Let us assume that

(A-1) $c_0(y) = c_\pm$ for $\pm y > 0$, where $c_\pm$ are constants such that $0 < c_- < c_+$.

(A-2) $\inf c(x, y) > 0$.

(A-3) There exist constants C, $\delta_0 > 0$ such that

$$|c(x, y) - c_0(y)| \leqslant Ce^{-\delta_0(|x| + |y|)}.$$

Note that no regularity is assumed on the perturbation $c(x, y) - c_0(y)$. In particular, it is allowed to be piecewise continuous.

As mentioned above, with this problem one can associate a family of unitary scattering matrices $\{\hat{S}(E); E > 0\}$, E being an energy parameter. The precise definition of $\hat{S}(E)$ is given in Section 2. Our aim is to show the following theorem.

THEOREM 1.1. *Under the assumptions* (A.1)–(A.3), *the perturbation* $c(x, y) - c_0(y)$ *is uniquely reconstructed from the scattering matrix of an arbitrarily fixed energy* $E > 0$.

For example, let us consider the wave propagation in the atmosphere under the following situation: The atmosphere has an interface Γ, above and below which the sound speed takes different constant values. Then Γ is uniquely reconstructed from the scattering matrix of an arbitrarily fixed energy provided Γ coincides with the plane $\{y = 0\}$ outside a bounded set.

Despite their appearance, the equations (1.1) require a rather involved analysis. In order to deal with the above problem in a Hilbert space $L^2(\mathbf{R}^{n+1}; dx\, dy)$, we employ the unitary transformations $U_0 f(x, y) = c_0(y) f(x, y)$, $Uf(x, y) = c(x, y) f(x, y)$, by which $-c_0(y)^2 \Delta_{x,y}$ and $-c(x, y)^2 \Delta_{x,y}$ are transformed into

$$H_0 = -c_0(y) \Delta_{x,y} c_0(y), \tag{1.2}$$

$$H = -c(x, y) \Delta_{x,y} c(x, y). \tag{1.3}$$

Next we note that with the aid of the invariance principle, studying the scattering matrix for the wave equations (1.1) is equivalent to doing so for the Schrödinger equations

$$i\partial u/\partial t = H_0 u, \qquad i\, \partial u/\partial t = Hu. \tag{1.4}$$

(See, e.g., Reed and Simon [17]).

Our basic strategy is to accommodate the Faddeev theory of inverse scattering [4] developed for Schrödinger operators to (1.4). We follow mainly our previous work [11, 12], in which the theory of Faddeev and the Green operator of Eskin and Ralston [3] were reformulated in terms of pseudo-differential calculus. In our case (1.4), the coefficient $c_0(y)$ has a singularity at the interface $\{y=0\}$, which makes it difficult to use the pseudo-differential calculus directly. We shall replace it in this paper by the calculus of commutators, which was developed in the study of N-body Schrödinger operators [6, 9, 10]. The basic observation is very simple. Let

$$A=\frac{1}{2i}(x\cdot\nabla_x+y\partial_y+\nabla_x\cdot x+\partial_y y). \tag{1.5}$$

Then $[A, c_0(y)]=0$; hence

$$i[H_0, A]=2H_0. \tag{1.6}$$

This simple relation enables us to transfer the results proved for N-body Schrödinger operators to our operators H_0 and H. In particular, one can introduce the radiation condition for H_0 in terms of $B=X^{-1/2}AX^{-1/2}$, $X=(1+|x|^2+y^2)^{1/2}$, which in turn is essential in studying the crucial tool of the direction dependent Green operator for H_0.

In Section 2, we represent the scattering matrix using the spectral representation of H_0. In Section 3, we summarize various facts on the resolvent of H_0 and the direction-dependent Green operator, leaving the proof until Sections 6 and 7. In Section 4, we construct the Faddeev scattering amplitude and the perturbation is reconstructed from this in Section 5.

The commutator calculus has now turned out to be one of the most powerful tools in the study of spectral and scattering theory in mathematical physics. It is therefore very plausible that the commutator method also has a wide range of applicability in the inverse scattering problem. We shall return to this subject elsewhere.

Let us also remark that by the terminology inverse scattering, we restrict ourselves in this paper to the reconstruction of the perturbations from the scattering matrices. There are of course many different formulations of inverse scattering, some of which can be seen in [1, 7, 18, 20].

We finally mention some notations used in this paper. $\mathbf{C}_\pm=\{z\in\mathbf{C};\ \pm\operatorname{Im} z>0\}$. For $\zeta=(\zeta_1, ..., \zeta_n)\in\mathbf{C}^n$, $\zeta^2=\sum_{j=1}^n\zeta_j^2$. For $x\in\mathbf{R}^n$, let $\langle x\rangle=(1+|x|^2)^{1/2}$. For two Banach spaces X and Y, $\mathbf{B}(X; Y)$ is the set of all bounded operators from X to Y, $\mathbf{B}(X)=\mathbf{B}(X; X)$. Cs denote various constants. $F(\cdots)$ denote the characteristic function of the set $\{\cdots\}$. For example, $F(t>0)$ is the Heaviside function. For a linear operator A, $\sigma_p(A)$ denotes the set of the eigenvalues of A.

2. PRELIMINARIES

2.1. *Spectral Properties of H*

Let H be as in (1.3). By the assumptions (A-1)–(A-3), it is self-adjoint with the domain

$$D(H) = \{u;\, c(x, y)\, u \in H^2(\mathbf{R}^{n+1})\}, \tag{2.1}$$

where $H^m(\mathbf{R}^{n+1})$ denotes the Sobolev space of order m. For $s \in \mathbf{R}$, the weighted Hilbert space $L^{2,s}$ is defined by

$$u \in L^{2,s} \Leftrightarrow \|u\|_s = \|(1 + |x| + |y|)^s\, u(x, y)\|_{L^2(\mathbf{R}^{n+1})} < \infty. \tag{2.2}$$

The following theorem was proved by Weder [21, p. 40, p. 59]. Let $R(z) = (H - z)^{-1}$.

THEOREM 2.1. *H has no eigenvalues. For any $E > 0$ and $s > 1/2$, the norm limit*

$$R(E \pm i0) = \lim_{\varepsilon \downarrow 0} R(E \pm i\varepsilon)$$

exists in $\mathbf{B}(L^{2,s}; L^{2,-s})$.

2.2. *Spectral Representation for H_0*

We recall the spectral representation for H_0 defined by (1.2). For the details see [22] or [2]. The generalized eigenfunctions for H_0 are given by

$$\varphi_j(x, y, E, \xi) = (2\pi)^{-n/2}\, e^{ix \cdot \xi} c_0(y)^{-1}\, a_j(E, \xi)\, \psi_j(y, E, \xi), \quad 1 \leqslant j \leqslant 3, \tag{2.3}$$

where $E > 0$, $\xi \in \mathbf{R}^n$, and ψ_j is a solution to the equation

$$\left(-\left(\frac{d}{dy}\right)^2 - Ec_0(y)^{-2} + \xi^2\right)\psi_j(y, E, \xi) = 0.$$

Explicitly, they have the following expressions. Let

$$\theta_\pm = \theta_\pm(E, \xi) = E/c_\pm^2 - \xi^2. \tag{2.4}$$

If $\xi^2 < E/c_+^2$ or $E/c_-^2 < \xi^2$,

$$\varphi_1(x, y, E, \xi) = 0.$$

If $E/c_+^2 < \xi^2 < E/c_-^2$,

$$a_1(E,\xi) = \theta_-^{1/4}(\pi E(c_+^2 - c_-^2))^{-1/2}\, c_+ c_-,$$

$$\psi_1(y,E,\xi) = \begin{cases} e^{-(-\theta_+)^{1/2}\, y}, & y>0, \\ \frac{1}{2}(1+i(-\theta_+/\theta_-)^{1/2})\, e^{i\theta_-^{1/2}y} + \frac{1}{2}(1-i(-\theta_+/\theta_-)^{1/2})\, e^{-i\theta_-^{1/2}y}, & y<0. \end{cases}$$

If $E/c_+^2 < \xi^2$,

$$\varphi_j(x,y,E,\xi) = 0, \qquad j=2,3.$$

If $\xi^2 < E/c_+^2$,

$$a_2(E,\xi) = \pi^{-1/2}\theta_+^{1/4}(\theta_+^{1/2} + \theta_-^{1/2})^{-1},$$

$$a_3(E,\xi) = \pi^{-1/2}\theta_-^{1/4}(\theta_+^{1/2} + \theta_-^{1/2})^{-1},$$

$$\psi_2(y,E,\xi) = \begin{cases} e^{-i\theta_-^{1/2}y}, & y<0, \\ \frac{1}{2}(1-(\theta_-/\theta_+)^{1/2})\, e^{i\theta_+^{1/2}y} + \frac{1}{2}(1+(\theta_-/\theta_+)^{1/2})\, e^{-i\theta_+^{1/2}y}, & y>0, \end{cases}$$

$$\psi_3(y,E,\xi) = \begin{cases} e^{i\theta_+^{1/2}y}, & y>0, \\ \frac{1}{2}(1+(\theta_+/\theta_-)^{1/2})\, e^{i\theta_-^{1/2}y} + \frac{1}{2}(1-(\theta_+/\theta_-)^{1/2})\, e^{-i\theta_-^{1/2}y}, & y<0. \end{cases}$$

Using these generalized eigenfunctions, we can obtain the following spectral representation for H_0. Let

$$\Phi_0 = (\varphi_1, \varphi_2, \varphi_3), \qquad \Phi_0^* = {}^t(\overline{\varphi_1}, \overline{\varphi_2}, \overline{\varphi_3}).$$

Define the auxiliary Hilbert space by

$$\begin{aligned} &\mathscr{H}(E) = {}^t(L^2(\Omega_1(E)), L^2(\Omega_2(E)), L^2(\Omega_3(E))), \\ &\Omega_1(E) = \{\xi;\, E/c_+^2 < \xi^2 < E/c_-^2\}, \\ &\Omega_2(E) = \Omega_3(E) = \{\xi;\, \xi^2 < E/c_+^2\}. \end{aligned} \tag{2.5}$$

We define the operator $\mathscr{F}_0(E)$ by

$$(\mathscr{F}_0(E)\, f)(\xi) = \int_{\mathbf{R}^{n+1}} \Phi_0^*(x,y,E,\xi)\, f(x,y)\, dx\, dy. \tag{2.6}$$

Then for any $E>0$ and $s>1/2$,

$$\mathscr{F}_0(E)=\mathbf{B}(L^{2,s};\mathscr{H}(E)).$$

$\mathscr{F}_0(E)^*\in\mathbf{B}(\mathscr{H}(E);L^{2,-s})$ is an eigenoperator of H_0 in the sense that

$$(H_0-E)\,\mathscr{F}_0(E)^*=0.$$

Moreover it satisfies

$$\frac{1}{2\pi i}(R_0(E+i0)-R_0(E-i0))=\mathscr{F}_0(E)^*\,\mathscr{F}_0(E). \tag{2.7}$$

Here $R_0(z)=(H_0-z)^{-1}$. We also introduce

$$\mathscr{H}={}^t(L^2(\Omega_1),L^2(\Omega_2),L^2(\Omega_3)),$$
$$\Omega_1=\{(E,\xi);E/c_+^2<\xi^2<E/c_-^2\},$$
$$\Omega_2=\Omega_3=\{(E,\xi);\xi^2<E/c_+^2\}.$$

It has a direct integral representation

$$\mathscr{H}=\int_{(0,\infty)}^{\oplus}\mathscr{H}(E)\,dE,$$

and for $g\in\mathscr{H}$,

$$\iint|g(E,\xi)|^2\,d\xi\,dE=\int_0^\infty\|g(E,\cdot)\|^2_{\mathscr{H}(E)}\,dE.$$

We define $(\mathscr{F}_0 f)(E,\xi)=(\mathscr{F}_0(E)f)(\xi)$ for $f\in L^{2,s}$, $s>1/2$. Then $\mathscr{F}_0$ is uniquely extended to a unitary operator from $L^2(\mathbf{R}^{n+1})$ to $\mathscr{H}$. It diagonalizes H_0:

$$(\mathscr{F}_0 H_0 f)(E,\xi)=E(\mathscr{F}_0 f)(E,\xi).$$

2.3. *Scattering Matrix*

The wave operators are define by

$$W_\pm=s\text{-}\lim_{t\to\pm\infty}e^{itH}e^{-itH_0}.$$

Let $S=W_+^*W_-$ be the scattering operator. We define

$$\hat{S}=\mathscr{F}_0 S\mathscr{F}_0^*.$$

Then it has the direct integral representation

$$\hat{S} = \int_{(0,\infty)}^{\oplus} \hat{S}(E)\, dE.$$

More precisely, for any $E > 0$ there exists a unitary operator $\hat{S}(E)$ on $\mathscr{H}(E)$ such that

$$(\hat{S}f)(E, \xi) = (\hat{S}(E)\, f(E, \cdot))(\xi), \qquad \forall f \in \mathscr{H}.$$

The unitary operator $\hat{S}(E)$ is called the scattering matrix and has the following representation.

