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On stability crossing curves for general systems with two delays

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Abstract

For the general linear scalar time-delay systems of arbitrary order with two delays, this article provides a detailed study on the stability crossing curves consisting of all the delays such that the characteristic quasipolynomial has at least one imaginary zero. The crossing set, consisting of all the frequencies corresponding to all the points in the stability crossing curves, are expressed in terms of simple inequality constraints and can be easily identified from the gain response curves of the coefficient transfer functions of the delay terms. This crossing set forms a finite number of intervals of finite length. The corresponding stability crossing curves form a series of smooth curves except at the points corresponding to multiple zeros and a number of other degenerate cases. These curves may be closed curves, open ended curves, and spiral-like curves oriented horizontally, vertically, or diagonally. The category of curves are determined by which constraints are violated at the two ends of the corresponding intervals of the crossing set. The directions in which the zeros cross the imaginary

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axis are explicitly expressed. An algorithm may be devised to calculate the maximum delay deviation without changing the number of right half plane zeros of the characteristic quasipolynomial (and preservation of stability as a special case).

Keywords: Time delay; Difference-differential equations; Stability; Quasipolynomial

1. Introduction

The study of time-delay systems began as early as the 18th century. The topic received substantial attention due to its prevalence in many practical problems in biology, ecology, chemistry, physics, and numerous engineering disciplines. Indeed, it is the main subject of many books over the last few decades, see, for example, [1,12,13,18,19,22].

In this article, we will study the stability of the class of systems described by the equation

$$\sum_{l=0}^{2}\sum_{k=0}^{n} p_{lk}\frac{d^k x(t-\tau_l)}{dt^k} = 0, \tag{1.1}$$

where the coefficients p_{lk}, $l = 0, 1, 2$; $k = 0, 1, 2, \dots, n$ are real, and $\tau_0 = 0$. The stability of such a system is completely determined by the zeros of its characteristic quasipolynomial

$$p(s) = p_0(s) + p_1(s)e^{-\tau_1 s} + p_2(s)e^{-\tau_2 s}, \tag{1.2}$$

where

$$p_l(s) = \sum_{k=0}^{n} p_{lk}s^k.$$

We will study the change of system stability as the delays τ_1 and τ_2 vary.

The distribution of zeros of the characteristic quasipolynomial of time-delay systems and its implications for stability have been described in detail in the book by Bellman and Cooke [1]. While delays have often been regarded as having the tendency to cause instability, Cooke and Grossman [7] demonstrated, through a number of simple systems, that for an arbitrarily given integer N, it is possible to construct a system such that it switches from being stable to unstable and back to being stable at least N times as the delay increases. Another interesting study on such switches was conducted recently by Beretta and Kuang [2] regarding systems whose coefficients depend on the delays.

One problem, which typically admits simpler formulation, is the stability independent of delays. Hale et al. [17] described the necessary and sufficient conditions for the zeros not to reach the imaginary axis (which they refer to as being *hyperbolic*) as the delays vary either independently or in a linearly dependent fashion. These conditions include delay-independent stability as special cases. Chen and Latchman [5] proposed a frequency-sweeping algorithm to check delay-independent stability.

For systems with commensurate delays, i.e., when the delays are multiples of some constant, Walton and Marshall [32] described an elimination procedure to identify all the delay values in which the zeros cross the imaginary axis, and thus identifying the intervals of delays such that the system is stable. In its appendix, [32] also gave a correction and a list of references on an alternative method, known as the pseudodelay technique, based on a bilinear transformation proposed by Rekasius [27]. This technique was also used in a more recent paper by Olgac and Sipahi [25]. For systems that are stable when the delays are set to zero, Chen, et al. [6] gave another method which is based on the Orlando Theorem and involves calculating generalized eigenvalues.

For systems with incommensurate delays, Neimark's D-subdivision approach [11,19] can be used for many situations. Cooke and van den Driessche [8] proposed a procedure to vary the delays one at a time and identify the crossing points, which in theory can determine the number of right half plane (RHP) zeros for any given delays. However, it requires the identification of all the real zeros of quasipolynomials, and therefore, is computationally challenging. For example applications of such systems, see [22] for combustion systems, and [20] for biological systems. A particular case with two delays as applied to communication systems is considered in [23].

The special case of a two-delay system described by

$$\dot{x}(t) + ax(t) + bx(t-\tau_1) + cx(t-\tau_2) = 0 \tag{1.3}$$

has been studied by a number of authors. See, for example, the work by Nussbaum [24]. Another interesting study of a similar system was conducted by Ryan and Wei [26]. Hale and Huang [16] conducted a thorough study of (1.3), and gave a very thorough characterization of the boundary of the stability region of (τ_1, τ_2) connected to the origin. Such diagrams showing the boundaries of stability region are often known as the *stability charts*. The importance of stability charts in practical applications is well illustrated by Stépán [29]. The book [28] contains a rich collection of such stability charts in the space of delays or other parameters. Bélair and Campbell [3] conducted another such study. Part of this article can be considered as an extension of [16] to the general systems of arbitrary order and with neutral delays. Our approach here is more geometric rather than purely algebraic, which allows us to provide a more explicit analysis and made the generalization possible.

It should be pointed out that some attempt is also made to characterize such curves in [30], and parallel results for a more general class of systems described by state-space equations in [31]. The main results in these two articles are the observation of periodicity of the curves and crossing directions.

The article is organized as follows. Section 2 describes the problem setup. Section 3 discusses how to identify frequencies and delays such that zeros of the characteristic quasipolynomials may cross the imaginary axis, which we will refer to as the crossing set and the stability crossing curves, respectively. Section 4 elaborates on the crossing set and gives an exhaustive description of all the possible general forms of the stability crossing curves. Section 5 studies the smoothness of the stability crossing curves. Section 6 discusses the direction in which the zeros cross the imaginary axis as the delays cross the stability crossing curves. The article is concluded in Section 7 with a brief summary and some discussions.

2. Problem setup and background

Our notation is rather standard. $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{R}_+$ denotes the set of nonnegative real numbers. $\mathbb{R}^n$ and $\mathbb{R}_+^n$ are the sets of n-dimensional vectors with components in $\mathbb{R}$ and $\mathbb{R}_+$, respectively. $\mathbb{C}$ denotes the set of complex numbers, and $\mathbb{C}_+$ is the set of complex numbers with nonnegative real parts. We will often refer to $\mathbb{C}_+$ as the right half plane (RHP). The complement of $\mathbb{C}_+$ in $\mathbb{C}$ is referred to as the left half plane (LHP), which is the set of complex numbers with strictly negative real parts.

We will study the change of the number of zeros of (1.2) on $\mathbb{C}_+$ as the delays (τ_1, τ_2) vary on $\mathbb{R}_+^2$. For this purpose, we often write $p(s)$ in (1.2) as $p(s, \tau_1, \tau_2)$. Since the main objective of this article is to identify the regions of (τ_1, τ_2) in $\mathbb{R}_+^2$ such that $p(s)$ is stable, we will first exclude some simple trivial cases and restrict ourselves to $p(s, \tau_1, \tau_2)$ which satisfy the following conditions:

(I) Existence of principal term:

$$\deg\bigl(p_0(s)\bigr) \geqslant \max\bigl\{\deg\bigl(p_1(s)\bigr), \deg\bigl(p_2(s)\bigr)\bigr\}. \tag{2.1}$$

(II) Zero frequency

$$p_0(0) + p_1(0) + p_2(0) \neq 0. \tag{2.2}$$

(III) The polynomials $p_0(s)$, $p_1(s)$ and $p_2(s)$ do not have any common zeros.

(IV) Restriction on difference operator:

$$\lim_{s\to\infty} \bigl(\bigl|p_1(s)/p_0(s)\bigr| + \bigl|p_2(s)/p_0(s)\bigr|\bigr) < 1. \tag{2.3}$$

Indeed, if item (I) is not satisfied, the quasipolynomial cannot be stable for any positive delays [1].

If (II) is violated, then 0 is a zero of $p(s)$ for any $(\tau_1, \tau_2) \in \mathbb{R}_+^2$ and therefore, it can never be stable.

(III) is natural. If it is not satisfied, there exists a common factor $c(s) \neq$ constant, such that $p_l(s) = c(s)q_l(s)$, $l = 0, 1, 2$. Choose $c(s)$ be the highest possible order, then $q_l(s)$, $l = 0, 1, 2$ do not have any common zeros, and the delay-differential equation can be decomposed to an ordinary differential equation with characteristic polynomial $c(s)$ and a delay-differential equation with characteristic quasipolynomial

$$q_0(s) + q_1(s)e^{-\tau_1 s} + q_2(s)e^{-\tau_1 s},$$

which satisfies condition (III).

