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Zeta Polynomials and the Möbius Function*

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Using the theory of zeta polynomials, chain generalizations of well-known Möbius function identities are obtained. The proofs of these identities are combinatorial, and the Möbius function identities follows as corollaries.

1. INTRODUCTION

The Möbius function of a partially ordered set is a central object in enumerative combinatorics. As such, there is a good deal of interest in being able to compute it easily. To do this many identities of the Möbius function have been proven. (See for example [3, 5, 8].) Frequently the proofs have relied on some structural properties of the poset, notably the assumption that it is a lattice.

The aim of this paper is to prove some identities of the Möbius function by simple chain counting arguments which are true in any poset. Central to our discussion will be the zeta polynomial of the poset, a polynomial which enumerates multichains. The zeta polynomial was first explicitly defined by Stanley [9, p. 200]. Interesting examples of zeta polynomials had been computed by Kreweras [6, 7]. See also [4].

In the next section, a brief development of the incidence algebra is presented along with the fundamental properties of the zeta polynomial. In the following section, identities of zeta polynomials are proven. These will specialize to some well-known identities of the Möbius function. In the last section we discuss an open problem suggested by our techniques.

2. THE INCIDENCE ALGEBRA AND THE ZETA POLYNOMIAL

Let P be a locally finite poset, that is every interval is finite. We define $\mathscr{I}(P)$, the incidence algebra of P, to be the algebra of all functions

$$f: P\times P \to R,$$

such that $x \not\leq y$ implies $f(x, y)=0$. $\mathscr{I}(P)$ is an algebra under pointwise addition, scalar multiplication, and convolution defined by

$$f*g(x, y) = \sum_{x\leq z\leq y} f(x, z)g(z, y).$$

One element in $\mathscr{I}(P)$ is the zeta function, $\zeta(x, y)$, defined by,

$$\zeta(x, y)=1 \quad \text{if } x\leq y.$$

From the definition of multiplication it is easy to deduce that

$$\zeta^k(x, y) = \text{the number of multichains,}$$

$$x=x_0\leq x_1\leq \cdots \leq x_k = y.$$

Another function in $\mathscr{I}(P)$ is $(\zeta-1)(x, y)$ where 1 is the identity element, i.e.,

$$1(x, y)=\begin{cases} 1 & \text{if } x=y \\ 0 & \text{otherwise.}\end{cases}$$

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Thus

$$(\zeta-1)(x, y)=\begin{cases}1 & \text{if } x<y\\ 0 & \text{otherwise.}\end{cases}$$

Hence

$$(\zeta-1)^k(x, y) = \text{the number of chains,}$$
$$x=x_0<x_1<\cdots<x_k=y.$$

Define $\mu(x, y)$, the Möbius function, by $\mu(x, y)=\zeta^{-1}(x, y)$. Observe that

$$\begin{aligned}\mu(x, y)&=1/\zeta(x, y)=1/(1-(1-\zeta)(x, y)\\ &=1(x, y)+(1-\zeta)(x, y)+(1-\zeta)^2(x, y)+\cdots.\end{aligned}$$

So

$$\mu(x, y)=1(x, y)-(\zeta-1)(x, y)+(\zeta-1)^2(x, y)-\cdots.$$

If we let $c_i=(\zeta-1)^i(x, y)$ we have

$$\mu(x, y)=c_0-c_1+c_2-\cdots,$$

a result due to Philip Hall (see [8, p. 346] for details). Note that this sum terminates since the interval $[x, y]$ is finite.

Suppose P has a maximum and minimum element, called $\hat{1}$ and $\hat{0}$ respectively. We define $Z(P; n)$, the zeta polynomial of P, by the identity

$$Z(P; n)=\zeta^n(\hat{0}, \hat{1}).$$

To show that this is a polynomial in n, we observe that

$$\zeta^n(\hat{0}, \hat{1})=(1+\zeta-1)^n(\hat{0}, \hat{1}),$$

and so

$$\zeta^n(\hat{0}, \hat{1})=\sum_{k=1}^{n}\binom{n}{k}(\zeta-1)^k(\hat{0}, \hat{1}),$$

hence

$$\zeta^n(\hat{0}, \hat{1})=\sum_{k=1}^{r}\binom{n}{k}c_k,$$

where r is the length of the longest chain in P.