Lemma 2.2.

$$\hat{S}(E) - I = -2\pi i E \mathscr{F}_0(E)(Q - E\tilde{Q}R(E + i0)\tilde{Q})\, \mathscr{F}_0(E)^*,$$

$$Q = 1 - \left(\frac{c_0}{c}\right)^2, \qquad \tilde{Q} = \frac{c}{c_0} - \frac{c_0}{c}.$$

Proof. Since $c_0/c - 1$ is H_0-compact, we have

$$W_{\pm} = s\text{-}\lim_{t \to \pm\infty} e^{itH} \frac{c_0}{c} e^{-itH_0}$$

$$= \frac{c_0}{c} + i \int_0^{\pm\infty} e^{itH} \tilde{Q} H_0 e^{-itH_0}\, dt.$$

Using this formula, one can argue in the same way as in [8, Lemma 3.1] to get

$$\hat{S}(E) - I = -2\pi i \mathscr{F}_0(E) \frac{c_0}{c} \tilde{Q} \mathscr{F}_0(E)^*$$

$$+ 2\pi i \mathscr{F}_0(E)\, H_0 \tilde{Q} R(E + i0)\, \tilde{Q} H_0 \mathscr{F}_0(E)^*.$$

Since $H_0 \mathscr{F}_0(E)^* = E\mathscr{F}_0(E)^*$, we obtain the lemma. ∎

3. GREEN OPERATORS

In this section, we summarize fundamental properties of the resolvent of H_0 and introduce the direction dependent Green operator. Most of the proofs are given in Sections 6 and 7.

3.1. *Resolvent of* H_0

We first introduce some notations. Let $p_x = -i\nabla_x$, $p_y = -i\partial/\partial y$, and $X = (1 + |x|^2 + |y|^2)^{1/2}$. Let B be the self-adjoint operator defined by

$$B = \tfrac{1}{2} X^{-1/2}(x \cdot p_x + p_x \cdot x + y \cdot p_y + p_y \cdot y) X^{-1/2}. \tag{3.1}$$

For $m \in \mathbf{R}$, $\mathscr{F}^m$ denotes the set of C^∞-functions $f(t)$ on $\mathbf{R}$ satisfying

$$|f^{(k)}(t)| \leqslant C_k(1 + |t|)^{m-k}, \qquad \forall k \geqslant 0.$$

For $a, m \in \mathbf{R}$, $\mathscr{F}^m_\pm(a)$ are the subsets of $\mathscr{F}^m$ such that

$$\mathscr{F}^m_+(a) = \{ f \in \mathscr{F}^m;\ \text{supp}\ f \subset (a, \infty) \},$$

$$\mathscr{F}^m_-(a) = \{ f \in \mathscr{F}^m;\ \text{supp}\ f \subset (-\infty, a) \}.$$

Let $R_0(z) = (H_0 - z)^{-1}$. The following theorem will be proved in Section 6.

THEOREM 3.1. (1) *For any* $E > 0$ *and* $s > 1/2$,

$$R_0(E \pm i0) \in \mathbf{B}(L^{2,s}; L^{2,-s}).$$

(2) *Let* $C_0(E) = \sqrt{E}/c_+$. *Then for any* $m > -1/2$, $t > 1$ *and* $F_\mp \in \mathscr{F}^0_\mp(\pm C_0(E))$,

$$F_\mp(B)\, R_0(E \pm i0) \in \mathbf{B}(L^{2,m+t}; L^{2,m}).$$

(3) *Let* $0 < \alpha < 1/2 < s < 1$ *and* $E > 0$. *Let* $u \in L^{2,-s}$ *satisfy* $c_0(y)u \in H^2_{loc}(\mathbf{R}^{n+1})$ *and* $H_0 u = Eu$. *Suppose there exists an* $\varepsilon > 0$ *such that* $F(B)\, u \in L^{2,-\alpha}$ *either for any* $F \in \mathscr{F}^0_-(\varepsilon)$ *or for any* $F \in \mathscr{F}^0_+(-\varepsilon)$. *Then* $u = 0$.

By the above theorem, one can see that when $u \in L^{2,-s}$ $(s > 1/2)$ satisfies

$$(H_0 - E)\, u = f \in L^{2,s},$$

u is written as $u = R_0(E + i0)\, f$ if and only if there exist $\varepsilon > 0$ and $0 < \alpha < 1/2$ such that

$$(C)_{\text{out}} \qquad F_-(B)\, u \in L^{2,-\alpha}, \qquad \forall F_- \in \mathscr{F}^0_-(\varepsilon),$$

and u is written as $u = R_0(E - i0) f$ if and only if

$$(C)_{\text{in}} \qquad F_+(B)\, u \in L^{2,-\alpha}, \qquad \forall F_+ \in \mathscr{F}^0_+(-\varepsilon),$$

for some $\varepsilon > 0$ and $0 < \alpha < 1/2$. Therefore, $(C)_{\text{out}}$ should be called the *outgoing radiation condition* and $(C)_{\text{in}}$ the *incoming radiation condition*. We shall also use the terminology outgoing (incoming) solution, the meaning of which is evident.

3.2. *Direction Dependent Green Operator*

For $\zeta \in \mathbf{C}^n$, let

$$H_0(\zeta) = c_0(y)((p_x + \zeta)^2 + p_y^2)\, c_0(y), \tag{3.2}$$

$$L_0(\zeta) = (p_x + \zeta)^2 + p_y^2 - V(y), \tag{3.3}$$

$$V(y) = Ec_0(y)^{-2}. \tag{3.4}$$

We then have formally

$$(H_0(\zeta) - E)^{-1} = c_0^{-1} L_0(\zeta)^{-1}\, c_0^{-1}.$$

We fix an arbitrary direction $\gamma \in S^{n-1}$ and construct $L_0(z\gamma)^{-1}$ for $z \in \mathbf{C}_+$. For $\varepsilon > 0$ let

$$D_\varepsilon = \{z \in \mathbf{C}_+;\ |\text{Re}\, z| < \varepsilon/2\}. \tag{3.5}$$

Let $\varphi_1(t) \in C^\infty(\mathbf{R})$ be such that $\varphi_1(t) = 1$ for $|t| > 2\varepsilon$, $\varphi_1(t) = 0$ for $|t| < \varepsilon$. Let

$$V_{\gamma,0}(E, z) = (\mathscr{F}_{x \to \xi})^{-1}\, ((\xi + z\gamma)^2 + p_y^2 - V(y))^{-1}\, \varphi_1(\gamma \cdot \xi)\, \mathscr{F}_{x \to \xi}, \tag{3.6}$$

where $\mathscr{F}_{x \to \xi}$ denotes the Fourier transformation

$$\mathscr{F}_{x \to \xi} f = (2\pi)^{-n/2} \int_{\mathbf{R}^n} e^{-ix \cdot \xi} f(x)\, dx.$$

Since $|\text{Im}(\xi + z\gamma)^2| = |2i\ \text{Im}\ z(\gamma \cdot \xi + \text{Re}\ z)| \geqslant \varepsilon\ \text{Im}\ z$, $V_{\gamma,0}(E, z)$ is a $\mathbf{B}(L^2(\mathbf{R}^{n+1}))$-valued analytic function on D_ε.

For $\varepsilon_1 > 0$, let

$$\Omega_{\varepsilon_1}^{(\pm)} = \mathbf{C}_\pm \cup \{z;\ |\text{Im}(z + E/c_+^2)|^2 \leqslant \varepsilon_1\, \text{Re}(z + E/c_+^2)\}. \tag{3.7}$$

For $s \in \mathbf{R}$, let

$$\mathscr{H}_s = \{u;\ e^{s(|x| + |y|)} u \in L^2(\mathbf{R}^{n+1})\}.$$

Without loss of generality, we take $\gamma = (1, 0, ..., 0)$ and let $x = (x_1, x')$. We define $\mathscr{H}'_s$ similarly to above on $\mathbf{R}_{x'}^{n-1} \times \mathbf{R}_y^1$. Then for any $\delta > 0$, there exists $\varepsilon_1 > 0$ such that $(p_{x'}^2 + p_y^2 - V(y) - z)^{-1}$ defined for $z \in \mathbf{C}_\pm$ has an analytic continuation on $\Omega_{\varepsilon_1}^{(\pm)}$ as a $\mathbf{B}(\mathscr{H}'_\delta; \mathscr{H}'_{-\delta})$-valued function (see Theorem 7.5), which we denote by $R'_\pm(z)$.

Let $\varphi_0(t) = 1 - \varphi_1(t)$ and

$$W_{\gamma,0}(E, z) = (\mathscr{F}_{x_1 \to \xi_1})^{-1} \{R'_+(-(\xi_1 + z)^2) F(\xi_1 < 0) \varphi_0(\xi_1) + R'_-(-(\xi_1 + z)^2) F(\xi_1 > 0) \varphi_0(\xi_1)\} \mathscr{F}_{x_1 \to \xi_1}, \tag{3.8}$$

where $\mathscr{F}_{x_1 \to \xi_1}$ denotes the partial Fourier transformation with respect to x_1. Here we recall that $F(\cdots)$ denotes the characteristic function of the set $\{\cdots\}$. For small $\varepsilon > 0$, $-(\xi_1 + z)^2 \in \Omega_{\varepsilon_1}^{(\pm)}$ if $\xi_1 \in \operatorname{supp} \varphi_0$, $z \in D_\varepsilon$. Therefore $W_{\gamma,0}(E, z)$ is analytic on D_ε as a $\mathbf{B}(\mathscr{H}_\delta; \mathscr{H}_{-\delta})$-valued function.

We finally define

$$U_{\gamma,0}(E, z) = V_{\gamma,0}(E, z) + W_{\gamma,0}(E, z). \tag{3.9}$$

We need to prepare some more notations. For $t \in \mathbf{R}$, let

$$M_\gamma^{(\pm)}(t) = (\mathscr{F}_{x \to \xi})^{-1} F(\pm \gamma \cdot (\xi - t\gamma) \geqslant 0) \mathscr{F}_{x \to \xi}. \tag{3.10}$$

We also let $x' = x - (\gamma \cdot x)\gamma$, $X' = (1 + |x'|^2 + y^2)^{1/2}$ and

$$B' = \tfrac{1}{2} X'^{-1/2} (x' \cdot p_{x'} + p_{x'} \cdot x' + y \cdot p_y + p_y \cdot y) X'^{-1/2}. \tag{3.11}$$

For $s \in \mathbf{R}$, $L_\perp^{2,s}$ is defined by

$$u \in L_\perp^{2,s} \Leftrightarrow \|u\|_{s,\perp} = \|(1 + |x'| + |y|)^s u(x, y)\|_{L^2(\mathbf{R}^{n+1})} < \infty.$$

We prove the following theorem in Section 7.