Regarding (IV), if the system is of retarded type, (2.3) is automatically satisfied since its left-hand side is zero. For systems of neutral type, let

$$c_l = \frac{p_{ln}}{p_{0n}} = \lim_{s\to\infty} p_l(s)/p_0(s), \quad l = 1, 2.$$

Then, it is known that the stability of the system (1.2) is possible only if the difference equation

$$x(t) + c_1 x(t - \tau_1) + c_2 x(t - \tau_2) = 0 \tag{2.4}$$

is exponentially stable [1]. The condition (2.3) guarantees the stability of (2.4). It is also a necessary condition for the continuity condition (Lemma 2.1 to be presented next) to be valid.

It can be argued that condition (I) is implied by (IV): if I is not satisfied, the left-hand side of (2.3) is infinite. It is also important to point out that a system satisfying (I) to (IV) is stable if and only if none of the zeros of the characteristic quasipolynomial is on the RHP.

What makes our discussions to follow meaningful is the continuity of the zeros with respect to the delay parameters as stated in the following lemma.

Lemma 2.1. *As the delays* (τ_1, τ_2) *continuously vary within* $\mathbb{R}_+^2$*, the number of zeros* (*counting multiplicity*) *of* $p(s, \tau_1, \tau_2)$ *on* $\mathbb{C}_+$ *can change only if a zero appears on or cross the imaginary axis.*

The lemma can be proven in a way very similar to [7]. From the Rouche theorem [21], all the finite zeros of $p(s, \tau_1, \tau_2)$ vary continuously with τ_1 and τ_2. Therefore, a root cannot suddenly disappear or appear or change its multiplicity at a finite point in the complex plane. Therefore, the only possibility of changing the number of RHP zeros without crossing the imaginary axis first is at ∞. However, this is not possible in our case: for either a retarded system or a neutral system satisfying (2.3), any zeros of sufficiently large magnitude have negative real parts for $\tau_1 \geqslant 0$, $\tau_2 \geqslant 0$.

In the special case that the system is stable when $\tau_1 = \tau_2 = 0$, and the stability region connected to the origin (such as the case discussed in [16]), we may also invoke the theorem due to Datko in [9]. It is also interesting to point out that the continuity in Lemma 2.1 no longer holds if (2.3) is violated as was shown in [9].

Due to the continuity, given $\tau_1 = \tau_1^0$ and $\tau_2 = \tau_2^0$, in principle, we may find the number of zeros of $p(s)$ on $\mathbb{C}_+$ using the following procedure:

(1) find the number of right half plane zeros of $p(s)$ with $\tau_1 = 0$ and $\tau_2 = 0$;
(2) form a curve in the τ_1–τ_2 plane within $\mathbb{R}_+^2$ initiating from the origin and ending at (τ_1^0, τ_2^0);
(3) find all the points of (τ_1, τ_2) in the curve such that there are zeros of $p(s)$ crossing the imaginary axis, and find the directions of crossing (from left to right, or the other way) as one moves along the curve.

By keeping a tally on the number of RHP zeros as we move along the curve, we can find the number of RHP zeros at $\tau_1 = \tau_1^0$ and $\tau_2 = \tau_2^0$, and, therefore, whether the system is stable or not with the given delays. For example, the method proposed in [8] uses the curve consisting of straight lines parallel to τ_1 and τ_2 axis, and the method proposed in [32] uses the straight line connecting the origin and (τ_1^0, τ_2^0). It is, therefore, of great interest to identify these crossing points, which will be the topic of the next section.

To conclude this section, we introduce the following definition.

Definition 2.2. Let $\mathcal{C}_k : [a, b] \to \mathbb{R}^2$, $k = 1, 2, \ldots$, be a series of curves satisfying

$$\mathcal{C}_k(b) - \mathcal{C}_k(a) = A, \quad k = 1, 2, \ldots,$$

where $A \in \mathbb{R}^2$ is a constant 2-dimensional vector independent of k, and

$$\mathcal{C}_{k+1}(a) = \mathcal{C}_k(b).$$

Then, the curve $\mathcal{C}$ formed by connecting all the curves $\mathcal{C}_k$, $k = 1, 2, \ldots,$

$$\mathcal{C} = \bigcup_{k=1}^{\infty} \mathcal{C}_k$$

is known as a *spiral-like* curve, and A is known as its *axis*. If in addition,

$$\mathcal{C}_{k+1}(\xi) = \mathcal{C}_k(\xi) + A \quad \text{for all } \xi \in [a, b],$$

then $\mathcal{C}$ is known as a *spiral*.

In other words, a spiral is forms by connecting identical curves head to tail. On the other hand, the composite curves in a spiral-like curve do not have to be identical. In this article, in a spiral-like curves, $\mathcal{C}_{k+1}$ can often be viewed as formed from $\mathcal{C}_k$ with a small deformation, which justifies the term "spiral-like curve."

3. Identification of crossing points

Let $\mathcal{T}$ denote the set of all the points of (τ_1, τ_2) in $\mathbb{R}_+^2$ such that $p(s)$ has at least one zero on the imaginary axis. Any $(\tau_1, \tau_2) \in \mathcal{T}$ is known as a *crossing point*. The set $\mathcal{T}$, which is the collection of all the crossing points, is known as the *stability crossing curves*. We will write

$$a_l(s) = p_l(s)/p_0(s), \quad l = 1, 2,$$

and

$$a(s, \tau_1, \tau_2) = 1 + a_1(s)e^{-\tau_1 s} + a_2(s)e^{-\tau_2 s}.$$

We will also write $a(s, \tau_1, \tau_2)$ as $a(s)$ when no confusion may arise. For given τ_1 and τ_2, as long as $p_0(s)$ does not have imaginary zeros, $p(s)$ and $a(s)$ share all the zeros in a neighborhood of the imaginary axis. Therefore, in general, we may obtain all the crossing points and directions of crossing (from LHP to RHP, for example) from the solutions of

$$a(s, \tau_1, \tau_2) = 0 \tag{3.1}$$

instead of $p(s, \tau_1, \tau_2) = 0$.

For each given $s = j\omega$, $\omega \neq 0$, we may consider the three terms in $a(j\omega, \tau_1, \tau_2)$ as three vectors in the complex plane, with the magnitudes 1, $|a_1(j\omega)|$, and $|a_2(j\omega)|$, respectively. Furthermore, if we adjust the values of τ_1 and τ_2, we may arbitrarily adjust the directions of the vectors represented by the second and third terms. Equation (3.1) means that if we put these vectors head to tail, they form a triangle as illustrated in Fig. 3.1. This allows us to conclude the following proposition.

Proposition 3.1. *For each ω, $\omega \neq 0$, $p_0(j\omega) \neq 0$, $s = j\omega$ can be a solution of $p(s, \tau_1, \tau_2) = 0$ for some $(\tau_1, \tau_2) \in \mathbb{R}_+^2$ if and only if*

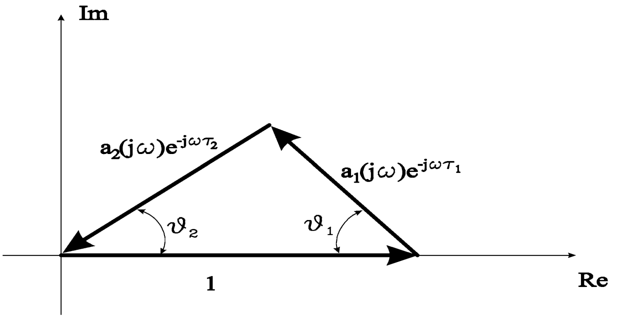

Fig. 3.1. Triangle formed by 1, $|a_1(j\omega)|$ and $|a_2(j\omega)|$.

$$\big|a_1(j\omega)\big| + \big|a_2(j\omega)\big| \geqslant 1, \tag{3.2}$$

$$-1 \leqslant \big|a_1(j\omega)\big| - \big|a_2(j\omega)\big| \leqslant 1. \tag{3.3}$$

For $\omega \neq 0$ satisfying $p_0(j\omega) = 0$, $s = j\omega$ can be a zero of $p(s, \tau_1, \tau_2)$ for some $(\tau_1, \tau_2) \in \mathbb{R}^2_+$ if and only if

$$\big|p_1(j\omega)\big| = \big|p_2(j\omega)\big|. \tag{3.4}$$

Proof. For $p_0(j\omega) \neq 0$, conditions (3.2) and (3.3) are obvious from the geometric point of view: a triangle can be formed by three line segments with arbitrary orientation if and only if the length of any one side does not exceed the sum of the other two sides. Notice also that $\angle[a_l(j\omega)e^{j\omega\tau_l}]$, $l = 1, 2$, can assume any value by adjusting τ_l, $l = 1, 2$, since $\omega \neq 0$. For the case of $p_0(j\omega) = 0$, the condition (3.4) is obvious considering again the fact that $e^{-j\omega\tau_l}$, $l = 1, 2$, can be used to change the directions but not the magnitudes of $p_l(j\omega)e^{-j\omega\tau_l}$, $l = 1, 2$. □