It is possible to define the zeta polynomial for any finite poset P regardless of whether it has a $\hat{0}$ and a $\hat{1}$. Let $\hat{c}_k$ be the number of chains in P of the form $x_1<x_2<\cdots<x_k$. Then define

$$\bar{Z}(P; n)=\sum_{k=1}^{r}\binom{n-2}{k-1}\hat{c}_k.$$

THEOREM 2.1. *If P has a $\hat{0}$ and a $\hat{1}$ then*

$$Z(P; n)=\bar{Z}(P; n).$$

PROOF. $Z(P; n)$ is defined to be $\zeta^n(\hat{0}, \hat{1})$. By a previous observation, this is the number of multichains in P of the form

$$0=x_0\leqslant x_1\leqslant\cdots\leqslant x_n=1.$$

To see that $\bar{Z}(P; n)$ counts the same thing, note that given an i-element chain in P, something counted by $\hat{c}_i$, there are $\binom{n-2}{i-1}$ different $n-1$ element multichains that contain those i elements: Then we add the $\hat{0}$ and $\hat{1}$ to make a chain of the form counted by $Z(P; n)$.

So, by Theorem 2.1 we can speak of the zeta polynomial for any finite poset. We conclude this section with a lemma which assembles some useful values of the zeta polynomial. The simple proof is omitted.

LEMMA 2.2. *Let P be a poset with a $\hat{0}$ and a $\hat{1}$. Then*

(i) $Z(P;1)=1$,
(ii) $Z(P;0)=0$,
(iii) $Z(P;-1)=\mu(\hat{0},\hat{1})$.

The idea of this paper is that it is sometimes easier to work with the zeta polynomial than with the Möbius function itself, because $Z(P;n)$ counts the number of elements in a certain finite set. The identities involving $Z(P;n)$ can be proved for all positive integers n by pure enumerative techniques. But a polynomial is determined by its values at positive integers, so we may substitute $n=-1$ to get an identity involving the Möbius function.

For instance, one can show by simple counting arguments that the zeta polynomial for a Boolean algebra of rank k is n^k. Hence the Möbius function satisfies $\mu(\hat{0},\hat{1})=(-1)^k$.

3. IDENTITIES OF ZETA POLYNOMIALS

In this section several identities of zeta polynomials are proven. By evaluating these identities at $n=-1$ certain Möbius function identities will follow.

We begin by establishing an identity of zeta polynomials across a Galois connection. Given two posets P and Q and a pair of order inverting maps $\rho: P\to Q$ and $\pi: Q\to P$, then ρ, π are said to form a Galois connection if $\rho\pi(q)\geqslant q$ for all $q\in Q$ and $\pi\rho(p)\geqslant p$ for all $p\in P$.

THEOREM 3.1. *Let P and Q be posets where Q has a $\hat{0}$. Let ρ, π be a Galois connection between P and Q. Then for all $y\in P$*

$$\sum_{a\geqslant y} Z([\hat{0},\rho(a)];n+1)\,\mu_P(y,a)=\sum_{\{x\mid\pi(x)=y\}} Z([\hat{0},x];n).$$

PROOF. Since $Z(n)=\zeta^n$, it is clear that

$$\sum_{x\leqslant\rho(y)} Z([\hat{0},x];n)=Z([\hat{0},\rho(y)];n+1).$$

Since π and ρ form a Galois connection we know that $x\leqslant\rho(y)$ if and only if $\pi(x)\geqslant y$. From these two observations we derive

$$\sum_{a\geqslant y}\left[\sum_{\pi(x)=a} Z([\hat{0},x];n)\right]=Z([\hat{0},\rho(y)];n+1).$$

By performing Möbius inversion over P the theorem is proven.

The following corollary is a strengthening of [8, Theorem 1] and first appeared in [3, Corollary 4].