THEOREM 3.2. (1) *For any $\delta > 0$, there exists $\varepsilon > 0$ such that as a $\mathbf{B}(\mathscr{H}_\delta; \mathscr{H}_{-\delta})$-valued function, $U_{\gamma,0}(E, z)$ is analytic on D_ε and for $s > 1/2$, there exists a constant $C > 0$ such that*

$$\|U_{\gamma,0}(E, i\tau)\|_{\mathbf{B}(L^{2,s}; L^{2,-s})} \leqslant C/\tau \qquad \text{if} \quad \tau > 1.$$

(2) *Let $I_\varepsilon = (-\varepsilon/2, \varepsilon/2)$. As $z \to t \in I_\varepsilon$, $U_{\gamma,0}(E, z)$ converges in $\mathbf{B}(\mathscr{H}_\delta; \mathscr{H}_{-\delta})$. Hence so does $V_{\gamma,0}(E, z)$. Let $V_{\gamma,0}(E, t) = \lim_{z \to t} V_{\gamma,0}(E, z)$. Actually $V_{\gamma,0}(E, z)$ converges to $V_{\gamma,0}(E, t)$ in $\mathbf{B}(L^{2,s}; L^{2,-s})$ and*

$$V_{\gamma,0}(E, t) \in \mathbf{B}(L^{2,s}; L^{2,-s}), \qquad s > 1/2, \quad t \in I_\varepsilon.$$

(3) *For $t \in I_\varepsilon$, $W_{\gamma,0}(E, t) \in \mathbf{B}(L_\perp^{2,s}; L_\perp^{2,-s})$, $s > 1/2$.*

(4) *Let*

$$G_1(E, t) = c_0^{-1} e^{itx \cdot \gamma} V_{\gamma, 0}(E, t)\, e^{-itx \cdot \gamma} c_0^{-1}, \tag{3.12}$$

$$G_0(E, t) = c_0^{-1} e^{itx \cdot \gamma} W_{\gamma, 0}(E, t)\, e^{-itx \cdot \gamma} c_0^{-1}, \tag{3.13}$$

$$R_{\gamma, 0}(E, t) = G_1(E, t) + G_0(E, t). \tag{3.14}$$

Then we have

$$(H_0 E)\, G_1(E, t) = \varphi_1(\gamma \cdot p_x - t),$$

$$(H_0 - E)\, G_0(E, t) = \varphi_0(\gamma \cdot p_x - t),$$

$$(H_0 - E)\, R_{\gamma, 0}(E, t) = I.$$

(5) *For any* $m > -1/2$ *and* $s > 1$, *we have*

$$F_\pm(B)\, M_\gamma^{(\pm)}(t)\, G_1(E, t) \in \mathbf{B}(L^{2, m+s}; L^{2, m}),$$

$$\forall F_\pm \in \mathscr{F}_\pm^0(\mp C_0(E)), \tag{3.15}$$

$$F_\pm(B')\, M_\gamma^{(\pm)}(t)\, G_0(E, t) \in \mathbf{B}(L_\perp^{2, m+s}; L_\perp^{2, m}),$$

$$\forall F_\pm \in \mathscr{F}_\pm^0(\mp C_\varepsilon(E)), \tag{3.16}$$

where $C_0(E) = \sqrt{E}/c_+$, $C_\varepsilon(E) = (E/c_+^2 - \varepsilon^2/4)^{1/2}$.

3.3. *Perturbed Green Operator*

We construct an operator satisfying

$$((p_x + z\gamma)^2 + p_y^2 - Ec(x, y)^{-2})\, U_\gamma(E, z) = I.$$

It should satisfy the equation

$$(1 - EU_{\gamma, 0}(E, z)(c^{-2} - c_0^{-2}))\, U_\gamma(E, z) = U_{\gamma, 0}(E, z).$$

In view of (A-3), we choose $\delta = \delta_0/2$, accordingly taking ε small enough so that $U_{\gamma, 0}(E, z)(c^{-2} - c_0^{-2})$ is a bounded operator on $\mathscr{H}_{-\delta}$.

DEFINITION 3.3 (Exceptional Points). $\mathscr{E}_\gamma(E)$ is the set of $z \in \overline{D_\varepsilon}$ such that $1 \in \sigma_p(EU_{\gamma, 0}(E, z)(c^{-2} - c_0^{-2}))$.

LEMMA 3.4. $\mathscr{E}_\gamma(E) \cap \mathbf{C}_+$ *is discrete and* $\mathscr{E}_\gamma(E) \cap \mathbf{R}$ *is a closed set of measure* 0.

Proof. As an operator on $\mathscr{H}_{-\delta}$, $U_{\gamma,0}(E,z)(c^{-2}-c_0^{-2})$ is compact and analytic. Moreover by Theorem 3.2 (1) for $\tau>1$

$$\|U_{\gamma,0}(E,i\tau)(c^{-2}-c_0^{-2})\|_{\mathbf{B}(\mathscr{H}_{-\delta})}\leqslant C/\tau.$$

The lemma then follows from the analytic Fredholm theorem. ∎

We define for $z\in\bar{D}_\varepsilon\setminus\mathscr{E}_\gamma(E)$,

$$U_\gamma(E,z)=(1-EU_{\gamma,0}(E,z)(c^{-2}-c_0^{-2}))^{-1}\,U_{\gamma,0}(E,z). \tag{3.17}$$

The following theorem is a direct consequence of Theorem 3.2 and (3.17).

THEOREM 3.5. (1) *As a* $\mathbf{B}(\mathscr{H}_\delta;\mathscr{H}_{-\delta})$*-valued function,* $U_\gamma(E,z)$ *is analytic on* $D_\varepsilon\setminus\mathscr{E}_\gamma(E)$*, continuous on* $\bar{D}_\varepsilon\setminus\mathscr{E}_\gamma(E)$*.*

(2) *There exists a constant* $C>0$ *such that*

$$\|U_\gamma(E,i\tau)\|_{\mathbf{B}(\mathscr{H}_\delta;\,\mathscr{H}_{-\delta})}\leqslant C/\tau$$

for $\tau>1$.

(3) *For* $t\in I_\varepsilon\setminus\mathscr{E}_\gamma(E)$*, let*

$$R_\gamma(E,t)=c^{-1}e^{itx\cdot\gamma}U_\gamma(E,t)\,e^{-itx\cdot\gamma}c^{-1}. \tag{3.18}$$

Then it satisfies

$$(H-E)\,R_\gamma(E,t)=I.$$

(4) (*Resolvent Equations*) *Let* $W=c_0/c$*. Then*

$$R_\gamma(E,t)=WR_{\gamma,0}(E,t)\,W+EWR_{\gamma,0}(E,t)(W-W^{-1})\,R_\gamma(E,t).$$

4. FADDEEV THEORY

$R_{\gamma,0}(E,t)$ introduced in (3.14) has the half-outgoing and half-incoming property, which is seen in the following formal formula:

$$R_{\gamma,0}(E,t)=R_0(E-i0)\,M_\gamma^{(+)}(t)+R_0(E+i0)\,M_\gamma^{(-)}(t). \tag{4.1}$$

Let us give the precise meaning to (4.1). We take $\chi(t)\in C^\infty(\mathbf{R})$ such that $\chi(t)=1$ for $|t|>2$, $\chi(t)=0$ for $|t|<1$, and let for small $\delta>0$

$$K_\delta=(\mathscr{F}_{x\to\xi})^{-1}\,\chi(\gamma\cdot(\xi-t\gamma)/\delta)\,\mathscr{F}_{x\to\xi}. \tag{4.2}$$

THEOREM 4.1. *For* $f \in L^{2,s}$, $s > 1/2$, *we have in* $L^{2,-s}$

$$R_{\gamma,0}(E,t)f = \lim_{\delta\to 0} (R_0(E-i0)\,M_\gamma^{(+)}(t) + R_0(E+i0)\,M_\gamma^{(-)}(t))K_\delta f.$$

Proof. We take $\gamma = (1, 0, ..., 0)$ and let $x = (x_1, x')$. Letting $\widetilde{K}_\delta = 1 - K_\delta$, we have for small $\delta > 0$

$$R_{\gamma,0}(E,t)\widetilde{K}_\delta = G_0(E,t)\widetilde{K}_\delta.$$

Therefore by Theorem 3.2 (3) for $f \in L^{2,s}$

$$\|X'^{-s}R_{\gamma,0}(E,t)\widetilde{K}_\delta f\| \leqslant C\,\|X'^{s}\widetilde{K}_\delta f\|,$$

which shows that as $\delta \to 0$

$$R_{\gamma,0}(E,t)\,\widetilde{K}_\delta f \to 0 \qquad \text{in } L^{2,-s}.$$

By the same reasoning we have

$$(R_0(E-i0)\,M_\gamma^{(+)}(t) + R_0(E+i0)\,M_\gamma^{(-)}(t))\,\widetilde{K}_\delta f \to 0 \qquad \text{in } L^{2,-s}.$$

Therefore we have only to show that for small $\delta > 0$

$$R_{\gamma,0}(E,t)K_\delta = (R_0(E-i0)\,M_\gamma^{(+)}(t) + R_0(E+i0)\,M_\gamma^{(-)}(t))K_\delta. \quad (4.3)$$

This is proved if we show that

$$M_\gamma^{(\pm)}(t)\,R_{\gamma,0}(E,t)K_\delta = M_\gamma^{(\pm)}(t)\,R_0(E \mp i0)K_\delta.$$

Let $u = M_\gamma^{(-)}(t)\,G_1(E,t)K_\delta f$. Then by Theorem 3.2 (4) and (5), u is the outgoing solution of

$$(H_0 - E)u = M_\gamma^{(-)}(t)\,\varphi_1(p_{x_1} - t)K_\delta f.$$

Therefore by Theorem 3.1

$$M_\gamma^{(-)}(t)\,G_1(E,t)K_\delta = R_0(E+i0)\,M_\gamma^{(-)}(t)\,\varphi_1(p_{x_1} - t)K_\delta. \qquad (4.4)$$

We next let $v = M_\gamma^{(-)}(t)\,c_0 G_0(E,t)\,c_0 K_\delta g$ with $c_0 g = f$, and let $w = \mathscr{F}_{x_1\to\xi_1} v$. Then by virtue of Theorem 3.2 (5), w is the outgoing solution of the equation

$$(p_y^2 - V(y) + p_{x'}^2 + \xi_1^2)\,w = \mathscr{F}_{x_1\to\xi_1} M_\gamma^{(-)}(t)\,\varphi_0(p_{x_1} - t)K_\delta g.$$

Therefore in view of Theorems 7.1–7.3 in Section 7, we have

$$w=(p_y^2-V(y)+p_{x'}^2+\xi_1^2-i0)^{-1}\,\mathscr{F}_{x_1\to\xi_1}M_\gamma^{(-)}(t)\,\varphi_0(p_{x_1}-t)\,K_\delta g.$$

Multiplying by $(\mathscr{F}_{x_1\to\xi_1})^{-1}$, we see that v is the outgoing solution of the equation

$$L_0(0)\,v=M_\gamma^{(-)}(t)\,\varphi_0(p_{x_1}-t)\,K_\delta g.$$

This implies that $M_\gamma^{(-)}(t)\,G_0(E,t)\,K_\delta f$ is the outgoing solution of the equation

$$(H_0-E)\,u=M_\gamma^{(-)}(t)\,\varphi_0(p_{x_1}-t)\,K_\delta f.$$

Therefore we have

$$M^{\mathrm{AAAF}-)}_\gamma(t)\,G_0(Et)\,K_\delta=R_0(E+i0)\,M_\gamma^{(-)}(t)\,\varphi_0(p_{AAHx_1}-t)\,K_\delta \tag{4.5}$$

The formulas (3.14), (4.4), and (4.5) imply tAAt

$$M_\gamma^{(-)}(t)\,R_{\gamma,0}(E,t)\,K_\delta=R_0(E+i0)\,M_\gamma^{(-)}(t)\,K_\delta.$$

In theAAame way one can show that

$$M_\gamma^{(+)}(t)\,R_{\gamma,0}(E,t)\,K_\delta=R_0(E-i0)\,M_\gamma^{(+)}(t)\,K_\delta. \quad \blacksquare$$

Let

$$T_\gamma(E)=2\pi i\mathscr{F}_0(E)^*\,F(\gamma\cdot(\xi-t\gamma)\geqslant 0)\,\mathscr{F}_0(E). \tag{4.6}$$

The following formula follows directly from Theorem 4.1 and (2.7).

LEMMA 4.2.

$$R_{\gamma,0}(E,t)=R_0(E+i0)-T_\gamma(E).$$

LEMMA 4.3. *Let $W=c_0(y)/c(x,y)$. Then we have*

$$R_\gamma=R-(W+ER(W-W^{-1}))\,T_\gamma(W+E(W-W^{-1})\,R_\gamma), \tag{4.7}$$

where we have used the abbreviation $R_\gamma=R_\gamma(E,t)$, $R=R(E+i0)$, $T_\gamma=T_\gamma(E)$.