Due to symmetry and (2.2), we only need to consider positive ω. Let Ω be the set of all $\omega > 0$ which satisfy (3.2) and (3.3) if $p_0(j\omega) \neq 0$ and (3.4) if $p_0(j\omega) = 0$. We will refer to Ω as the *crossing set*. It contains all the ω such that some zero(s) of $p(s, \tau_1, \tau_2)$ may cross the imaginary axis at $j\omega$. Then, for any given $\omega \in \Omega$, $p_l(j\omega) \neq 0$, $l = 0, 1, 2$, one may easily find all the pairs of (τ_1, τ_2) satisfying (3.1) as follows:

$$\tau_1 = \tau_1^{u\pm}(\omega) = \frac{\angle a_1(j\omega) + (2u-1)\pi \pm \theta_1}{\omega} \geqslant 0, \quad u = u_0^{\pm}, u_0^{\pm} + 1, u_0^{\pm} + 2, \ldots, \tag{3.5}$$

$$\tau_2 = \tau_2^{v\pm}(\omega) = \frac{\angle a_2(j\omega) + (2v-1)\pi \mp \theta_2}{\omega} \geqslant 0, \quad v = v_0^{\pm}, v_0^{\pm} + 1, v_0^{\pm} + 2, \ldots, \tag{3.6}$$

where $\theta_1, \theta_2 \in [0, \pi]$ are the internal angles of the triangle in Fig. 3.1, and can be calculated by the law of cosine as

$$\theta_1 = \cos^{-1}\left(\frac{1 + |a_1(j\omega)|^2 - |a_2(j\omega)|^2}{2|a_1(j\omega)|}\right), \tag{3.7}$$

$$\theta_2 = \cos^{-1}\left(\frac{1+|a_2(j\omega)|^2-|a_1(j\omega)|^2}{2|a_2(j\omega)|}\right), \tag{3.8}$$

and $u_0^+, u_0^-, v_0^+, v_0^-$ are the smallest possible integers (may be negative and may depend on ω) such that the corresponding $\tau_1^{u_0^+ +}, \tau_1^{u_0^- -}, \tau_2^{v_0^+ +}, \tau_2^{v_0^- -}$ calculated are nonnegative. Notice, $u_0^+ \leqslant u_0^-$, $v_0^+ \geqslant v_0^-$. The position in Fig. 3.1 corresponds to $(\tau_1^{u+}, \tau_2^{v+})$. The position corresponding to $(\tau_1^{u-}, \tau_2^{v-})$ is its mirror image about the real axis.

Let $\mathcal{T}_{\omega,u,v}^+$ and $\mathcal{T}_{\omega,u,v}^-$ be the singletons defined by

$$\mathcal{T}_{\omega,u,v}^{\pm} = \left\{\left(\tau_1^{u\pm}(\omega), \tau_2^{v\pm}(\omega)\right)\right\},$$

and define

$$\mathcal{T}_\omega = \left(\bigcup_{\substack{u\geqslant u_0^+\\ v\geqslant v_0^+}} \mathcal{T}_{\omega,u,v}^+\right) \cup \left(\bigcup_{\substack{u\geqslant u_0^-\\ v\geqslant v_0^-}} \mathcal{T}_{\omega,u,v}^-\right).$$

Then $\mathcal{T}_\omega$ is the set of all (τ_1, τ_2) such that $p(s)$ has a zero at $s = j\omega$. In the following remark, we will discuss the degenerate cases of $p_k(j\omega) = 0$ for at least one k.

Remark 3.2. If $p_0(j\omega) = 0$, $\omega \in \Omega$. Then $p(j\omega) = 0$ and assumption (III) imply $|p_1(j\omega)| = |p_2(j\omega)| \neq 0$. In this case, $\mathcal{T}_\omega$ consists of the solutions of

$$\angle p_1(j\omega) - \omega\tau_1 + 2\pi u = \angle p_2(j\omega) - \omega\tau_2 + 2\pi v + \pi$$

in $\mathbb{R}_+^2$ for integers u, v. Instead of isolated points, $\mathcal{T}_\omega$ now consists of an infinite number of straight lines of slope 1 of equal distance.

On the other hand, if $p_0(j\omega) \neq 0$, $\omega \in \Omega$, and $p_1(j\omega) = 0$, then $a_1(j\omega) = 0$ and $|a_2(j\omega)| = 1$, we have $\theta_2 = 0$, and θ_1 can assume all the values in $[0, \pi]$, and $\mathcal{T}_{\omega,u,v}^{\pm}$ contains all the points calculated by (3.5) and (3.6) with $\theta_1 \in [0, \pi]$, $\theta_2 = 0$. The corresponding $\mathcal{T}_\omega$ is a series of horizontal lines. Similarly, for $\omega \in \Omega$ satisfying $p_0(j\omega) \neq 0$, $p_2(j\omega) = 0$, the corresponding $\mathcal{T}_{\omega,u,v}^{\pm}$ contains all the points calculated by (3.5) and (3.6) with $\theta_1 = 0$, $\theta_2 \in [0, \pi]$, and $\mathcal{T}_\omega$ is a series of vertical lines.

Obviously,

$$\mathcal{T} = \{\mathcal{T}_\omega \mid \omega \in \Omega\}.$$

Since the behavior of the degenerate cases discussed in the above remark is easily understood, for the clarity of presentation, we will exclude these degenerate cases from our discussions, and make the following nondegeneracy condition as our standing assumption unless otherwise pointed out.

(V) Nondegeneracy

$$p_l(j\omega) \neq 0 \quad \text{for all } \omega \in \Omega \text{ and } l = 0, 1, 2. \tag{3.9}$$

4. Stability crossing curves

In this section, we make some observations on the crossing set Ω and the stability crossing curves $\mathcal{T}$.

Proposition 4.1. *The crossing set Ω consists of a finite number of intervals of finite length, including the cases which may violate* (3.9).

Proof. Obviously, the number of points in Ω violating (3.9) is finite. Therefore, we only need to show that the set of points satisfying (3.2) and (3.3) consists of a finite number of intervals of finite size. First, we observe that there can only be a finite number of ω satisfying

$$\left|a_1(j\omega)\right| + \left|a_2(j\omega)\right| = 1 \tag{4.1}$$

since all such ω's are included in the solution set of the equation

$$\left|a_1(j\omega)\right|^2 + 2\left|a_1(j\omega)a_2(j\omega)\right| + \left|a_2(j\omega)\right|^2 = 1$$

which are in turn included in the solution set of the equation

$$4\left|a_1(j\omega)a_2(j\omega)\right|^2 = \left(1 - \left|a_1(j\omega)\right|^2 - \left|a_2(j\omega)\right|^2\right)^2,$$

which can be written as a polynomial equation of variable ω^2. This implies that the set of ω satisfying (3.2) consists of a finite number of intervals. Similarly, we can show that the set satisfying (3.3) also consists of a finite number of intervals since

$$\left|a_1(j\omega)\right| - \left|a_2(j\omega)\right| = 1 \quad \text{and} \tag{4.2}$$

$$\left|a_2(j\omega)\right| - \left|a_1(j\omega)\right| = 1 \tag{4.3}$$

can only have a finite number of solutions. The intersections are again a finite number of intervals. Furthermore, due to (2.3), any sufficiently large ω violates (3.2). Therefore, the lengths of all intervals have to be finite. □

Let these intervals be Ω_k, $k = 1, 2, \ldots, N$, arranged in such an order that the left end point of Ω_k increases with increasing k. Then

$$\Omega = \bigcup_{k=1}^{N} \Omega_k.$$

It is worth clarifying that $0 \notin \Omega$ by definition even if $\omega = 0$ satisfies (3.2) and (3.3). Indeed, if (3.2) and (3.3) are satisfied for $\omega = 0$ and sufficiently small positive values of ω, then, $\Omega_1 = (0, \omega_1^r]$, and we will let $\omega_1^l = 0$ in this case. Otherwise, $\Omega_1 = [\omega_1^l, \omega_1^r]$, $\omega_1^l \neq 0$. For $k \geqslant 2$, $\Omega_k = [\omega_k^l, \omega_k^r]$. We will subdivide the intervals if necessary so that for any $\omega \in (\omega_k^l, \omega_k^r)$, none of the three equations (4.1), (4.2) and (4.3) is satisfied.