COROLLARY 3.2. *For all $y\in P$*

$$\sum_{\{a\mid\rho(a)=\hat{0}\}}\mu_P(y,a)=\sum_{\{x\mid\pi(x)=y\}}\mu_Q(\hat{0},x).$$

PROOF. Evaluating the result from Theorem 3.1 at $n=-1$ all the terms on the left vanish except those where $\rho(a)=\hat{0}$. This follows from Lemma 2.2. If $\rho(a)=\hat{0}$, then $Z([\hat{0}, \rho(a)]; n)=1$ for all n. On the right-hand side $Z([\hat{0}, x]; -1)=\mu_Q(\hat{0}, x)$. The corollary follows.

COROLLARY 3.3 (WEISNER). *Let $a>\hat{0}$ in a finite lattice L. Then, for any $b \in L$*

$$\sum_{x \vee a=b} \mu(\hat{0}, x)=0.$$

PROOF. Let $P=[a, 1]$ and $Q=L$. Define $\pi: Q \rightarrow P$ by $\pi(q)=q \vee a$ and $\rho: P \rightarrow Q$ by $\rho(p)=p$. Then ρ, π form a Galois connection between P^* and Q, where P^* is the dual of P. Applying Corollary 3.2 setting $y=b$, gives the required result.

Next we wish to prove a zeta polynomial analogue to Rota's cross-cut theorem [8, Theorem 3]. Recall that a subset $X=\{x_1, x_2, \ldots, x_k\}$ of a poset P is called a cutset if every maximal chain intersects X. A cutset is a cross-cut if it is an antichain not $\hat{0}$ or $\hat{1}$. Let X be a cutset and $S \subset X$. Define

$$P_S=\{y \mid y \leqslant x_i \text{ for all } x_i \in S \quad \text{or} \quad y \geqslant x_i \text{ for all } x_i \in S\}.$$

THEOREM 3.4. *If X is a cutset of P then*

$$Z(P; n)=\sum_{\substack{S \subset X \\ |S| \geqslant 1}}(-1)^{|S|+1} Z(P_S; n).$$

PROOF. The proof is by inclusion–exclusion on multichains of size n. Let $N_=(S)$, $S \subset X$, be the number of multichains which are ordered with respect to S but not to any larger subset of X. That is, every element in the multichain is comparable to every element in S, but this is not true for any set larger than S. Define $N_{\geqslant}(S)=\sum_{T \supset S} N_=(T)$. Then by inclusion–exclusion we have

$$N_=(\varnothing)=\sum_S(-1)^{|S|} N_{\geqslant}(S).$$

Since X is a cutset, $N_=(\varnothing)=0$. $N_{\geqslant}(S)$ is just $Z(P_S; n)$ since a multichain is ordered with respect to S if and only if it lies in P_S. Observing that $N_{\geqslant}(\varnothing)=Z(P; n)$ the theorem follows.

COROLLARY 3.5 (THE CROSS-CUT THEOREM). *Let X be a cross-cut of a lattice L. Define q_i to be the number of i-subsets of X whose join is $\hat{1}$ and whose meet is $\hat{0}$. Then $\mu(\hat{0}, \hat{1})=q_2-q_3+\cdots(-1)^k q_k$.*

PROOF. If $S=\{s_1, s_2, \ldots, s_m\}$ is a subset of X, let $\bigvee S=s_1 \vee s_2 \cdots \vee s_m$ and $\bigwedge S=s_1 \wedge s_2 \cdots \wedge s_m$. Note that $P_S=\{y \mid y \leqslant \bigwedge S$ or $y \geqslant \bigvee S\}$. Evaluating the zeta polynomial expression of Theorem 3.4 we have

$$\mu_L(\hat{0}, \hat{1})=\sum_{\substack{S \subset X \\ |S| \geqslant 1}} \mu_{P_S}(\hat{0}, \hat{1})(-1)^{|S|+1}.$$

Unless $\bigvee S=\hat{1}$ and $\bigwedge S=\hat{0}$, P_S will have an element $x \neq \hat{0}$ or $\hat{1}$ which is comparable to every other element. From first principles it is evident that this implies $\mu_{P_S}(\hat{0}, \hat{1})=0$. Hence the only non-zero terms in the above expression are when $\bigvee S=\hat{1}$ and $\bigwedge S=\hat{0}$. This means that P_S is the 2-element chain. Hence $\mu_{P_S}(\hat{0}, \hat{1})=-1$. The corollary is immediate.