Proof. Using the resolvent equations,

$$R = WR_0 W + ER(W - W^{-1}) R_0 W,$$

$$R_\gamma = WR_{\gamma,0} W + EWR_{\gamma,0}(W - W^{-1}) R_\gamma,$$

we have

$$\begin{aligned} &-(W + ER(W - W^{-1}))\, T_\gamma (W + E(W - W^{-1}) R_\gamma) \\ &\quad = (W + ER(W - W^{-1}))\, R_{\gamma,0}(W + E(W - W^{-1}) R_\gamma) \\ &\qquad - (W + ER(W - W^{-1}))\, R_0(W + E(W - W^{-1}) R_\gamma) \\ &\quad = (1 + E(1 - W^{-2}))\, R_\gamma - R(1 + E(1 - W^{-2}) R_\gamma) \\ &\quad = R_\gamma - R, \end{aligned}$$

which proves the lemma. ∎

Let us define the operators in $\mathbf{B}(L^{2,s}; \mathscr{H}(E))$, $s > 1/2$, by

$$\mathscr{F}(E) = \mathscr{F}_0(E)(W + E(W - W^{-1})\, R(E + i0)^*), \tag{4.8}$$

$$\mathscr{F}_\gamma(E, t) = \mathscr{F}_0(E)(W + E(W - W^{-1})\, R_\gamma(E, t)^*). \tag{4.9}$$

They are eigenoperators of H in the sense that

$$(H - E)\, \mathscr{F}(E)^* = 0, \qquad (H - E)\, \mathscr{F}_\gamma(E, t)^* = 0.$$

We define the physical scattering amplitude $A(E)$ and the Faddeev scattering amplitude $A_\gamma(E, t)$ by

$$A(E) = \mathscr{F}_0(E)(W^{-1} - W)\, \mathscr{F}(E)^*, \tag{4.10}$$

$$A_\gamma(E, t) = \mathscr{F}_0(E)(W^{-1} - W)\, \mathscr{F}_\gamma(E, t)^*. \tag{4.11}$$

By Lemma 2.2, we have

$$\hat{S}(E) = 1 - 2\pi i E A(E).$$

Let

$$F_\gamma = F(\gamma \cdot (\xi - t\gamma) \geqslant 0). \tag{4.12}$$

THEOREM 4.4.

$$\mathscr{F}_\gamma(E, t)^* = \mathscr{F}(E)^* + 2\pi i E \mathscr{F}(E)^* F_\gamma A_\gamma(E, t), \tag{4.13}$$

$$A_\gamma(E, t) = A(E) + 2\pi i E A(E) F_\gamma A_\gamma(E, t). \tag{4.14}$$

Proof. By using Lemma 4.3, we have

$$\begin{aligned} \mathscr{F}_\gamma(E, t)^* &= W\mathscr{F}_0(E)^* + ER_\gamma(W - W^{-1}) \mathscr{F}_0(E)^* \\ &= \mathscr{F}(E)^* - E(W + ER(W - W^{-1})) T_\gamma(W - W^{-1}) \mathscr{F}_\gamma(E, t)^*. \end{aligned}$$

Using the definition of T_γ, we get (4.13). By multiplying (4.13) by $\mathscr{F}_0(E)(W - W^{-1})$, we get (4.14). ∎

THEOREM 4.5. *Let $K = 2\pi i E A(E) F_\gamma$. Then*

$$t \in \mathscr{E}_\gamma(E) \Leftrightarrow 1 \in \sigma_p(K).$$

Proof. One can easily check that

$$t \in \mathscr{E}_\gamma(E) \Leftrightarrow 1 \in \sigma_p(EWR_{\gamma, 0}(W - W^{-1})).$$

Letting $\tilde{K} = 2\pi i E \mathscr{F}(E)^* F_\gamma \mathscr{F}_0(E)(W^{-1} - W)$, we have

$$1 - EWR_{\gamma, 0}(W - W^{-1}) = (1 - EWR_0(W - W^{-1}))(1 - \tilde{K}). \tag{4.15}$$

In fact, by the resolvent equation and (4.8) we have

$$\mathscr{F}(E)^* = (W^{-1} - ER_0(W - W^{-1})) \mathscr{F}(E)^*. \tag{4.16}$$

This and Lemma 4.2 imply that

$$\begin{aligned} &(1 - EWR_0(W - W^{-1}))(1 - \tilde{K}) \\ &\quad = 1 - EWR_0(W - W^{-1}) + EWT_\gamma(W - W^{-1}) \\ &\quad = 1 - EWR_{\gamma, 0}(W - W^{-1}). \end{aligned}$$

Since $E \notin \sigma_p(H)$, $1 - EWR_0(W - W^{-1})$ is a bijection. Therefore

$$1 \in \sigma_p(EWR_{\gamma, 0}(W - W^{-1})) \Leftrightarrow 1 \in \sigma_p(\tilde{K}).$$

Letting $S_1 = \mathscr{F}_0(E)(W^{-1} - W)$, $S_2 = 2\pi i E \mathscr{F}(E)^* F_\gamma$, we have

$$K = S_1 S_2, \qquad \tilde{K} = S_2 S_1.$$

As can be easily seen, $\sigma_p(S_1 S_2)\backslash\{0\} = \sigma_p(S_2 S_1)\backslash\{0\}$, which proves that

$$1 \in \sigma_p(\tilde{K}) \Leftrightarrow 1 \in \sigma_p(K). \quad \blacksquare$$

By Theorem 4.5 and (4.14), for $t \in I_\varepsilon \backslash \mathscr{E}_\gamma(E)$ one can construct the Faddeev scattering amplitude $A_\gamma(E, t)$ from the physical scattering amplitude:

$$A_\gamma(E, t) = (1 - K)^{-1} A(E). \tag{4.17}$$

5. RECONSTRUCTION PROCEDURE

We shall prove Theorem 1.1 in this section. Suppose we are given the scattering amplitude $A(E)$ for a fixed energy $E > 0$. By virtue of (4.17), for an arbitrary direction $\gamma \in S^{n-1}$, one can construct the Faddeev scattering amplitude $A_\gamma(E, t)$ for $t \in I_\varepsilon \backslash \mathscr{E}_\gamma(E)$. This is an operator-valued 3×3 matrix,

$$A_\gamma(E, t) = (A_\gamma^{(jk)}(E, t)),$$

where each $A_\gamma^{(jk)}(E, t)$ is in $\mathbf{B}(L^2(\Omega_k(E)); L^2(\Omega_j(E)))$, and has the following expression:

$$A_\gamma^{(kj)}(E, t) = \mathscr{F}_j(E)\, Q \mathscr{F}_k(E)^* - E \mathscr{F}_j(E)\, \tilde{Q} R_\gamma(E, t)\, \tilde{Q} \mathscr{F}_k(E)^*,$$

$$Q = 1 - \left(\frac{c_0}{c}\right)^2, \qquad \tilde{Q} = \frac{c}{c_0} - \frac{c_0}{c},$$

$$(\mathscr{F}_j(E)\, f)(\xi) = \int_{\mathbf{R}^{n+1}} \overline{\varphi_j(x, y, E, \xi)}\, f(x, y)\, dx\, dy.$$

This has a continuous kernel $A_\gamma^{(jk)}(E, t; \xi, \xi')$, $\xi \in \Omega_j(E)$, $\xi' \in \Omega_k(E)$. We are going to extend $A_\gamma^{(jk)}(E, t; \xi, \xi')$ meromorphically with respect to t by restricting ξ and ξ' to some affine spaces. Let

$$\lambda_+(t) = \left(\frac{E}{c_+^2} - t^2\right)^{1/2},$$

and for $\omega, \omega' \in \mathbf{R}^n$ satisfying $|\omega| < 1$, $|\omega'| < 1$, and $\omega \cdot \gamma = \omega' \cdot \gamma = 0$, let

$$B(t; \omega, \omega') = A_\gamma^{(33)}(E, t; \xi(t, \omega), \xi(t, \omega')),$$

$$\xi(t, \omega) = \lambda_+(t)\, \omega + t\gamma, \qquad \xi(t, \omega') = \lambda_+(t)\, \omega' + t\gamma.$$

By virtue of the expression of $\varphi_3(x, y, E, \xi)$ and Theorem 3.5 (3), it has the expression

$$B(t; \omega, \omega') = B_1(t; \omega, \omega') - EB_2(t; \omega, \omega'),$$

$$B_1(t; \omega, \omega') = \int e^{-i\lambda_+(t)(\omega - \omega')x} \overline{\Psi_3(y, \xi(t, \omega))}$$

$$\times \Psi_3(y, \xi(t, \omega'))\, Q(x, y)\, c_0(y)^{-2}\, dx\, dy,$$

$$B_2(t; \omega, \omega') = \int e^{-i\lambda_+(t)\,\omega x} \overline{\Psi_3(y, \xi(t, \omega))}\, \tilde{Q}(x, y)\, c_0(y)^{-1}\, c(x, y)^{-1}$$

$$\times U_\gamma(E, t)(\tilde{Q} e^{i\lambda_+(t)\,\omega'} \cdot \Psi_3(\cdot, \xi(t, \omega'))\, c_0^{-1} c^{-1})\, dx\, dy,$$

$$\Psi_3(y, \xi) = (2\pi)^{-n/2}\, a_3(E, \xi)\, \psi_3(y, E, \xi).$$

Since $U_\gamma(E, t)$ has a meromorphic extension on D_ε by Theorem 3.5, $B(t; \omega, \omega')$ has a unique meromorphic continuation on D_ε by the well-known theorem.

We reconstruct the perturbation $c(x, y) - c_0(y)$ from the asymptotic behavior of $B_1(i\tau; \omega, \omega')$ as $\tau \to \infty$. We take $\bar{p} = (p, p_{n+1}) \in \mathbf{R}^{n+1}$ with $p \neq 0$ and $p_{n+1} \neq 0$ arbitrarily. We choose $\gamma \in S^{n-1}$, $\bar{\eta} = (\eta, \eta_{n+1}) \in S^n$ such that

$$\bar{p} \cdot \bar{\eta} = p \cdot \gamma = \eta \cdot \gamma = 0, \qquad \eta_{n+1} > 0.$$

In fact, we first take $\gamma \in S^{n-1}$ orthogonal to p. Then $\bar{\eta} = c(p, -|p|^2/p_{n+1})$ satisfies the above conditions by the suitable choice of c.

For sufficiently large $\tau > 0$, let

$$(\omega, \omega_{n+1}) = \left(1 - \frac{\bar{p}^2}{4\tau^2}\right)^{1/2} \bar{\eta} + \frac{\bar{p}}{2\tau},$$

$$(\omega', \omega'_{n+1}) = \left(1 - \frac{\bar{p}^2}{4\tau^2}\right)^{1/2} \bar{\eta} - \frac{\bar{p}}{2\tau}.$$

Then we have

$$\omega = \eta + \frac{p}{2\tau} + O(\tau^{-2}), \qquad \omega_{n+1} = \eta_{n+1} + \frac{p_{n+1}}{2\tau} + O(\tau^{-2}),$$

$$\omega' = \eta - \frac{p}{2\tau} + O(\tau^{-2}), \qquad \omega'_{n+1} = \eta_{n+1} - \frac{p_{n+1}}{2\tau} + O(\tau^{-2}).$$

We also have for any $(\omega, \omega_{n+1}) \in S^n$ with $\omega_{n+1} > 0$

$$\theta_+(E, \xi(t, \omega))^{1/2} = \left(\frac{E}{c_+^2} - \xi(t, \omega)^2\right)^{1/2} = \lambda_+(t)\, \omega_{n+1},$$

$$\theta_-(E, \xi(t, \omega))^{1/2} = \left(-t^2\omega_{n+1}^2 + \frac{E}{c_-^2} - \frac{E}{c_+^2}\, \omega^2\right)^{1/2}.$$

Therefore as $\tau \to \infty$,

$$\theta_+(E, \xi(i\tau, \omega))^{1/2} = \tau\omega_{n+1} + O(\tau^{-1}),$$

$$\theta_-(E, \xi(i\tau, \omega))^{1/2} = \tau\omega_{n+1} + O(\tau^{-1}).$$

We have therefore

$$\Psi_3(y, \xi(i\tau, \omega)) = 2^{-1/2}(2\pi)^{-(n+1)/2}\, (\tau\omega_{n+1})^{-1/4}\, e^{i\tau\omega_{n+1}y} + O(\tau^{-5/4}).$$

Choosing ω, ω' as above we have

$$\begin{aligned}
&\overline{\Psi_3(y, \xi(i\tau, \omega))}\, \Psi_3(y, \xi(i\tau, \omega')) \\
&\quad = 2^{-1}(2\pi)^{-(n+1)}\, (\tau\eta_{n+1})^{-1/2}\, e^{-i\tau(\omega_{n+1} - \omega'_{n+1})y} + O(\tau^{-3/2}) \\
&\quad = 2^{-1}(2\pi)^{-(n+1)}\, (\tau\eta_{n+1})^{-1/2}\, e^{-ip_{n+1}y} + O(\tau^{-3/2}).
\end{aligned}$$

Since $\lambda_+(i\tau)(\omega - \omega') = p + O(\tau^{-1})$, we have

$$e^{-i\lambda_+(i\tau)(\omega - \omega') \cdot x} = e^{-ip \cdot x} + O(\tau^{-1}).$$

We have therefore

$$\begin{aligned}
&B_1(i\tau; \omega, \omega') \\
&\quad \sim 2^{-1}(2\pi)^{-(n+1)}\, (\tau\eta_{n+1})^{-1/2} \int e^{-i(p \cdot x + p_{n+1}y)} Q(x, y)\, c_0^{-2}\, dx\, dy.
\end{aligned}$$

On the other hand, we have by Theorem 3.5 (2) that

$$\tau^{1/2} B_2(i\tau; \omega, \omega') \to 0.$$

This shows that the perturbation $c(x, y) - c_0(y)$ is uniquely reconstructed from the limit of $\tau^{1/2} B(i\tau; \omega, \omega')$ as $\tau \to \infty$.