Let

$$\mathcal{T}_{u,v}^{\pm k} = \bigcup_{\omega \in \Omega_k} \mathcal{T}_{\omega,u,v}^{\pm} = \left\{\left(\tau_1^{u\pm}(\omega), \tau_2^{v\pm}(\omega)\right) \;\middle|\; \omega \in \Omega_k\right\},$$

and

$$\mathcal{T}^k = \bigcup_{u=-\infty}^{\infty} \bigcup_{v=-\infty}^{\infty} \left(\mathcal{T}_{u,v}^{+k} \cup \mathcal{T}_{u,v}^{-k}\right) \cap \mathbb{R}_+^2 = \bigcup_{\omega \in \Omega_k} \mathcal{T}_\omega.$$

Then,

$$\mathcal{T} = \bigcup_{k=1}^{N} \mathcal{T}^k.$$

Note that we allow part of $\mathcal{T}_{u,v}^{+k}$ or $\mathcal{T}_{u,v}^{-k}$ to be outside of $\mathbb{R}_+^2$ in some cases for the convenience of discussions. We should, however, keep in mind that the part of $\mathcal{T}_{u,v}^{+k}$ or $\mathcal{T}_{u,v}^{-k}$ outside of $\mathbb{R}_+^2$ no longer represents the boundary of a meaningful change of the number of RHP zeros of $p(s)$. As is well known, $p(s)$ has an infinite number of RHP zeros if τ_1 or τ_2 assumes a negative value [1].

We will not restrict $\angle a_l(j\omega)$ to be within a range of 2π but make it a continuous function of ω within each Ω_k. This is always possible due to the way Ω_k is defined. As a result, for a fixed pair of integers (u, v), each $\mathcal{T}_{u,v}^{+k}$ or $\mathcal{T}_{u,v}^{-k}$ is a continuous curve. To study how each $\mathcal{T}_{u,v}^{+k}$ or $\mathcal{T}_{u,v}^{-k}$ is connected in $\mathcal{T}^k$ at the ends of Ω_k, we make the following observation: under our standing nondegenerate assumption (3.9), the end points of the intervals, ω_k^l, $k = 2, 3, \ldots$, and ω_k^r, $k = 1, 2, \ldots$, must satisfy one and only one of the three equations (4.1), (4.2) and (4.3). Accordingly, we can classify these end points into three types according to which equation $\omega = \omega_k^l$ or $\omega = \omega_k^r$ satisfies. The left end of Ω_1 may have an additional type if $\omega_1^l = 0$. A careful examination of Eqs. (3.5) and (3.6) allows us to arrive at the following list:

Type 1. (4.2) is satisfied. In this case, $\theta_1 = 0$, $\theta_2 = \pi$, and $\mathcal{T}_{u,v}^{+k}$ is connected with $\mathcal{T}_{u,v-1}^{-k}$ at this end.

Type 2. (4.3) is satisfied. In this case, $\theta_1 = \pi$, $\theta_2 = 0$, and $\mathcal{T}_{u,v}^{+k}$ is connected with $\mathcal{T}_{u+1,v}^{-k}$ at this end.

Type 3. (4.1) is satisfied. In this case, $\theta_1 = \theta_2 = 0$, and $\mathcal{T}_{u,v}^{+k}$ is connected with $\mathcal{T}_{u,v}^{-k}$ at this end.

Type 0. $\omega_k^l = 0$. This requires that $\omega = 0$ satisfy (3.2) and (3.3). In this case, as $\omega \to 0$, $\mathcal{T}_{u,v}^{+k}$ and $\mathcal{T}_{u,v}^{-k}$ approach ∞ with asymptotes passing through the points $(\hat{a}_1 \pm \hat{\theta}_1, \hat{a}_2 \mp \hat{\theta}_2)$ with slopes of

$$\frac{\tau_2^{v\pm}}{\tau_1^{u\pm}} \to \kappa_{u,v}^{\pm} = \frac{\angle a_2(0) + (2v-1)\pi \mp \theta_2(0)}{\angle a_1(0) + (2u-1)\pi \pm \theta_1(0)}, \tag{4.4}$$

where $\theta_1(0)$ and $\theta_2(0)$ are evaluated by (3.7) and (3.8) using $a_1(0)$ and $a_2(0)$, respectively, and

$$\hat{a}_l = \frac{d}{d\omega}\left[\angle a_l(j\omega)\right]_{\omega=0}, \tag{4.5}$$

$$\hat{\theta}_l = \frac{d}{d\omega}\theta_l(j\omega)|_{\omega=0}. \tag{4.6}$$

Correspondingly, we say an interval Ω_k is of type lr if the left end of Ω_k is of type l and its right end is of type r. There are a total of $4 \times 3 = 12$ possible types of such intervals.

Example 4.2. Consider a system with

$$a_1(s) = \frac{2.5}{s^2 + 2\zeta_1 s + 1}, \tag{4.7}$$

$$a_2(s) = \frac{1}{3s^2 + 6\zeta_2 s + 1}, \tag{4.8}$$

where $\zeta_1 = 1/\sqrt{2}$, $\zeta_2 = 0.1$. Figure 4.1 plots $|a_1(j\omega)| + |a_2(j\omega)|$ and $|a_1(j\omega)| - |a_2(j\omega)|$ against ω. The crossing set Ω can be easily identified from Fig. 4.1, it contains two intervals:

$$\Omega_1 = [0.346, 0.758] \quad \text{of type 11}, \qquad \Omega_2 = [1.333, 1.650] \quad \text{of type 13}.$$

Example 4.3. Figure 4.2 plots $|a_1(j\omega)| + |a_2(j\omega)|$ and $|a_1(j\omega)| - |a_2(j\omega)|$ against ω with

$$a_1(s) = \frac{3}{s^2 + 2s + 1}, \tag{4.9}$$

$$a_2(s) = \frac{9s + 1}{s^2 + 2s + 1}. \tag{4.10}$$

In this case, Ω contains two intervals:

$$\Omega_1 = [0.188, 0.453] \quad \text{of type 12}, \qquad \Omega_2 = [8.532, 9.217] \quad \text{of type 23}.$$

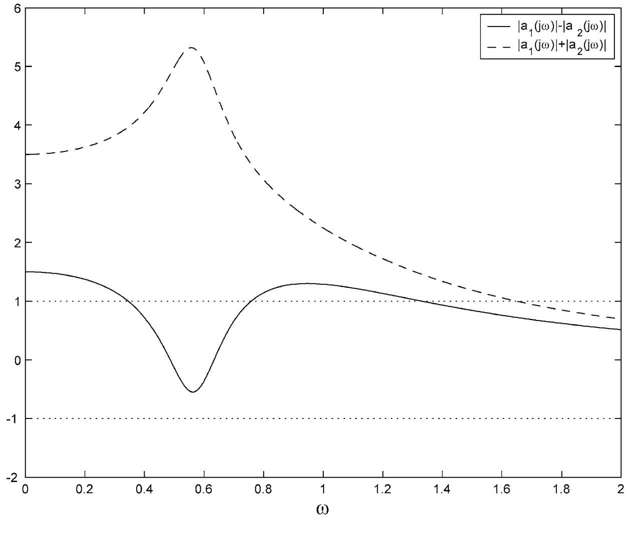

Fig. 4.1. $|a_1(j\omega)| \pm |a_2(j\omega)|$ versus ω for system represented by (4.7) and (4.8).

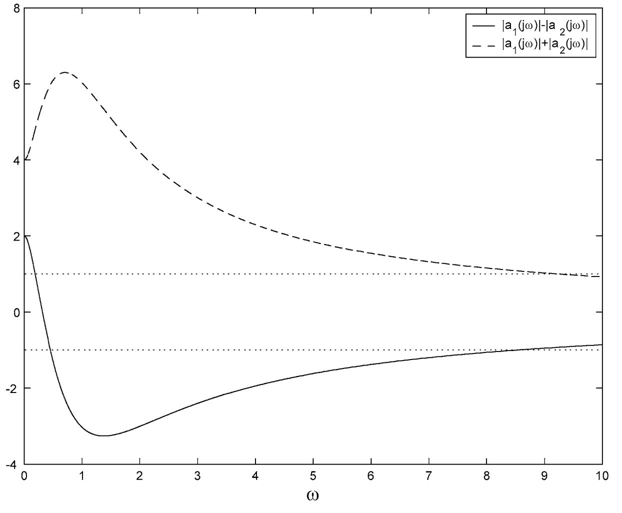

Fig. 4.2. $|a_1(j\omega)| \pm |a_2(j\omega)|$ versus ω for system represented by (4.9) and (4.10).