Note that the proof of Corollary 3.5 is sufficient to prove the stronger theorem of Björner [1, Theorem 2.5]. The details are left to the reader.

Theorem 3.4 can be abstracted in the following way: Let $P_1, P_2, \ldots, P_r$ be a collection of subsets of a poset P, unequal to P, such that every chain of P is contained in at least one of them. Let L be the poset of all intersections of the $\{P_i\}$ ordered by reverse inclusion. Note that L will have a $\hat{0}$, the null intersection.

THEOREM 3.6. *$0=\sum_{Q\in L}\mu(\hat{0}, Q)\, Z(Q; n)$ where L is described above.*

PROOF. The proof follows that of Theorem 3.4, only Möbius inversion over L is substituted for inclusion–exclusion.

4. The Crapo Complementation Theorem

The only major identity of Möbius functions not proven in the previous section is the Crapo complementation theorem [2]:

THEOREM. *Let L be a lattice. Fix $a\in L$. Then*

$$\mu_L(\hat{0}, \hat{1}) = \sum_{y,z} \mu(\hat{0}, y)\, \zeta(y, z)\, \mu(z, \hat{1}),$$

where the sum is over all ordered pairs (y, z) of complements of a in L.

In this section we present partial results toward that end. We will require the following lemma.

LEMMA 4.1. *Let P be a poset with $\hat{0}$ and $\hat{1}$. Then*

$$Z(P; -n) = \mu^n(\hat{0}, \hat{1}).$$

PROOF. See [9, p. 210].

Recall that a subset F of a poset P is called a filter if $x\in F$ implies that for all $y\geq x$, $y\in F$.

THEOREM 4.2. *Let F be a filter of a poset P. Let $b\in F$ and $a\leq b$ $a\notin F$. Then*

$$Z([a, b]; n) = \sum_{z\in F} Z([z, b]; n)\left[-\sum_{\substack{y\leq z\\ y\notin F}} Z([y, z]; -n) Z([a, y]; n)\right].$$

PROOF. We begin by establishing the equation

$$Z([a, b]; n) = \sum_{y,z\in F} Z([a, y]; n)\, Z([y, z]; -n)\, Z([z, b]; n). \qquad (*)$$

Summing over y first in the right-hand side of (*) we get

$$\sum_{y\in F} Z([a, y]; n) \sum_{y\leq z} Z([y, z]; -n)\, Z([z, b]; n). \qquad (**)$$

By Lemma 4.1, $Z([y, z]; -n) = \mu^n(y, z)$ and $Z([z, b]; n) = \zeta^n(z, b)$ and μ^n and ζ^n are inverses. Hence the second sum in (**) is $1(y, b)$. This proves equation (*). Equation (*) is the zeta polynomial analogue of Theorem 2 in [2].

If we sum over z first in the right-hand side of (*) we get

$$\sum_{z \in F} Z([z, b]; n) \sum_{\substack{y \leq z \\ y \in F}} Z([a, y]; n) Z([y, z]; -n). \tag{***}$$

Since ζ^n is the inverse of μ^n, the second sum of (***) can be rewritten as

$$-\sum_{\substack{y \leq z \\ y \notin F}} Z([a, y]; n)\, Z([y, z]; -n)$$

and the theorem is proved.

Evaluating Theorem 4.2 at $n = -1$ we get

$$\mu(a, b) = \sum_{z \in F} \mu(z, b)\Bigl(-\sum_{\substack{y \leq z \\ y \notin F}} \mu(a, y)\Bigr).$$

Using Crapo's notation, the sum

$$-\sum_{\substack{y \leq z \\ y \notin F}} \mu(a, y) = \mu_f(a, z)$$

by the recurrence formula for the Möbius function. So the result in Theorem 4.2 specializes to Theorem 1 of [2]. However, it is not a particularly aesthetic generalization. Moreover, it seems powerless in proving a generalization of the complementation theorem itself. It would be nice if such a generalization could be found.

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