6. PROPERTIES OF $(H_0 - E \mp i0)^{-1}$

We shall prove Theorem 3.1 in this section. The method we use here is the combination of the classical idea of integration by parts and the commutator calculus developed in the study of N-body Schrödinger operators. The basic ideas have already been proposed in the papers [5, 6, 9], and the summary of the applications to the N-body problem was presented in [10]. We state below essential requisites to reproduce the arguments in [5, 6, 9, 10] but sometimes omit the details.

6.1. *Mourre Estimate*

Let $p_x = -i\nabla_x$, $p_y = -i\partial/\partial y$ and

$$A = (x \cdot p_x + y p_y + p_x \cdot x + p_y y)/2. \tag{6.1}$$

A is essentially self-adjoint on $\mathscr{S}$ = the space of rapidly decreasing functions. Let $\mathbf{R}^{n+1}_{\pm} = \{(x, y) \in \mathbf{R}^{n+1};\ \pm y > 0\}$. Let $\mathscr{D}_0$ be the set of functions u such that

$$u \in H^1(\mathbf{R}^{n+1}_+) \cap H^1(\mathbf{R}^{n+1}_-), \qquad x \cdot p_x u,\ y p_y u \in L^2(\mathbf{R}^{n+1}_+) \cap L^2(\mathbf{R}^{n+1}_-).$$

By the cut-off argument and the standard mollifier technique, one can show that $\mathscr{D}_0 \subset D(A)$. A direct calculation shows that for $u, v \in D(H_0) \cap \mathscr{D}_0$

$$i\{(Au, H_0 v) - (H_0 u, Av)\} = 2(H_0 u, v). \tag{6.2}$$

Let $E > 0$ and let $\varphi(t) \in C_0^\infty(\mathbf{R})$ be such that $\varphi(t) = 1$ for $|t - E| < \delta$, $\varphi(t) = 0$ for $|t - E| > 2\delta$. Then it follows from (6.2) that for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varphi(H_0)\, i[H_0, A]\, \varphi(H_0) \geqslant 2(E - \varepsilon)\, \varphi(H_0)^2. \tag{6.3}$$

Let $R_0(z) = (H_0 - z)^{-1}$. Then by (6.3) and the well-known Mourre theory [15, 16], we can show that

$$R_0(E \pm i0) \in \mathbf{B}(L^{2,s}; L^{2,-s}), \qquad \forall s > 1/2. \tag{6.4}$$

This is an alternative proof of Theorem 3.1 (1).

6.2. *Commutator Calculus*

Let $X = (1 + |x|^2 + y^2)^{1/2}$ and

$$B = X^{-1/2} A X^{-1/2}. \tag{6.5}$$

For $m \in \mathbf{R}$, we let $\mathscr{V}_m$ be the set of smooth functions $v(x, y)$ such that $|\partial_x^\alpha \partial_y^\beta v| \leqslant C_{\alpha,\beta} X^{m-|\alpha|-|\beta|}$, $\forall \alpha, \beta$. $\mathscr{P}_{k,m}$ denotes the set of differential operators of order k with coefficients $\in \mathscr{V}_m$. For two operators P and A, we define their multiple commutators by

$$ad_0(P, A) = P,$$

$$ad_n(P, A) = [ad_{n-1}(P, A), A], \qquad n \geqslant 1.$$

Straightforward manipulations show the following lemma.

Lemma 6.1. (1) $[A, c_0(y)] = [B, c_0(y)] = 0$.

(2) $P \in \mathscr{P}_{k,m} \Rightarrow [P, B] \in \mathscr{P}_{k,m-1}$.

(3) $ad_n(X, B) \in \mathscr{P}_{0,1-n}$, $n \geqslant 0$.

(4) $i[H_0, B] = 2X^{-1/2}(H_0 - c_0 B^2 c_0) X^{-1/2} + c_0 X^{-1/2} P X^{-1/2} c_0, P \in \mathscr{P}_{1,-1}$.

(5) $ad_n(H_0, B) = c_0 P c_0$, $\quad P \in \mathscr{P}_{2,-n}$, $\quad n \geqslant 0$.

We now define the following class of operators.

Definition 6.2. For $m \in \mathbf{R}$, $\mathscr{OP}^m(X)$ is the set of operators P satisfying $X^\alpha ad_n(P, B) X^\beta \in \mathbf{B} = \mathbf{B}(L^2; L^2)$, $\forall \alpha, \beta$, $\forall n \geqslant 0$ such that $\alpha + \beta = n - m$.

The following lemma is proved easily by the above definition and a simple computation.

Lemma 6.3. (1) $P \in \mathscr{OP}^m(X) \Leftrightarrow P = X^m P_0$ *for some* $P_0 \in \mathscr{OP}^0(X)$.

(2) $P \in \mathscr{OP}^m(X) \Rightarrow [P, B] \in \mathscr{OP}^{m-1}(X)$.

(3) $P \in \mathscr{OP}^m(X) \Rightarrow X^k P X^l \in \mathscr{OP}^{m+k+l}(X)$, $\forall k, l \in \mathbf{R}$.

(4) $P \in \mathscr{OP}^m(X) \Rightarrow P^* \in \mathscr{OP}^m(X)$.

(5) $P \in \mathscr{OP}^m(X)$, $Q \in \mathscr{OP}^n(X) \Rightarrow PQ \in \mathscr{OP}^{m+n}(X)$.

(6) $P \in \mathscr{OP}^m(X) \Rightarrow c_0 P \in \mathscr{OP}^m(X)$.

Let $\mathscr{F}^m$ be the set introduced in Section 3. Representing $f \in \mathscr{F}^m$ by its almost analytic extension, one can show the following lemma (see [5, Lemma 2.4] or [10, Lemma 2.4]).

Lemma 6.4. (1) $f(X) \in \mathscr{OP}^m(X)$ *for* $f \in \mathscr{F}^m$, $m \in \mathbf{R}$.

(2) $f(H_0), f(B) \in \mathscr{OP}^0(X)$ *for* $f \in \mathscr{F}^m$, $m < 0$.

Here in proving (2), one must use the following inequality

$$\|c_0(y)(H_0-z)^{-1} f\|_{H^2} \leqslant C\,\frac{(1+|z|)}{|\mathrm{Im}\, z|}\,\|f\|.$$

LEMMA 6.5. (1) $B^N(H_0+i)^{-N} \in \mathbf{B}$, $\forall N \geqslant 0$.

(2) $f(B)\,\varphi(H_0) \in \mathbf{B}$ *if* $f \in \mathscr{F}^m$, $m \in \mathbf{R}$, $\varphi \in C_0^\infty(\mathbf{R})$.

Proof of (1). By an induction one can show that $B^N(H_0+i)^{-N}$ is written as a finite sum of the terms of the form

$$B_1(H_0+i)^{-p_1} \cdots B_k(H_0+i)^{-p_k}, \qquad B_i = ad_{n_i}(H_0, B).$$

The assertion (1) then follows from Lemma 6.1 (5). The assertion (2) follows easily from (1). ∎

The following lemma is proved by the formulas (2.3) and (2.4) of [10].

LEMMA 6.6. *For* $P \in \mathcal{OP}^m(X)$, $f \in \mathscr{F}^n$, $m, n \in \mathbf{R}$, *we have the following asymptotic expansion*

$$[P, f(B)] \sim \sum_{k \geqslant 1} (-1)^{k-1}/k!\; ad_k(P, B)\, f^{(k)}(B), \qquad ad_k(P, B) \in \mathcal{OP}^{m-k}(X).$$

6.3. *Uniqueness Theorem*

One can now prove the following uniqueness theorem which is well known for the Laplacian.

THEOREM 6.7. *Let* $0 < \alpha < 1/2$, $E > 0$. *Suppose* $u \in L^{2,-\alpha}$ *satisfies* $c_0(y)\,u \in H^2_{loc}(\mathbf{R}^{n+1})$ *and* $H_0 u = Eu$. *Then* $u = 0$.

Proof. Let $\varphi(t) \in C^\infty(\mathbf{R})$ be such that $\varphi(t) = 1$ for $|t-E| < \delta$, $\varphi(t) = 0$ for $|t-E| > 2\delta$, where $0 < \delta < E/2$. Since $u = \varphi(H_0)\,u$, we have only to show $\varphi(H_0)\,u \in L^2$.

Let $v = \varphi(H_0)\,u$, $v_\varepsilon = (1+\varepsilon X)^{-\alpha}\, v$. Then

$$(H_0 - E)\, v_\varepsilon = 2i\alpha\varepsilon(1+\varepsilon X)^{-1}\, c_0 B c_0 v_\varepsilon + K_\varepsilon v,$$
$$|K_\varepsilon(x, y)| \leqslant C X^{-2},$$

where the constant C is independent of $\varepsilon > 0$. A direct calculation shows that

$$i([H_0, A]\, v_\varepsilon, v_\varepsilon) = 4\alpha(X Y_\varepsilon B c_0 v_\varepsilon, B c_0 v_\varepsilon) + (Q_\varepsilon c_0 v, c_0 v),$$

where $Y_\varepsilon=\varepsilon(1+\varepsilon X)^{-1}$, $|Q_\varepsilon(x, y)|\leqslant CX^{-2}$. Noting that $XY_\varepsilon\leqslant 1$, we have

$$\begin{aligned}2(H_0 v_\varepsilon, v_\varepsilon) &\leqslant 4\alpha\, \|Bc_0 v_\varepsilon\|^2 + C\, \|X^{-1}v\|^2\\ &\leqslant 4\alpha\, \|\nabla c_0 v_\varepsilon\|^2 + C\, \|X^{-1}v\|^2\\ &\leqslant 4\alpha(H_0 v_\varepsilon, v_\varepsilon)+C.\end{aligned}$$

Since $4\alpha<2$, we have $(H_0 v_\varepsilon, v_\varepsilon)\leqslant C$. This implies that

$$(H_0\varphi(H_0)^2\,(1+\varepsilon X)^{-\alpha}\, u, (1+\varepsilon X)^{-\alpha}\, u)\leqslant C.$$

Letting $\varepsilon\to 0$, we have $v\in L^2$. ∎

Let us prove Theorem 3.1 (3).

THEOREM 6.8. *Let $0<\alpha<1/2<s<1$ and $E>0$. Let $u\in L^{2,-s}$ satisfy $c_0(y)\,u\in H^2_{loc}(\mathbf{R}^{n+1})$ and $H_0 u=Eu$. Suppose there exists an $\varepsilon>0$ such that $F(B)\,u\in L^{2,-\alpha}$ either for any $F\in\mathscr{F}^0_-(\varepsilon)$ or for any $F\in\mathscr{F}^0_+(-\varepsilon)$. Then $u=0$.*

Proof. We shall assume that $F(B)\,u\in L^{2,-\alpha}$ for any $F\in\mathscr{F}^0_-(\varepsilon)$ and prove $u=0$. Let $v=c_0(y)\,u$. Then

$$(-\Delta-V)\,u=0,\qquad V=c_0(y)^{-2}\,E.$$

We take $\chi(\rho)\in C_0^\infty(\mathbf{R})$ such that $\chi(\rho)=1$ if $|\rho|<1$, $\chi(\rho)=0$ if $|\rho|>2$ and let

$$\tilde{\chi}_t(X)=\int_X^\infty k^{-2\alpha}\chi(k/t)^2\,dk,\qquad t>0.$$

Using the identity

$$i[-\Delta_{x,y}, \tilde{\chi}_t]=2\,\mathrm{Re}\left(\nabla_x\tilde{\chi}_t\cdot p_x+\frac{\partial}{\partial y}\,\tilde{\chi}_t\cdot p_y\right),$$

we have

$$\mathrm{Re}(X^{-2\alpha}\chi_t^2\,Bv, v)=0,$$

where $\chi_t=\chi(X/t)$. Since $[B, c_0(y)]=0$, we have

$$\mathrm{Re}(c_0(y)^2\,X^{-2\alpha}\chi_t^2\,Bu, u)=0. \tag{6.6}$$

Let $\varphi(t)\in C_0^\infty(\mathbf{R})$ be such that $\varphi(t)=1$ for $|t-E|<\delta$, $\varphi(t)=0$ for $|t-E|>2\delta$, δ being a small constant. Since $H_0 u=Eu$, we have $u=\varphi(H_0)\,u$.