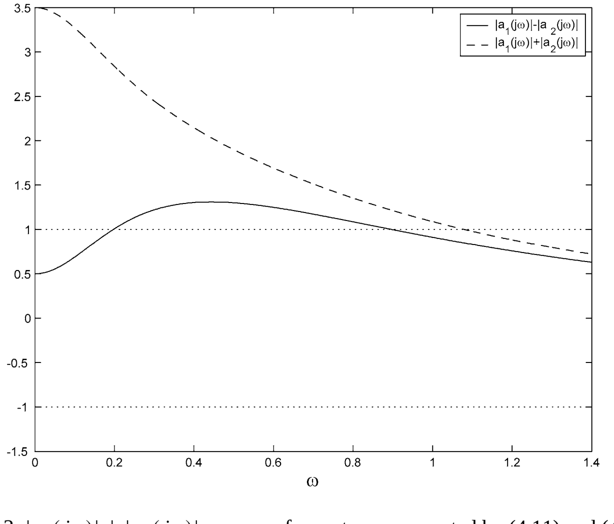

Fig. 4.3. $|a_1(j\omega)| \pm |a_2(j\omega)|$ versus ω for system represented by (4.11) and (4.12).

Example 4.4. Figure 4.3 plots $|a_1(j\omega)| + |a_2(j\omega)|$ and $|a_1(j\omega)| - |a_2(j\omega)|$ against ω with

$$a_1(s) = \frac{2}{s^2 + 2s + 1}, \tag{4.11}$$

$$a_2(s) = \frac{1.5}{16s^2 + 8s + 1}. \tag{4.12}$$

In this case, Ω contains two intervals:

$$\Omega_1 = (0, 0.197] \quad \text{of type 01}, \qquad \Omega_2 = [0.898, 1.079] \quad \text{of type 13}.$$

According to the types of Ω_k, $\mathcal{T}^k$ may have different shapes, as specified in the following proposition.

Proposition 4.5. *Under the standing assumption* (3.9), *the stability crossing curves* $\mathcal{T}^k$ *corresponding to* Ω_k *must be an intersection of* $\mathbb{R}_+^2$ *with a series of curves belonging to one of the following categories*:

(A) *A series of closed curves.*
(B) *A series of spiral-like curves with axes oriented either horizontally, vertically, or diagonally.*
(C) *A series of open ended curves with both ends approaching* ∞.

The rest of this section will be devoted to showing the validity of the above proposition by an exhaustive list, providing additional details as well as illustrative examples.

As an illustration of a series of *closed curves*, we examine $\mathcal{T}^k$ corresponding to Ω_k of type 11. In this case, for given u and v, $\mathcal{T}_{u,v}^{+k}$ and $\mathcal{T}_{u,v-1}^{-k}$ are connected on both ends to form a closed curve. As u and v vary, a series of deformed versions of such closed curves are generated along the horizontal and vertical directions. $\mathcal{T}^k$ is the intersection of $\mathbb{R}_+^2$ with this series of closed curves. Plotted in Fig. 4.4 is $\mathcal{T}^1$ of the system described in Example 4.2.

It is easily shown that a $\mathcal{T}^k$ corresponding to Ω_k of type 22 or type 33 also form a similar series of closed curves. In the case of type 22, a closed curved is formed by connecting both ends of $\mathcal{T}_{u,v}^{+k}$ and $\mathcal{T}_{u+1,v}^{-k}$. For type 33, a closed curve is formed by connecting both ends of $\mathcal{T}_{u,v}^{+k}$ and $\mathcal{T}_{u,v}^{-k}$.

To illustrate the case of *spiral-like curves with axes oriented diagonally*, consider $\mathcal{T}^k$ corresponding to Ω_k of type 12. In this case, $\mathcal{T}_{u,v}^{+k}$ is connected to $\mathcal{T}_{u+1,v}^{-k}$ at ω_k^r, and the

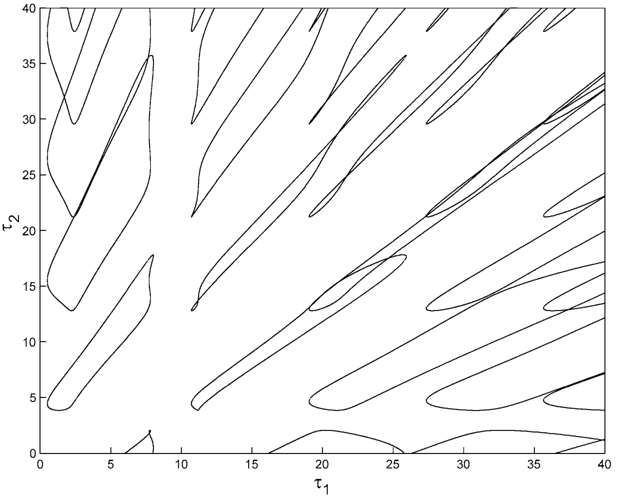

Fig. 4.4. $\mathcal{T}^1$ of the system in Example 4.2.

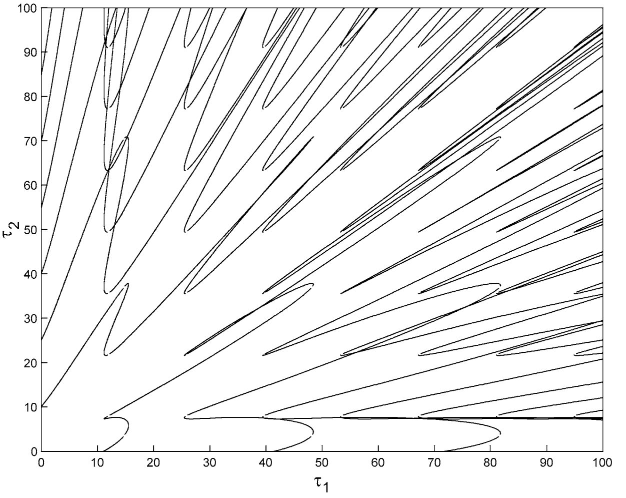

Fig. 4.5. $\mathcal{T}^1$ of the system in Example 4.3.

other end of $\mathcal{T}_{u+1,v}^{-k}$ is connected to $\mathcal{T}_{u+1,v+1}^{+k}$ at ω_k^l, which is again connected to $\mathcal{T}_{u+2,v+2}^{-k}$ at ω_k^r, and so on. According to Definition 2.2, with $\mathcal{C}_k = \mathcal{T}_{u,v}^{+k} \cup \mathcal{T}_{u+1,v}^{-k}$, it can be easily verified that this forms a spiral-like curve with the axis

$$\begin{aligned} A &= \left(\tau_1^{u+1-}\left(\omega_k^l\right) - \tau_1^{u+}\left(\omega_k^l\right), \tau_2^{v-}\left(\omega_k^l\right) - \tau_2^{v+}\left(\omega_k^l\right)\right) \\ &= \left(\frac{2p}{\omega_k^l}, \frac{2p}{\omega_k^l}\right), \quad \text{independent of } u \text{ (or } v\text{)}, \end{aligned}$$

forming a 45° from the horizontal. This spiral-like curve is repeated an infinite number of times in a deformed form as the difference between u and v changes. Shown in Fig. 4.5 is $\mathcal{T}^1$ for the system in Example 4.3.

We can observe that a $\mathcal{T}^k$ corresponding to Ω_k of type 21 also form such a series spiral-like curves with axes oriented diagonally. In this case $\mathcal{T}_{u,v}^{+k}$ is connected to $\mathcal{T}_{u+1,v}^{-k}$ at ω_k^l instead, and $\mathcal{T}_{u+1,v}^{-k}$ is connected to $\mathcal{T}_{u+1,v+1}^{+k}$ at ω_k^r, and so on.

To illustrate the case of *spiral-like curves with vertical axes*, consider $\mathcal{T}^k$ corresponding to Ω_k of type 13. In this case, $\mathcal{T}_{u,v}^{+k}$ is connected to $\mathcal{T}_{u,v}^{-k}$ at ω_k^r, and the other end of $\mathcal{T}_{u,v}^{-k}$ is connected to $\mathcal{T}_{u,v+1}^{+k}$ at ω_k^l, and so on. This forms a spiral-like curve with a vertical axis. This spiral-like curve is repeated in deformed form along the horizontal direction as u changes. Shown in Fig. 4.6 is $\mathcal{T}^2$ for the system in Example 4.2.

It is easily shown that $\mathcal{T}^k$ corresponding to Ω_k of type 31 is also in the form of a series of vertically oriented spiral-like curves, with $\mathcal{T}_{u,v}^{+k}$ and $\mathcal{T}_{u,v}^{-k}$ connected at ω_k^l, and $\mathcal{T}_{u,v}^{-k}$ and $\mathcal{T}_{u,v+1}^{+k}$ connected at ω_k^r, and so on.

The curves of $\mathcal{T}^k$ corresponding to Ω_k of type 23 and 32 are in the form of a series of *spiral-like curves with horizontal axes*. For type 23, $\mathcal{T}_{u,v}^{+k}$ is connected to $\mathcal{T}_{u+1,v}^{-k}$ at ω_k^l, and the other end of $\mathcal{T}_{u+1,v}^{-k}$ is connected to $\mathcal{T}_{u+1,v}^{+k}$ at ω_k^r, and so on. For type 32, $\mathcal{T}_{u,v}^{+k}$ and $\mathcal{T}_{u+1,v}^{-k}$ are connected at ω_k^r, and $\mathcal{T}_{u+1,v}^{-k}$ and $\mathcal{T}_{u+1,v}^{+k}$ are connected at ω_k^l, and so on.

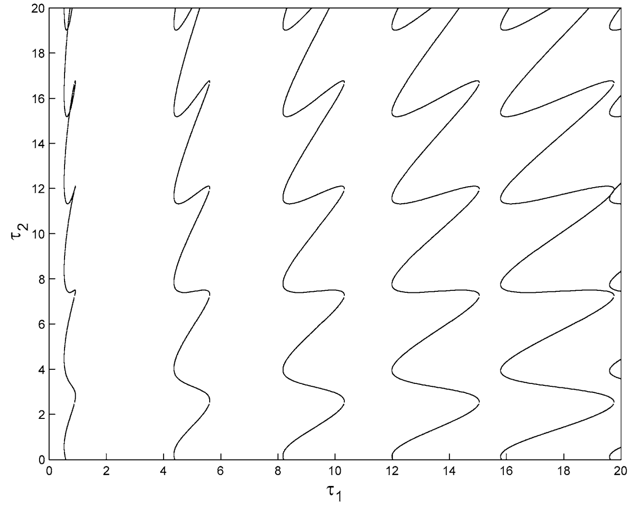

Fig. 4.6. $\mathcal{T}^2$ of the system in Example 4.2.

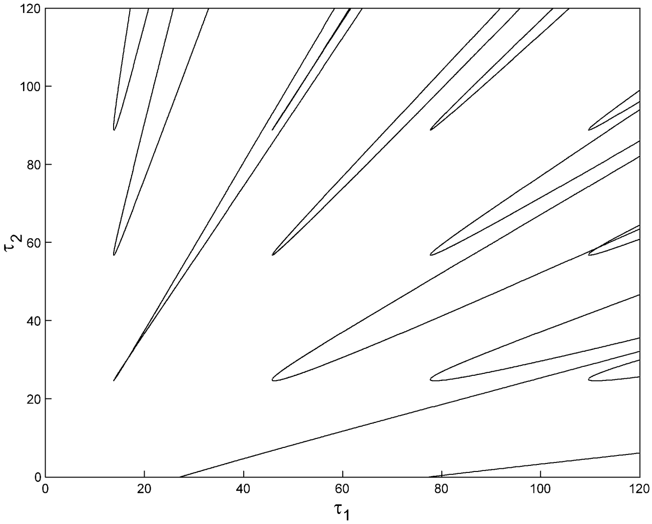

Fig. 4.7. $\mathcal{T}^1$ of the system in Example 4.4.

Corresponding to $\Omega_1 = (0, \omega_1^r]$, $\mathcal{T}^1$ is a series of *open-ended curves*. For type 01, $\mathcal{T}_{u,v}^{-1}$ and $\mathcal{T}_{u,v+1}^{+1}$ are connected at ω_1^r. The other end of $\mathcal{T}_{u,v}^{-1}$ extends to infinity with asymptote of a slope $\kappa_{u,v}^-$ passing through the point $(\hat{a}_1 - \hat{\theta}_1, \hat{a}_2 + \hat{\theta}_2)$. The other end of $\mathcal{T}_{u,v+1}^{+1}$ extends to infinity with asymptote of a slope $\kappa_{u,v+1}^+$ passing through the point $(\hat{a}_1 + \hat{\theta}_1, \hat{a}_2 - \hat{\theta}_2)$. This pattern is repeated in a deformed form in both horizontal and vertical directions. Note also that the slopes also change for different u and v. Shown in Fig. 4.7 is $\mathcal{T}^1$ of the system described in Example 4.4.

It is easy to show that $\mathcal{T}^1$ corresponding to Ω_1 of type 02 and type 03 also forms open-ended curves. For type 02, $\mathcal{T}_{u,v}^{+1}$ and $\mathcal{T}_{u+1,v}^{-1}$ are connected at ω_1^r. The other ends of $\mathcal{T}_{u,v}^{+1}$ and $\mathcal{T}_{u+1,v}^{-1}$ extend to infinity with slopes $\kappa_{u,v}^{+}$ and $\kappa_{u+1,v}^{-}$, respectively. For type 03, $\mathcal{T}_{u,v}^{+1}$ and $\mathcal{T}_{u,v}^{-1}$ are connected at ω_1^r. The other ends of $\mathcal{T}_{u,v}^{+1}$ and $\mathcal{T}_{u,v}^{-1}$ extend to infinity with slopes $\kappa_{u,v}^{+}$ and $\kappa_{u,v}^{-}$, respectively.

Thus far, we have exhausted all 12 types of Ω_k. Therefore, the proof of Proposition 4.5 is complete.

5. Tangents and smoothness

In this section, for a given k, we will discuss the smoothness of the curves in $\mathcal{T}^k$ and thus $\mathcal{T} = \bigcup_{k=1}^{N} \mathcal{T}^k$. We will understand a k is given and will refer to $\mathcal{T}^k$ without further comments. In addition to the explicit formulas (3.5) and (3.6), we will also use an approach similar to the one described in [10, Chapter 11] based on the implicit function theorem. For this purpose, we consider τ_1 and τ_2 as implicit functions of $s = j\omega$ defined by (3.1). As s moves along the imaginary axis, $(\tau_1,\tau_2) = (\tau_1^{u\pm}(\omega), \tau_2^{v\pm}(\omega))$ moves along $\mathcal{T}^k$. For a given $\omega \in \Omega_k$, let

$$\begin{aligned} R_0 &= \mathrm{Re}\left(\frac{j}{s}\frac{\partial a(s,\tau_1,\tau_2)}{\partial s}\right)_{s=j\omega} \\ &= \frac{1}{\omega}\mathrm{Re}\left(\left[a_1'(j\omega) - \tau_1 a_1(j\omega)\right]e^{-j\tau_1\omega} + \left[a_2'(j\omega) - \tau_2 a_2(j\omega)\right]e^{-j\tau_2\omega}\right), \qquad (5.1) \\ I_0 &= \mathrm{Im}\left(\frac{j}{s}\frac{\partial a(s,\tau_1,\tau_2)}{\partial s}\right)_{s=j\omega} \\ &= \frac{1}{\omega}\mathrm{Im}\left(\left[a_1'(j\omega) - \tau_1 a_1(j\omega)\right]e^{-j\tau_1\omega} + \left[a_2'(j\omega) - \tau_2 a_2(j\omega)\right]e^{-j\tau_2\omega}\right), \qquad (5.2) \end{aligned}$$

and

$$R_l = -\mathrm{Re}\left(\frac{1}{s}\frac{\partial a(s,\tau_1,\tau_2)}{\partial \tau_k}\right)_{s=j\omega} = \mathrm{Re}\left(a_k(j\omega)e^{-j\tau_k\omega}\right), \qquad (5.3)$$

$$I_l = -\mathrm{Im}\left(\frac{1}{s}\frac{\partial a(s,\tau_1,\tau_2)}{\partial \tau_k}\right)_{s=j\omega} = \mathrm{Im}\left(a_k(j\omega)e^{-j\tau_k\omega}\right), \qquad (5.4)$$

for $l = 1, 2$. Then, since $a(s,\tau_1,\tau_2)$ is an analytic function of s, τ_1 and τ_2, the implicit function theorem indicates that the tangent of $\mathcal{T}^k$ can be expressed as

$$\begin{pmatrix} \frac{d\tau_1}{d\omega} \\ \frac{d\tau_2}{d\omega} \end{pmatrix} = \begin{pmatrix} R_1 & R_2 \\ I_1 & I_2 \end{pmatrix}^{-1} \begin{pmatrix} R_0 \\ I_0 \end{pmatrix} = \frac{1}{R_1 I_2 - R_2 I_1}\begin{pmatrix} R_0 I_2 - I_0 R_2 \\ I_0 R_1 - R_0 I_1 \end{pmatrix}, \qquad (5.5)$$

provided that

$$R_1 I_2 - R_2 I_1 \neq 0. \qquad (5.6)$$

It follows from a well-known result [4,15] that $\mathcal{T}^k$ is smooth everywhere except possibly at the points where either (5.6) is not satisfied, or when

$$\frac{d\tau_1}{d\omega} = \frac{d\tau_2}{d\omega} = 0. \tag{5.7}$$

A careful examination of these cases allows us to conclude that

Proposition 5.1. *Under the standing assumptions including* (3.9), *the curves in* $\mathcal{T}^k$ *are smooth everywhere except possibly at the degenerate points corresponding to* ω *in any one of the following three cases*:

Case 1. $s = j\omega$ *is a multiple solution of* $a(s) = 0$.
Case 2. ω *is a type* 3 *end point of* Ω_k, *and* $\frac{d}{d\omega}(|a_1(j\omega)| + |a_2(j\omega)|) = 0$.
Case 3. ω *is a type* 1 *or type* 2 *end point of* Ω_k, *and* $\frac{d}{d\omega}(|a_1(j\omega)| - |a_2(j\omega)|) = 0$.