We take $F_{\pm}(t) \in C^{\infty}(\mathbf{R})$ such that $F_{+}(t) + F_{-}(t) = 1$, $F_{-}(t) = 1$ if $t < \varepsilon/2$, $F_{-}(t) = 0$ if $t > \varepsilon$. Since

$$\sup_{t \geqslant 1} |(c_0(y)^2 X^{-2\alpha} \chi_t^2 BF_{-}(B) \varphi(H_0)u, \varphi(H_0)u)| < \infty,$$

by the assumption of the theorem, we have by using (6.6) and $F_{+}(B) + F_{-}(B) = 1$

$$\sup_{t \geqslant 1} \operatorname{Re}(c_0(y)^2 X^{-2\alpha} \chi_t^2 BF_{+}(B) \varphi(H_0 u, \varphi(H_0)u) < \infty. \tag{6.7}$$

Let $g_{+}(t) = (F_{+}(t))^{1/2}$ and $g_{-}(t) = 1 - g_{+}(t)$. (6.7) implies that $\sqrt{B}\, g_{+}(B)\, \varphi(H_0)u \in L^{2,-\alpha}$, from which one can show $g_{+}(B)u \in L^{2,-\alpha}$. Therefore we have $u = g_{+}(B)u + g_{-}(B)u \in L^{2,-\alpha}$, which implies $u = 0$ by the previous theorem. ∎

6.4. *Resolvent Estimates*

We turn to the proof of Theorem 3.1 (2). Let $\varphi(t) \in C_0^{\infty}(\mathbf{R})$ be such that $\varphi(t) = 1$ for $|t - E| < \delta$, $\varphi(t) = 0$ for $|t - E| > 2\delta$. It follows from Lemma 6.1 (4) that

$$\varphi(H_0)\, X^{1/2} i[H_0, B]\, X^{1/2} \varphi(H_0) = 2\varphi(H_0)(H_0 - c_0 B^2 c_0 + K)\, \varphi(H_0),$$

K being a compact operator. Therefore, for any $\varepsilon > 0$ there exists a $\delta_0 > 0$ such that for any $\delta < \delta_0$

$$\varphi(H_0)\, X^{1/2} i[H_0, B]\, X^{1/2} \varphi(H_0) \geqslant 2\varphi(H_0)(E - \varepsilon - c_{+}^2 B^2)\, \varphi(H_0).$$

This is the basic inequality used to estimate the commutator $i[H_0, B]$ from below.

For a small $\varepsilon_1 > 0$, we take $F_0(t) \in \mathscr{F}_{-}^{0}(\sqrt{E}/c_{+})$ satisfying

$$\begin{cases} F_0(t) = 0 & \text{if} \quad t > \sqrt{E}/c_{+} - \varepsilon_1, \\ F_0(t) = 1 & \text{if} \quad t < \sqrt{E}/c_{+} - 2\varepsilon_1, \\ F_0(t) \geqslant 0, & \sqrt{F_0(t)} \in C^{\infty}, \\ F_0'(t) \leqslant 0, & \sqrt{-F_0'(t)} \in C^{\infty}. \end{cases}$$

We take $\varepsilon > 0$ such that

$$\frac{\sqrt{E}}{c_{+}} - \varepsilon_1 < \frac{1}{c_{+}} \sqrt{E - 2\varepsilon} =: C_1(E),$$

and let

$$F_m(t) = (C_1(E) - t)^m F_0(t),$$
$$\widetilde{F_{2m+1}}(t) = (C_1(E) - t) F_m(t)^2.$$

LEMMA 6.9. *Let* $m > -1/2$. *With* $F_m(t)$ *and* $\varphi(t)$ *introduced above, we define* $P_m = X^m F_m(B) \varphi(H)$. *Then there exists a constant* $C_0 > 0$ *such that*

$$-\mathrm{Re}\, \varphi(H_0)\, i[H_0, X^{2m+1}\widetilde{F_{2m+1}}(B)]\, \varphi(H_0) \geqslant C_0 P_m^* P_m + (*),$$

where $(*)$ *denotes an operator having the following asymptotic expansion*:

$$\sum_{n \geqslant 2} P_n f_n(B), \qquad P_n \in \mathcal{OP}^{2m+1-n}(X),$$
$$f_n \in \mathscr{F}^0_-(\sqrt{E}/c_+), \qquad \mathrm{supp}\, f_n \subset \mathrm{supp}\, F_0.$$

Proof. The proof of this lemma is essentially the same as that of [5, Lemma 3.2] or [10, Lemma 3.2]. We give only the sketch of the proof. First we note that

$$i[H_0, X^{2m+1}\widetilde{F_{2m+1}}(B)]$$
$$= i[H_0, X^{2m+1}]\, \widetilde{F_{2m+1}}(B) + iX^{2m+1}[H_0, \widetilde{F_{2m+1}}(B)].$$

Applying Lemma 6.6 formally, we have

$$i[H_0, X^{2m+1}] = 2(2m+1)\, c_0^2 B X^{2m} + \cdots,$$
$$i[H_0, \widetilde{F_{2m+1}}(B)] = i[H_0, B]\, \widetilde{F'_{2m+1}}(B) + \cdots,$$

where $\cdots$ denotes the *lower order* terms in X. Noting that

$$-\widetilde{F'_{2m+1}}(t) = (2m+1) F_m(t)^2 + G(t),$$
$$G(t) = -2(C_1(E) - t)^{2m+1} F_0'(t) F_0(t),$$

we have

$$-\mathrm{Re}\, \varphi(H_0)\, i[H_0, X^{2m+1}\widetilde{F_{2m+1}}(B)]\, \varphi(H_0)$$
$$= 2(2m+1)\, \varphi(H_0) F_m(B) X^m (c_0^2 B^2 - C_1(E)\, c_0^2 B) X^m F_m(B)\, \varphi(H_0)$$
$$+ (2m+1)\, \varphi(H_0) F_m(B) X^{m+1/2} i[H_0, B] X^{m+1/2} F_m(B)\, \varphi(H_0)$$
$$+ \varphi(H_0) X^m \sqrt{G(B)} X^{1/2} i[H_0, B] X^{1/2} \sqrt{G(B)} X^m \varphi(H_0) + (*).$$

Taking account of Lemma 6.1 (4), we have

$$\begin{aligned}
&-\mathrm{Re}\,\varphi(H_0)\, i[H_0, \widetilde{X^{2m+1}F_{2m+1}}(B)]\,\varphi(H_0) \\
&\quad = 2(2m+1)\, P_m^*(H_0 - C_1(E)\, c_0^2 B) P_m \\
&\qquad + 2\varphi(H_0) X^m \sqrt{G(B)}\,(H_0 - c_0^2 B^2) \sqrt{G(B)}\, X^m \varphi(H_0) + (*).
\end{aligned}$$

We now use

$$\begin{aligned}
&c_- \leqslant c_0(y) \leqslant c_+, \\
&H_0 \geqslant E - \varepsilon \qquad \text{on} \quad \mathrm{supp}\, \varphi(H_0),
\end{aligned}$$

the latter of which holds if $2\delta \leqslant \varepsilon$, to get

$$\begin{aligned}
&P_m^*(H_0 - C_1(E)\, c_0^2 B) P_m \\
&\quad \geqslant P_m^*(E - \varepsilon - C_1(E)\, c_0^2 B) P_m + (*) \\
&\quad \geqslant P_m^* \frac{c_0}{c_+} (E - \varepsilon - c_+^2\, C_1(E) B) \frac{c_0}{c_+} P_m + (*).
\end{aligned}$$

Noting that

$$E - \varepsilon - c_+^2\, C_1(E) B \geqslant \varepsilon \qquad \text{on supp}\, F_m(B),$$

we have

$$\begin{aligned}
P_m^*(H_0 - C_1(E)\, c_0^2 B) P_m &\geqslant \varepsilon P_m^* \left(\frac{c_0}{c_+}\right)^2 P_m + (*) \\
&\geqslant \varepsilon \left(\frac{c_-}{c_+}\right)^2 P_m^* P_m + (*).
\end{aligned}$$

Since

$$E - \varepsilon - c_+^2 B^2 \geqslant 0 \qquad \text{on} \quad \mathrm{supp}\, G(B),$$

we also have

$$\begin{aligned}
&\varphi(H_0) X^m \sqrt{G(B)}\,(H_0 - c_0^2 B^2) \sqrt{G(B)}\, X^m \varphi(H_0) \\
&\quad \geqslant \varphi(H_0)\, X^m \sqrt{G(B)}\,(E - \varepsilon - c_+^2 B^2) \sqrt{G(B)}\, X^m \varphi(H_0) + (*) \\
&\quad \geqslant (*).
\end{aligned}$$

We have thus proven that

$$-\operatorname{Re} \varphi(H_0)\, i[H_0, X^{2m+1}\widetilde{F_{2m+1}}(B)]\, \varphi(H_0)$$
$$\geqslant 2(2m+1)\, \varepsilon \left(\frac{c_-}{c_+}\right)^2 P_m^* P_m + (*),$$

which completes the proof of the lemma. ∎

Once we have proved Lemma 6.9, we can prove the following theorem in the same way as in Theorem 3.4 of [5] or Theorem 3.4 of [10]. We have only to estimate the quadratic form

$$-\operatorname{Re}(\varphi(H_0)\, i[H_0, X^{2m+1}\widetilde{F_{2m+1}}(B)]\, \varphi(H_0)\, R_0(z) f, R_0(z) f)$$

from above and below.

THEOREM 6.10. *Let* $R_0(z) = (H_0 - z)^{-1}$, $m > -1/2$, $t > 1$. *Let* $F \in \mathscr{F}^0_-(\sqrt{E}/c_+)$. *Then we have*

$$X^m F(B)\, R_0(E + i0) X^{-m-t} \in \mathbf{B}.$$

The proof for $R(E - i0)$ is obtained similarly.

7. PROPERTIES OF $L_0(\zeta)^{-1}$

We shall study various properties of $L_0(\zeta)^{-1}$ in this section. We begin with establishing the same results as in Theorem 3.1 for $L_0 = L_0(0) = p_x^2 + p_y^2 - V(y)$. Since $[V(y), A] = 0$ and $V(y) \geqslant E/c_+^2$, we have

$$i[L_0, A] = 2(p_x^2 + p_y^2) \geqslant 2L_0 + 2E/c_+^2. \tag{7.1}$$

THEOREM 7.1. *Let* $s > 1/2$. *Then for* $\lambda > -E/c_+^2$,

$$(L_0 - \lambda \mp i0)^{-1} \in \mathbf{B}(L^{2,s}; L^{2,-s}).$$

Moreover there exists a constant $C > 0$ *such that*

$$\|(L_0 - \lambda \mp i\varepsilon)^{-1}\|_{\mathbf{B}(L^{2,s};\, L^{2,-s})} \leqslant C\lambda^{-1/2},$$

for $\lambda > 1$ *and* $\varepsilon \geqslant 0$.

Proof. This theorem follows from (7.1) and the Mourre theory. The high-energy estimate is proved by the same method as in [13] (see Theorem 4.2 of [13]). ▮

For a small $\delta > 0$, let $\varphi(t) \in C_0^\infty(\mathbf{R})$ be such that $\varphi(t) = 1$ for $|t| < \delta$, $\varphi(t) = 0$ for $|t| > 2\delta$. Then by a direct calculation we have

$$\varphi(L_0) X^{1/2} i[L_0, B] X^{1/2} \varphi(L_0) = 2\varphi(L_0)(L_0 - B^2 + V + K)\, \varphi(L_0),$$

K being a compact operator. Therefore for any $\varepsilon > 0$, by choosing δ small enough we have

$$\varphi(L_0) X^{1/2} i[L_0, B] X^{1/2} \varphi(L_0) \geqslant 2\varphi(L_0)(E/c_+^2 - B^2 - \varepsilon)\, \varphi(L_0). \tag{7.2}$$

This inequality (7.2) should be compared with (6.8). Using (7.2) one can repeat the same argument as in the previous section to show the following theorem.

THEOREM 7.2. *Let* $C_0(E) = \sqrt{E}/c_+$. *Then for* $m > -1/2$, $t > 1$ *and* $F_\mp \in \mathscr{F}^0_\mp(\pm C_0(E))$, *we have*

$$F_\mp(B)(L_0 \mp i0)^{-1} \in \mathbf{B}(L^{2, m+t}; L^{2, m}).$$

The proof of Theorem 6.8 actually shows the following theorem.

THEOREM 7.3. *Let* $0 < \alpha < 1/2 < s < 1$. *Let* $u \in L^{2, -s}$ *satisfy* $L_0 u = 0$. *Suppose there exists* $\varepsilon > 0$ *such that* $F(B)\, u \in L^{2, -\alpha}$ *either for any* $F \in \mathscr{F}^0_-(\varepsilon)$ *or for any* $F \in \mathscr{F}^0_+(-\varepsilon)$. *The* $u = 0$.

In particular Theorems 3.1 and 7.1–7.3 imply

COROLLARY 7.4. $R_0(E \pm i0) = c_0^{-1}(L_0 \mp i0)^{-1}\, c_0^{-1}$.

We study the analytic continuation of $(L_0 - z)^{-1}$.

THEOREM 7.5. *For any* $\delta > 0$, *there exists an* $\varepsilon > 0$ *such that* $(L_0 + E/c_+^2 - z)^{-1}$ *defined on* $\mathbf{C}_\pm$ *have analytic continuations across the positive real axis* $(0, \infty)$ *into the region* $\{z \in \mathbf{C};\, |\mathrm{Im}\, z|^2 \leqslant \varepsilon\, \mathrm{Re}\, z\}$ *as* $\mathbf{B}(\mathscr{H}_\delta; \mathscr{H}_{-\delta})$*-valued functions.*

Proof. Let $A = -\Delta_x$, $B = -(\partial/\partial y)^2 - V(y) + E/c_+^2$. Let $\mathscr{F}_A(\lambda)$, $\lambda > 0$, and $\mathscr{F}_B(k)$, $k > \sigma$, $\sigma = E/c_+^2 - E/c_-^2$, be the spectral representations for A and B respectively. $\mathscr{F}_A(\lambda)$ is the trace on the sphere of radius $\sqrt{\lambda}$ of the usual Fourier transformation. Therefore if $e^{\delta|x|} f(x) \in L^2(\mathbf{R}^n)$, $\mathscr{F}_A(\lambda)\, f$ is analytic on $\{\mathrm{Re}\sqrt{\lambda} > 0,\ |I_m \sqrt{\lambda}| < \varepsilon_0\}$ for some $\varepsilon_0 > 0$. By solving the differential equation $(-d^2/dy^2 - V(y) + E/c_+^2)\, u = ku$ explicitly (see, e.g.

[21], p. 165), we see that the generalized eigenfunctions of B associated with the energy $k>0$ are linear combinations of $\exp(\pm i\sqrt{k}\,y)$ and $\exp(\pm i\sqrt{k-\sigma}\,y)$, both of which are analytic with respect to $\sqrt{k}$ in the right-half plane. Therefore if $e^{\delta|y|}f(y)\in L^2(\mathbf{R})$, $\mathscr{F}_B(k)f$ is analytic on $\{\mathrm{Re}\sqrt{k}>0, |\mathrm{Im}\sqrt{f}|<\varepsilon_0\}$ for some $\varepsilon_0>0$.

Since $\mathscr{F}_A(\lambda)\otimes\mathscr{F}_B(k)$ diagonalizes $A\otimes 1+1\otimes B$, we have for $\mathrm{Im}\, z\neq 0$,

$$(A\otimes 1+1\otimes B-z)^{-1}=\iint_{\lambda>0,\,k>\sigma}\frac{\mathscr{F}_1(\lambda)^*\otimes\mathscr{F}_B(k)^*\,\mathscr{F}_A(\lambda)\otimes\mathscr{F}_B(k)}{\lambda+k-z}\,d\lambda\,dk.$$

The analytic continuation of this operator is obtained by deforming the path of integration. For $a, \varepsilon>0$, let us introduce the following sets:

$$\Omega_\varepsilon=\{z\in\mathbf{C};\,|\mathrm{Im}\,z|^2\leqslant\varepsilon\,\mathrm{Re}\,z\},$$

$$\Omega_{a,\varepsilon}=\{z\in\Omega_\varepsilon;\,5a<\mathrm{Re}\,z<6a\}.$$

Elementary computations show the following facts:

$$z\in\Omega_\varepsilon\Rightarrow|\mathrm{Im}\sqrt{z}|\leqslant\sqrt{\varepsilon},$$

$$z\in\Omega_{a,\varepsilon},\,k<4a\Rightarrow|\mathrm{Im}\sqrt{z-k}|\leqslant\sqrt{6\varepsilon}.$$

We split the integral for $(A\otimes 1+1\otimes B-z)^{-1}$ into two parts:

$$\iint_{\lambda+k<4a\text{ or }\lambda+k>8a}\cdots\,d\lambda\,dk+\iint_{4a<\lambda+k<8a}\cdots\,d\lambda\,dk=:I_1(z)+I_2(z).$$

$I_1(z)$ clearly has an analytic continuation on $\Omega_{a,\varepsilon}$ as a $\mathbf{B}(L^2;L^2)$-valued function. $I_2(z)$ is again split into two parts:

$$\iint_{4a<\lambda+k<8a,\,k<4a}\cdots\,d\lambda\,dk+\iint_{4a<\lambda+k<8a,\,k>4a}\cdots\,d\lambda\,dk=:I_3(z)+I_4(z).$$

We factor the denominator if $I_3(z)$ as $(\sqrt{\lambda}-\sqrt{z-k})(\sqrt{\lambda}+\sqrt{z-k})$. Since $|\mathrm{Im}\sqrt{z-k}|\leqslant\sqrt{6\varepsilon}$ if $k<4a$, $z\in\Omega_\varepsilon$, by deforming the path of $\sqrt{\lambda}$-integration we see that $I_3(z)$ has an analytic continuation on $\Omega_{a,\varepsilon}$ as $\mathbf{B}(\mathscr{H}_\delta;\mathscr{H}_{-\delta})$-valued function for sufficiently small $\varepsilon>0$. On the integrand of $I_4(z)$, $|\mathrm{Im}\sqrt{z-\lambda}|\leqslant\sqrt{6\varepsilon}$, since $\lambda<4a$, $z\in\Omega_{a,\varepsilon}$. Therefore the deformation of the path of $\sqrt{k}$-integration implies that $I_4(z)$ has an analytic continuation on $\Omega_{a,\varepsilon}$. ∎

We turn to the proof of Theorem 3.2. The main part of the proof is the convergence of $V_{\gamma,0}(E,z)$ as $z \to t \in I_\varepsilon = (-\varepsilon/2, \varepsilon/2)$. Let $z = t + i\tau$, $t \in I_\varepsilon$, $0 < \tau < 1$, and let

$$\begin{aligned}\tilde{V}_{\gamma,0}(E,z) &= e^{itx\cdot\gamma} V_{\gamma,0}(E,z)\, e^{-itx\cdot\gamma} \\ &= (\mathscr{F}_{x\to\xi})^{-1} \left((\xi + i\tau\gamma)^2 + p_y^2 - V(y)\right)^{-1} \varphi_1(\gamma\cdot(\xi - t\gamma))\, \mathscr{F}_{x\to\xi}.\end{aligned} \tag{7.3}$$

For $f \in L^{2,s}$, we put

$$u_\pm(\tau) = (\mathscr{F}_{x\to\xi})^{-1} F(\pm\gamma\cdot(\xi - t\gamma) \geqslant 0)\, \mathscr{F}_{x\to\xi} \tilde{V}_{\gamma,0}(E,z)\, f. \tag{7.4}$$

By the support property of φ_1, they satisfy

$$\operatorname{supp}_\xi \mathscr{F}_{x\to\xi} u_\pm(\tau) \subset \{\xi;\ \pm\gamma\cdot\xi \geqslant \varepsilon/2\}, \tag{7.5}$$

$$L_0(i\tau\gamma)\, u_\pm(\tau) = f_\pm, \tag{7.6}$$

$$f_\pm = (\mathscr{F}_{x\to\xi})^{-1} F(\pm\gamma\cdot(\xi - t\gamma) \geqslant 0)\, \varphi_1(\gamma\cdot(\xi - t\gamma))\, \mathscr{F}_{x\to\xi} f. \tag{7.7}$$

LEMMA 7.6. *Let $m > -1/2$ and $F_m(B)$, $\widetilde{F_{2m+1}}(B)$ be as in Lemma 6.9. Let $\varphi(t)$ be as in (7.2). We take $\psi(t) \in C^\infty(\mathbf{R})$ such that $\psi(t) = 1$ for $t < -\varepsilon/4$, $\psi(t) = 0$ for $t > -\varepsilon/8$. Then there exist a constant $C_0 > 0$ and $Q \in \mathscr{OP}^{2m-1}(X)$ such that for $0 < \tau < 1$*

$$\begin{aligned}-\operatorname{Re} \varphi(L_0)\, \psi(\gamma\cdot p_x)\, i\{&L_0(i\tau\gamma)^*\, X^{2m+1}\widetilde{F_{2m+1}}(B) \\ &- X^{2m+1}\widetilde{F_{2m+1}}(B)\, L_0(i\tau\gamma)\}\, \psi(\gamma\cdot p_x)\, \varphi(L_0) \\ \geqslant{}& C_0 P_m^* P_m + Q,\end{aligned}$$

where $P_m = X^m F_m(B)\, \psi(\gamma\cdot p_x)\, \varphi(L_0)$.

Proof. We shall estimate

$$\begin{aligned}-i(L_0(i\Gamma\gamma)^*\, &X^{2m+1}\widetilde{F_{2m+1}}(B) - X^{2m+1}\widetilde{F_{2m+1}}(B)\, L_0(i\tau\gamma)) \\ &= -i[L_0, X^{2m+1}\widetilde{F_{2m+1}}(B)] \\ &\quad - 2\tau(X^{2m+1}\widetilde{F_{2m+1}}(B)\, \gamma\cdot p_x + \gamma\cdot p_x X^{2m+1}\widetilde{F_{2m+1}}(B)).\end{aligned}$$

Using (7.2) and arguing in the same way as in the proof of Lemma 6.9, we have

$$\begin{aligned}-\operatorname{Re} \psi(\gamma\cdot p_x)\, \varphi(L_0)\, &i[L_0, X^{2m+1}\widetilde{F_{2m+1}}(B)]\, \varphi(L_0)\, \psi(\gamma\cdot p_x) \\ &\geqslant C_0 P_m^* P_m + Q\end{aligned}$$

with $C_0 > 0$, $Q \in \mathscr{OP}^{2m-1}(X)$.

We let

$$T = -\varphi(L_0)\,\psi(\gamma\cdot p_x)\,S\psi(\gamma\cdot p_x)\,\varphi(L_0),$$
$$S = \mathrm{Re}(X^{2m+1}\widetilde{F_{2m+1}}(B)\,\gamma\cdot p_x + \gamma\cdot p_x X^{2m+1}\widetilde{F_{2m+1}}(B)),$$

and estimate T from below by an operator in $\mathcal{OP}^{2m-1}(X)$.