Furthermore, if the point is not among the three cases, then the tangents of the curves in $\mathcal{T}^k$ *can be expressed as*

$$\frac{d\tau_2}{d\tau_1} = \begin{cases} \frac{1/\tan\varphi_0 - 1/\tan\varphi_1}{1/\tan\varphi_0 - 1/\tan\varphi_2}, & \omega \in (\omega_k^l, \omega_k^r), \\ -\frac{|a_1(j\omega)|}{|a_2(j\omega)|}, & \omega \text{ is a type 3 end point of } \Omega_k, \\ \frac{|a_1(j\omega)|}{|a_2(j\omega)|}, & \omega \text{ is a type 1 or 2 end point of } \Omega_k, \end{cases} \tag{5.8}$$

where

$$\varphi_0 = \angle\big(\big[a_1'(j\omega) - \tau_1 a_1(j\omega)\big]e^{-j\tau_1\omega} + \big[a_2'(j\omega) - \tau_2 a_2(j\omega)\big]e^{-j\tau_2\omega}\big),$$
$$\varphi_k = \angle\big(a_k(j\omega)e^{-j\tau_k\omega}\big), \quad k = 1, 2.$$

Proof. It is sufficient to show that

(I) If (5.7) is satisfied, then $s = j\omega$ is a multiple solution of $a(s) = 0$.
(II) Condition (5.6) is satisfied for all $\omega \in (\omega_k^l, \omega_k^r)$. Furthermore, the tangent can be written as the first expression of (5.8) if it is not case 1.
(III) If (5.6) is not satisfied, but it is not among the three cases, then the curves are still smooth at these points, the tangent still exists and can be written as the second or the third expression of (5.8).

To show (I), we observe that (5.7) can be true only if $R_0 = I_0 = 0$. This means that

$$\left.\frac{\partial a(s, \tau_1, \tau_2)}{\partial s}\right|_{s=j\omega} = 0, \tag{5.9}$$

in view of the expression of R_0 and I_0. But (5.9) and $a(j\omega, \tau_1, \tau_2) = 0$ implies that $j\omega$ is a multiple solution of $a(j\omega) = 0$.

Now consider (II). Let $s = j\omega_0$ satisfy

$$R_1 I_2 - R_2 I_1 = 0. \tag{5.10}$$

From the definition of R_k and I_k, $k = 1, 2$, (5.10) means that $a_1(j\omega_0)e^{-j\tau_1\omega_0}$ and $a_2(j\omega_0)e^{-j\tau_2\omega_0}$ are either in the same or opposite directions. But these two vectors have to add up to -1 in order to satisfy $a(j\omega_0, \tau_1, \tau_2) = 0$, and therefore, they are both real and satisfy one and only one of the relations expressed by (4.1)–(4.3), i.e., ω_0 is an end point of Ω_k of type 1, 2 or 3. Thus, we have shown that (5.6) is satisfied for all $\omega \in (\omega_k^l, \omega_k^r)$. Furthermore, according to (5.5), we can express the tangent as

$$\frac{d\tau_2}{d\tau_1} = \frac{I_0R_1 - R_0I_1}{R_0I_2 - I_0R_2} = \left(\frac{I_1}{I_2}\right)\left(\frac{R_1/I_1 - R_0/I_0}{R_0/I_0 - R_2/I_2}\right),$$

which can be written as the first expression of (5.8) in view of

$$\tan\varphi_k = I_k/R_k, \quad k = 0, 1, 2,$$

and the fact that (3.1) implies

$$I_1 = -I_2.$$

Thus we have proven (II).

To show (III), since (5.6) is not satisfied, it has to be an end point of Ω_k of type 1, 2, or 3. To be specific, let $\omega_0 = \omega_k^l$ be a type 3 left end point of Ω_k. For a sufficiently small $\varepsilon > 0$, $R_1I_2 - R_2I_1 \neq 0$ for $\omega \in (\omega_0, \omega_0 + \varepsilon)$. Since it is not a point of Case 1, either $R_0(j\omega_0) \neq 0$ or $I_0(j\omega_0) \neq 0$. Without loss of generality, assume that $I_0(j\omega_0) \neq 0$. Then for $\omega \in (\omega_0, \omega_0 + \varepsilon)$,

$$\frac{d\tau_2}{d\tau_1} = \frac{1/\tan\varphi_0 - 1/\tan\varphi_1}{1/\tan\varphi_0 - 1/\tan\varphi_2},$$

and

$$\lim_{\omega\to\omega_0^+} 1/\tan\varphi_0 < \infty.$$

We will show that

$$\lim_{\omega\to\omega_0^+} \frac{d\tau_2}{d\tau_1}$$

exists and approaches the second expression of (5.8). Since

$$\varphi_1 = \angle\left[a_1(j\omega)e^{-j\tau_1\omega}\right] = -(2u - 1)\pi \mp \theta_1,$$
$$\varphi_2 = \angle\left[a_2(j\omega)e^{-j\tau_2\omega}\right] = -(2v - 1)\pi \pm \theta_2,$$

and

$$\lim_{\omega\to\omega_0} \tan\varphi_k = 0,$$

we have

$$\lim_{\omega\to\omega_0^+} \frac{d\tau_2}{d\tau_1} = \lim_{\omega\to\omega_0^+} \frac{\tan\varphi_2}{\tan\varphi_1} = -\lim_{\omega\to\omega_0^+} \frac{\theta_2}{\theta_1}.$$

For $\omega \in (\omega_0, \omega_0 + \varepsilon)$, let

$$\left|a_k(j\omega)\right| = \left|a_k(j\omega_0)\right| + \delta_k, \quad k = 1, 2.$$

Then we can show from (3.7) and (3.8) that

$$\cos\theta_k(j\omega) = 1 - \frac{|a_{2-k}(j\omega_0)|}{|a_k(j\omega_0)|}(\delta_1+\delta_2) + o\big(|\delta_1|+|\delta_2|\big).$$

Since it is not Case 2, $(\delta_1+\delta_2)$ is an infinitesimal of the same order as $\omega-\omega_0$, and therefore $o(|\delta_1|+|\delta_2|)$ is indeed higher order infinitesimal than $(\delta_1+\delta_2)$. On the other hand, using Taylor expansion, we have

$$\cos\theta_k(j\omega) = 1 - \frac{\theta_k^2}{2} + o\big(\theta_k^2\big).$$

This allows us to conclude that

$$\frac{\theta_k^2}{2} = \frac{|a_{2-k}(j\omega_0)|}{|a_k(j\omega_0)|}(\delta_1+\delta_2) + o\big(|\delta_1|+|\delta_2|\big).$$

From the above equation and the fact that θ_1 and θ_2 are of the same sign, we conclude that

$$\lim_{\omega\to\omega_0}\frac{\theta_2}{\theta_1} = \frac{|a_1(j\omega_0)|}{|a_2(j\omega_0)|}.$$

We have thus concluded the proof of (III) for the case of ω_0 being a type 3 left end point. The idea is very similar for the other cases, and the details are omitted here. □

From the proof, we see that an alternative expression of $\frac{d\tau_2}{d\tau_1}$ is

$$\frac{d\tau_2}{d\tau_1} = \frac{I_0R_1 - R_0I_1}{R_0I_2 - I_0R_2}, \tag{5.11}$$

which is still valid even when (5.10) is satisfied. The above proposition indicates that, except for the three degenerate cases, the closed curves, the spiral-like curves, or the open ended curves are all smooth curves even though the parameterization in terms of ω reverses direction at each end of $\mathcal{T}_{u,v}^{+k}$ or $\mathcal{T}_{u,v}^{-k}$. Another interesting observation from the second and the third expression of (5.8) is that, corresponding to the same end of Ω_k, the tangents of all the curves $\mathcal{T}_{u,v}^{+k}$ (or $\mathcal{T}_{u,v}^{-k}$) have the identical slope, i.e., they are independent of u or v, as long as it does not approach ∞ (type 0), i.e., as long as it is connected with another section of curve $\mathcal{T}_{u',v'}^{-k}$ for some u' and v' (type 1, 2, or 3). This strongly suggests that all $\mathcal{T}_{u,v}^{+k}$ (or $\mathcal{T}_{u,v}^{-k}$) have similar topological structure for all u and v. The extent of such similarity, such as the intersections between different sections represented by $\mathcal{T}_{u,v}^{+k}$ and $\mathcal{T}_{u,v}^{-k}$ for different u and v is an interesting topic of further investigation, and has important implications on the stability analysis.