Let

$$T_1 = (-\gamma\cdot p_x)^{1/2}\,\psi(\gamma\cdot p_x)\,\varphi(L_0),$$
$$T_2 = X^{m+1/2}\widetilde{F_{2m+1}}(B)\,X^{m+1/2}.$$

We show that

$$T - 2T_1T_2T_1 \in \mathcal{OP}^{2m-1}(X).$$

In fact, using the relation

$$\mathrm{Re}\,X^{2m+1}\widetilde{F_{2m+1}}(B) = T_1 + \tfrac{1}{2}[[\widetilde{F_{2m+1}}(B), X^{m+1/2}], X^{m+1/2}],$$

we have

$$T \equiv -\varphi(L_0)\,\psi(\gamma\cdot p_x)(T_2\gamma\cdot p_x + \gamma\cdot p_x T_2)\,\psi(\gamma\cdot p_x)\,\varphi(L_0) \quad \mathrm{mod}\ \mathcal{OP}^{2m-1}(X).$$

Commuting $\varphi(L_0)\,\psi(\gamma\cdot p_x)$ and T_2, we get

$$T \equiv T_2T_1^2 + T_1^2T_2 \qquad \mathrm{mod}\ \mathcal{OP}^{2m-1}(X).$$

We finally commute T_1 and T_2 to get

$$T \equiv 2T_1T_2T_1 \qquad \mathrm{mod}\ \mathcal{OP}^{2m-1}(X).$$

Since $T_1T_2T_1 \geqslant 0$, we have $T \geqslant 0 \ \mathrm{mod}\ \mathcal{OP}^{2m-1}(X)$, which completes the proof. ∎

LEMMA 7.7. *Let $m > -1/2$ and P_m be as in Lemma* 7.6. *Then there exists a constant $C > 0$ such that for $0 < \tau < 1$,*

$$\|P_m u_-(\tau)\| \leqslant C(\|f\|_{m+1} + \|u_-(\tau)\|_{m-1/2}).$$

Proof. Let us first note that $u_-(\tau) = \psi(\gamma\cdot p_x)\,u_-(\tau)$ by virtue of (7.5). Using Lemma 7.6, we have

$$\begin{aligned} -\mathrm{Re}\, i\{&(X^{2m+1}\widetilde{F_{2m+1}}(B)\,\varphi(L_0)\,u_-(\tau), \varphi(L_0)\,f_-) \\ &- (\varphi(L_0)\,f_-, \ \widetilde{F_{2m+1}}(B)\,X^{2m+1}\varphi(L_0)\,u_-(\tau))\} \\ &\geqslant C_0\,\|P_m u_-(\tau)\|^2 - C\,\|u_-(\tau)\|^2_{m-1/2}. \end{aligned}$$

Noting that $[\widetilde{F_{2m+1}}(B), X^{2m+1}]\,\varphi(L_0)\in\mathcal{OP}^{2m}(X)$, one can see that the left-hand side is estimated from above by

$$2|(X^{2m+1}\widetilde{F_{2m+1}}(B)\,\varphi(L_0)\,u_-(\tau),\varphi(L_0)\,f_-)|+C(\|f\|^2_{m+1}+\|u_-(\tau)\|^2_{m-1}).$$

Since $\varphi(L_0)\,X^{2m+1}\,\widetilde{F_{2m+1}}\,(B)\,\varphi(L_0)\,u_-(\tau)=X^{m+1}QP_m u_-(\tau)$ with $Q\in\mathcal{OP}^0(X)$, this is estimated from above by

$$\varepsilon\,\|P_m u_-(\tau)\|^2+C_\varepsilon(\|f\|^2_{m+1}+\|u_-(\tau)\|^2_{m-1}),$$

ε being an arbitrary constant >0. This completes the proof. ∎

Lemma 7.8. *Let* $1/2<s<1$ *and* $\alpha=1-s$. *Let* $F_-\in\mathcal{F}^0_-(C_0(E))$. *Then there exists a constant* $C>0$ *such that*

$$\|F_-(B)\,u_-(\tau)\|_{-\alpha}\leqslant C(\|f\|_s+\|u_-(\tau)\|_{s-3/2})$$

for $0<\tau<1$.

Proof. Let $\varphi(t)$ be as in (7.2) and $\tilde\varphi(t)=1-\varphi(t)$. Then since $(L_0+i\tau\gamma\cdot p_x)\,\tilde\varphi(L_0)\,u_-(\tau)=\tilde\varphi(L_0)\,f$, we easily have

$$\|\tilde\varphi(L_0)\,u_-(\tau)\|_s\leqslant C\,\|f\|_s$$

with a constant $C>0$ independent of $0<\tau<1$.

Let $F_-\in\mathcal{F}^0_-(C_0(E))$. Choosing $P_{-\alpha}$ suitably as in Lemma 7.6, we have

$$X^{-\alpha}F_-(B)\,\varphi(L_0)\,u_-(\tau)=QP_{-\alpha}u_-(\tau)$$

with $Q\in\mathcal{OP}^0(X)$. Hence, by Lemma 7.7, letting $m=-\alpha=s-1$

$$\|F_-(B)\,\varphi(L_0)\,u_-(\tau)\|_{-\alpha}\leqslant C(\|f\|_s+\|u_-(\tau)\|_{s-3/2}),$$

which proves the lemma. ∎

Lemma 7.9. *Let* $1/2<\beta<s<3/4$. *Then there exist constants* $C>0$, $\varepsilon>0$ *such that*

$$\left\|F\left(\frac{X}{t}>2\right)u_-(\tau)\right\|_{-\beta}\leqslant Ct^{-\varepsilon}(\|u_-(\tau)\|_{-\beta}+\|f\|_s),$$

for $0<\tau<1$, $t>1$.

Proof. We take $\rho \in C^\infty(\mathbf{R})$ such that $\rho(k)=1$ for $k>2$ and $\rho(k)=0$ for $k<1$ and put $\rho_t(k)=\rho(k/t)$. We define

$$\chi_t(k)=\int_k^\infty x^{-2\beta}\rho_t(x)^2\,dx.$$

It is easy to see that

$$|\chi_t(k)| \leqslant Ct^{1-2\beta}.$$

By a direct calculation we have, letting $v=u_-(\tau)$,

$$-i([L_0, \chi_t(X)]v, v)=2(BX^{-\beta}\rho_t(X)v, X^{-\beta}\rho_t(X)v).$$

On the other hand, using the relation $L_0(i\tau\gamma\cdot p_x)v=f_-$ we have

$$-i([L_0, \chi_t(X)]v, v)=-2i\,\mathrm{Im}(f_-, \chi_t(X)v)+4\tau\,\mathrm{Re}(\chi_t(X)\gamma \cot p_x v, v).$$

By (7.5) and Gårding's inequality

$$\mathrm{Re}(\chi_t(X)\,\gamma\cdot p_x v, v) \leqslant Ct^{1-2\beta}\,\|u\|^2_{-1} \leqslant Ct^{1-2\beta}\,\|u\|^2_{-\beta}.$$

We also have

$$|(\chi_t(X)v, f_-)| \leqslant Ct^{1-2\beta}(\|v\|^2_{-\beta}+\|f\|^2_s).$$

We have therefore

$$(BX^{-\beta}\rho_t(X)v, X^{-\beta}\rho_t(X)v) \leqslant Ct^{1-2\beta}(\|v\|^2_{-\beta}+\|f\|^2_s). \tag{7.8}$$

Here we take $F_-(t) \in \mathscr{F}^0_-(C_0(E))$ such that $F_-(t)=1$ for $t<C_0(E)/2$ and let $F_+(t)=1-F_-(t)$. By Lemma 7.8 we have for some $\varepsilon>0$

$$\|F_-(B)\,X^{-\beta}\rho_t(X)v\| \leqslant Ct^{-\varepsilon}(\|v\|_{-\beta}+\|f\|_s). \tag{7.9}$$

Using $F_+(B)+F_-(B)=1$ and (7.9), one can also show that

$$\|F_+(B)\,X^{-\beta}\rho_t(X)v\| \leqslant Ct^{-\varepsilon}(\|v\|_{-\beta}+\|f\|_s). \tag{7.10}$$

The inequalities (7.9) and (7.10) prove the lemma. ∎

We let

$$Z^{(\pm)}_{\gamma,0}(E,z)=(\mathscr{F}_{x\to\xi})^{-1}\,F(\pm\gamma\cdot(\xi-t\gamma)\geqslant 0)\,\mathscr{F}_{x\to\xi}\,\widetilde{V}_{\gamma,0}(E,z). \tag{7.11}$$

Theorem 7.10. *Let $1/2<\beta<s<3/4$. Then there exists a constant $C>0$ such that*

$$\|Z^{(\pm)}_{\gamma,0}(E, t+i\tau) f\|_{-\beta} \leqslant C\|f\|_s,$$

for $t \in I_\varepsilon$, $0<\tau<1$.

Proof. We shall prove this theorem for the $(-)$ case. Suppose this is not true. Then there exist $t_n+i\tau_n$, $f_n \in L^{2,s}$ such that $t_n+i\tau_n \to t \in I_\varepsilon$ and $u_n = Z^{(-)}_{\gamma,0}(E, t_n+i\tau_n) f_n$ satisfies

$$\|u_n\|_{-\beta}=1, \qquad \|f_n\|_s \to 0.$$

By Lemma 7.9 and Rellich's compactness theorem, $\{u_n\}_{n=1}^{\infty}$ contains a subsequence convergent to some $u \in L^{2,-\beta}$. Hence $\|u\|_{-\beta}=1$, $L_0 u=0$. By Lemma 7.8, $F_-(B) u \in L^{2,-\alpha}$ for any $F_- \in \mathscr{F}^0_-(C_0(E))$. Theorem 7.3 then implies that $u=0$, which is a contradiction. ∎

Theorem 7.11. *Let $J=\{z=t+i\tau;\, t \in I_\varepsilon, 0<\tau<1\}$. Let $1/2<\beta<s<3/4$. Then as a $\mathbf{B}(L^{2,s}; L^{2,-\beta})$-valued function, $Z^{(\pm)}_{\gamma,0}(E,z)$ is uniformly continuous on J.*

Proof. Let $\varphi \in C_0^\infty(\mathbf{R})$ be such that $\varphi(t)=1$ near $t=0$. We have only to consider $\varphi(L_0)\, Z^{(-)}_{\gamma,0}(E,z)$. Suppose this is not uniformly continuous. Then there exist $\varepsilon_0>0$, $z_n, z'_n \in J$ and $f_n \in L^{2,s}$ such that

$$\|f_n\|_s=1, \qquad z_n, z'_n \to t \in I_\varepsilon,$$

$$\|(Z^{(-)}_{\gamma,0}(E,z_n)-Z^{(-)}_{\gamma,0}(E,z'_n))\, \varphi(L_0) f_n\|_{-\beta} \geqslant \varepsilon_0.$$

Let $u_n = Z^{(-)}_{\gamma,0}(E,z_n)\,\varphi(L_0) f_n$, $v_n = Z^{(-)}_{\gamma,0}(E,z'_n)\,\varphi(L_0) f_n$. Take $1/2<s'<s$. By the compactness, one can assume that $\varphi(L_0) f_n$ converges in $L^{2,s'}$. Let $w_n=u_n-v_n$. Then $\|w_n\|_{-\beta} \geqslant \varepsilon_0$. By Lemma 7.9 and Rellich's compactness theorem, one can assume that w_n converges to some w in $L^{2,-\beta}$. Hence $\|w\|_{-\beta} \geqslant \varepsilon_0$. On the other hand, one can see that $L_0 w=0$. By the same reasoning as in the proof of Theorem 7.10, one is led to the contradiction. ∎

Theorem 7.12. *Let $s>1/2$. Then $V_{\gamma,0}(E,z)$ is uniformly continuous in $\mathbf{B}(L^{2,s}; L^{2,-s})$ with respect to $z \in J$ and there exists a constant $C>0$ such that*

$$\|V_{\gamma,0}(E,z) f\|_{-s} \leqslant C\|f\|_s, \qquad z \in J.$$

Proof. We choose β, s' such that $1/2 < \beta < s' < s$ and consider

$$X^{-s}\tilde{V}_{\gamma,0}(E,z)\,X^{-s} = X^{-s+\beta}\cdot X^{-\beta}Z^{(+)}_{\gamma,0}(E,z)\,X^{-s'}\cdot X^{-(s-s')} + X^{-s+\beta}\cdot X^{-\beta}Z^{(-)}_{\gamma,0}(E,z)\,X^{-s'}\cdot X^{-(s-s')}$$

We then apply Theorems 7.10 and 7.11 to conclude the theorem. ∎

We now prove Theorem 3.2. The analyticity of $U_{\gamma,0}(E,z)$ in z is obvious by the definition. It is easy to show that $\| V_{\gamma,0}(E,i\tau)\|_{\mathbf{B}(L^2;L^2)} \leqslant C/\tau$ for $\tau > 1$. On the other hand Theorem 7.1 implies that $\|W_{\gamma,0}(E,i\tau)\|_{\mathbf{B}(L^{2,s};L^{2,-s})} \leqslant C/\tau$ for $\tau > 1$, which proves that $U_{\gamma,0}(E,i\tau)$ satisfies the same inequality. Theorem 7.12 proves (2). Theorem 7.1 proves (3). The assertion (4) follows from a direct computation. Finally, the assertion (3.15) can be proved by applying the same arguments as in the proof of Theorem 6.10 to $\tilde{V}_{\gamma,0}(E,z)$. The assertion (3.16) follows from Theorem 7.2.

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