6. Direction of crossing

Next, we will discuss the direction in which the solutions of (3.1) cross the imaginary axis as (τ_1,τ_2) deviates from a curve in $\mathcal{T}^k$. We will call the direction of the curve that corresponds to increasing ω the *positive direction*. Notice, as the curve passes through the points corresponding to the end points of Ω_k, the positive direction is reversed. We will also call the region on the left-hand side as we head in the positive direction of the curve *the region on the left*. Again, due to the possible reversion of parameterization, the same region may be considered on the left with respect to one point of the curve, and be considered as on the right on another point of the curve.

For the purpose of discussing the direction of crossing, we need to consider τ_1 and τ_2 as functions of $s = \sigma + j\omega$, i.e., functions of two real variables σ and ω, and partial derivative notation needs to be adopted instead. Since the tangent of $\mathcal{T}^k$ along the positive direction is $(\partial\tau_1/\partial\omega, \partial\tau_2/\partial\omega)$, the normal to $\mathcal{T}^k$ pointing to the left-hand side of the positive direction is $(-\partial\tau_2/\partial\omega, \partial\tau_1/\partial\omega)$. Also, as a pair of complex conjugate solutions of (3.1) cross the imaginary axis to the RHP, (τ_1, τ_2) moves along the direction $(\partial\tau_1/\partial\sigma, \partial\tau_2/\partial\sigma)$. We can therefore conclude that if the inner product of these two vectors are positive, i.e.,

$$\left[\frac{\partial\tau_1}{\partial\omega}\frac{\partial\tau_2}{\partial\sigma} - \frac{\partial\tau_2}{\partial\omega}\frac{\partial\tau_1}{\partial\sigma}\right]_{s=j\omega} > 0, \tag{6.1}$$

the region on the left of $\mathcal{T}^k$ at ω has two more solutions on the RHP. On the other hand, if the inequality in (6.1) is reversed, then the region on the left of $\mathcal{T}^k$ has two fewer solutions on the right-hand side of the complex plane. We can very easily express, parallel to (5.5), that,

$$\begin{aligned}\begin{pmatrix} \frac{\partial\tau_1}{\partial\sigma} \\ \frac{\partial\tau_2}{\partial\sigma} \end{pmatrix}_{s=j\omega} &= \begin{pmatrix} R_1 & R_2 \\ I_1 & I_2 \end{pmatrix}^{-1} \begin{pmatrix} I_0 \\ -R_0 \end{pmatrix} \\ &= \frac{1}{R_1 I_2 - R_2 I_1} \begin{pmatrix} R_0 R_2 + I_0 I_2 \\ -R_0 R_1 - I_0 I_1 \end{pmatrix},\end{aligned} \tag{6.2}$$

where R_l and I_l, $l = 0, 1, 2$, are defined in (5.1) to (5.4). This allows us to arrive at the following proposition.

Proposition 6.1. *Let* $\omega \in (\omega_k^l, \omega_k^r)$ *and* $(\tau_1, \tau_2) \in \mathcal{T}^k$ *such that* $j\omega$ *is a simple solution of* $a(j\omega, \tau_1, \tau_2) = 0$, *and*

$$a(j\omega', \tau_1, \tau_2) \neq 0 \quad \text{for any } \omega' > 0,\ \omega' \neq \omega. \tag{6.3}$$

Then as (τ_1, τ_2) *moves from the region on the right to the region on the left of the corresponding curve in* $\mathcal{T}^k$, *a pair of solutions of* (3.1) *cross the imaginary axis to the right if*

$$\operatorname{Im}\big(a_1(j\omega)a_2(-j\omega)e^{j\omega(\tau_2-\tau_1)}\big) = R_2 I_1 - R_1 I_2 > 0. \tag{6.4}$$

The crossing is in the opposite direction if the inequality is reversed.

Proof. Direct calculation shows that

$$\left[\frac{\partial\tau_1}{\partial\omega}\frac{\partial\tau_2}{\partial\sigma} - \frac{\partial\tau_2}{\partial\omega}\frac{\partial\tau_1}{\partial\sigma}\right]_{s=j\omega} = \big(R_0^2 + I_0^2\big)(R_2 I_1 - R_1 I_2)/(R_1 I_2 - R_2 I_1)^2.$$

Therefore, (6.1) can be written as

$$R_2 I_1 - R_1 I_2 > 0,$$

which is (6.4). $\square$

The condition (6.3) means that (τ_1, τ_2) is not an intersection point of two curves or different sections of a single curve in $\mathcal{T}$. It can be verified that the expression on the left hand side of (6.4) reaches zero as ω reaches an end point of Ω_k of type 1, 2 or 3. From there on, as the curve continues, ω reverses direction, and the left hand side of (6.4) changes sign. This is expected since for the same region, the "left-hand side" becomes "right-hand side" as the curve goes through this point.

Any given direction, (d_1, d_2), is to the left-hand side of the curve if its inner product with the left-hand side normal $(-\partial\tau_2/\partial\omega, \partial\tau_1/\partial\omega)$ is positive, i.e.,

$$-d_1\partial\tau_2/\partial\omega + d_2\partial\tau_1/\partial\omega > 0, \tag{6.5}$$

from which we have the following result.

Corollary 6.2. *Let ω, τ_1 and τ_2 satisfy the same condition as Proposition* 6.1. *Then as (τ_1, τ_2) crosses the curve along the direction (d_1, d_2), a pair of solutions of* (3.1) *cross the imaginary axis to the right if*

$$d_1(R_0I_1 - I_0R_1) + d_2(R_0I_2 - I_0R_2) > 0. \tag{6.6}$$

The crossing is in the opposite direction if the inequality is reversed.

Proof. Writing out the left-hand side, then (6.5) becomes

$$\big[d_1(R_0I_1 - I_0R_1) + d_2(R_0I_2 - I_0R_2)\big]/(R_2I_1 - R_1I_2) > 0. \tag{6.7}$$

If (d_1, d_2) is in the same side as the left-hand side normal, then, as we move along the (d_1, d_2) direction, the crossing is from the LHP to the RHP if the left-hand sides of (6.7) and (6.4) have the same sign, i.e., their product is positive. □

7. Conclusions and discussions

A detailed study is conducted regarding the details of the stability crossing curves in the delay parameter space for the two-delay case. The set of frequencies with possible crossing can be expressed by three constraints. This set forms a finite number of intervals Ω_k, $k = 1, 2, \ldots, N$, of finite length. Other than a few degenerate cases, the set of delay parameters form a series of smooth curves. These curves may be closed, open ended, and spiral-like with axis in the horizontal, vertical, or diagonal directions. The category of curves are determined by which constraints are violated at the two ends of Ω_k. The invariance of tangents in different curves or different parts of spiral-like curves strongly suggests similar topological structure of these series of curves. The condition for each direction of crossing is explicitly expressed, and found to be invariant in some special directions for all curves in the series.

Based on the results, an algorithm to calculate the maximum deviation of delays without changing the number of RHP zeros of characteristic quasipolynomial can be devised as discussed in detail in [14].

There are a number of interesting topics worth further investigation. The topological structure of the series of curves can be further studied. The details of the cases with more than two delays remains challenging for practical calculation.

It is interesting to look at all the degenerate cases, listed as follows:

1. $p_0(j\omega) = 0$, $|p_1(j\omega)| = |p_2(j\omega)| \neq 0$;
2. $p_1(j\omega) = 0$, $|p_0(j\omega)| = |p_2(j\omega)| \neq 0$;
3. $p_2(j\omega) = 0$, $|p_0(j\omega)| = |p_1(j\omega)| \neq 0$;
4. $a(j\omega, \tau_1, \tau_2) = 0$, $a'(j\omega, \tau_1, \tau_2) = 0$ (multiple solution case);
5. $|a_1(j\omega)| + |a_2(j\omega)| = 1$, $\frac{d}{d\omega}(|a_1(j\omega)| + |a_2(j\omega)|) = 0$;
6. $||a_1(j\omega)| - |a_2(j\omega)|| = 1$, $\frac{d}{d\omega}(|a_1(j\omega)| - |a_2(j\omega)|) = 0$.

The first three cases represent three equations and one variable ω. The fourth case represents 4 equations with three variables ω, τ_1 and τ_2. The last two cases represent two equations with one variable ω. Therefore, all the cases are generically not present since the number of equations exceeds the number of variables. They typically represent points of bifurcation. Also, the fourth to the sixth case have codimension 1. In other words, if the system depends on one parameter in addition to the two delays, then we generically should expect these degenerate points to appear, and the geometry of $\mathcal{T}$ changes as the parameter passes through these points. The first three cases are of codimension 2, it requires at least two additional parameters for them to appear generically.

There are other structural changes which are not represented by these conditions, for example, the intersection of curves from different branches of $\mathcal{T}_k$. These bifurcation of global nature cannot be detected using local analysis adopted here.

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