Existence of positive periodic solutions of impulsive functional differential equations with two parameters

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Abstract

In this paper, we employ a well-known fixed-point index theorem to study the existence and non-existence of positive periodic solutions for the periodic impulsive functional differential equations with two parameters. Several existence and non-existence results are established.

Keywords: Periodic solution; Impulse effect; Functional differential equation

1. Introduction and preliminaries

The theory and applications of impulsive functional differential equations are emerging as an important area of investigation, since it is far richer than the corresponding theory of non-impulsive functional differential equations. Various population models, which are characterized by the fact that per sudden changing of their state and process under depends on their prehistory at each moment of time, can be expressed by impulsive differential equations with deviating argument, as population dynamics, ecology and epidemic, etc. We note that the difficulties dealing with such models are that such equations have deviating arguments and theirs states are discontinuous. In [4], Cushing pointed out that it is necessary and important to consider the models with the parameters or perturbations. This might be quite naturally exposed, for example, those due to seasonal effects of weather, food supply, mating habit, etc.
Very recently, Cheng and Zhang [1] and Wang [19] study respectively the existence, multiplicity and non-existence of positive periodic solutions for periodic differential equations without impulsive effect

\[ y'(t) = -a(t)y(t) + \lambda h(t)f(y(t - \tau(t))) \]  

\( (*) \)

and

\[ y'(t) = a(t)g(y(t))y(t) - \lambda h(t)f(y(t - \tau(t))). \]  

\( (**) \)

In mathematical ecology, Eqs. (\( *) \)) and (\( ** \)) include various models.

Cooke and Kaplan [2], Cooke and Yorke [3] studied respectively the equation

\[ y'(t) = -g(y(t)) + f(y(t - \tau)) \]

for epidemic and population growth and gonorrhea epidemics.

The equation

\[ N'(t) = -\mu N(t) + pe^{-\gamma N(t-\tau)} \]

for model of the survival of red blood cells in an animal and the equation

\[ p'(t) = -\gamma p(t) + \beta_0 \theta_n \theta_n + p(t - \tau) \]

for model of hematopoiesis (blood cell production) were studied respectively by Wazewska-Czyzewska and Lasota [20], Mackey and Glass [17], also see Gopalsamy [7], Gyori and Ladas [10] and Kuang [13].

In this paper, we investigate the existence of positive periodic solutions for the periodic impulsive functional differential equation with two parameters

\[ y'(t) = h(t, y(t)) - \lambda f(t, y(t - \tau(t))), \quad t \in R \text{ and } t \neq t_k, \]

\( (1.1)_a \)

\[ y(t_k^+) - y(t_k) = \mu I_k(t_k, y(t_k - \tau(t_k))), \quad k \in Z. \]

\( (1.1)_b \)

Our motivation to study (1.1) came from [1,19]. For some other relative works, see, for example, [6,9,11,14–16,18,21] and references cited therein.

Let \( R = (-\infty, \infty), R_+ = [0, \infty) \) and \( Z \) is the set of all integers, \( \omega > 0 \). In this paper, the following assumptions will be used:

(A1) \( \lambda > 0 \) and \( \mu \geq 0 \) are parameters.

(A2) \( \{t_k\}, k \in Z \), is an increasing sequence of real number with \( \lim_{k \to \pm \infty} t_k = \pm \infty \).

(A3) \( h \) and \( f: R \times R_+ \to R_+ \) satisfy Caratheodory conditions, that is, \( h(t, y), f(t, y) \) are locally Lebesgue measurable in \( t \) for each fixed \( y \) and are continuous in \( y \) for each fixed \( t \), are \( \omega \)-periodic functions in \( t \). Moreover, \( f(t, y) > 0 \) for all \( t \) and \( y > 0 \). \( \tau: R \to R \) is locally bounded Lebesgue measurable \( \omega \)-periodic function.

(A4) There exist \( \omega \)-periodic functions \( a_1 \) and \( a_2: R \to R_+ \) which are locally bounded Lebesgue measurable so that \( a_1(t)y \leq h(t, y) \leq a_2(t)y \) for all \( y > 0 \) and \( \lim_{y \to 0^+} \frac{h(t, y)}{y} \) exists, \( \int_0^\omega a_1(t) \, dt > 0 \).

(A5) \( I_k: R \times [0, \infty) \to R, k \in Z \), satisfy Caratheodory conditions and are \( \omega \)-periodic functions in \( t \) and there exists an integer \( \rho \) such that \( I_{k+\rho}(t_k+\rho, y) = I_k(t_k, y), t_{k+\rho} = t_k + \omega, k \in Z \). Moreover, \( I_k(t, 0) = 0 \) for all \( k \in Z \).
Definition. A function \( y : \mathbb{R} \rightarrow (0, \infty) \) is said to be a positive solution of (1.1) \(((1.1)_a \text{ and } (1.1)_b)\) if the following conditions are satisfied:

(i) \( y(t) \) is absolutely continuous on each \( (t_k, t_{k+1}) \);
(ii) for each \( k \in \mathbb{Z} \), \( y(t_k^+) \) and \( y(t_k^-) \) exist and \( y(t_k^-) = y(t_k) \);
(iii) \( y(t) \) satisfies \((1.1)_a\) for almost everywhere in \( \mathbb{R} \) and \( y(t_k) \) satisfies \((1.1)_b\) at impulsive point \( t_k, k \in \mathbb{Z} \).

The paper is organized as follows. In Section 2, we give some lemmas to prove the main results of this paper. In Section 3, existence theorems for one or two positive periodic solutions of (1.1) are established by using a well-known fixed-point index theorem due to Krasnoselskii under the conditions that \( f \) and \( I_k \) have the same limits. In Section 4, non-existence theorem are obtained for positive periodic solutions under the conditions that \( f \) and \( I_k \) have different limits. The results imply that impulsive effects can destroy the existence of positive periodic solutions of the corresponding non-impulsive equations.

2. Some lemmas

Throughout this paper, we will use the following notations:

\[
\delta_i = e^{-\int_0^\omega a_i(t)\,dt}, \quad i = 1, 2, \quad \alpha = \frac{\delta_2}{1 - \delta_2}, \quad \beta = \frac{1}{1 - \delta_1}, \quad \sigma = \frac{\alpha}{\beta},
\]

where \( a_i, i = 1, 2 \), are defined in \((A_4)\).

Let \( E \) be the Banach space \( \{y(t) : \mathbb{R} \rightarrow \mathbb{R} \text{ is continuous in } (t_k, t_{k+1}), y(t_k^+) \text{ and } y(t_k^-) \text{ exist, } y(t_k^-) = y(t_k), k \in \mathbb{Z}, \text{ and } y(t + \omega) = y(t) \} \) with \( \|y\| = \sup_{0 \leq t \leq \omega} |y(t)| \). Define \( K \) be a cone in \( E \) by

\[
K = \{y \in E : y(t) \geq \sigma \|y\|, t \in [0, \omega]\}.
\]

One may readily verify that \( K \) is a cone. That is, \( K \) is a closed non-empty subset of \( E \), \( \alpha u + \beta v \in K \) for all \( u, v \in K \) and \( \alpha, \beta \geq 0 \), and \( u, -u \in K \) imply \( u = 0 \). We also define, for a positive number \( r \), \( \Omega_r \) by

\[
\Omega_r = \{y \in K : \|y\| < r\} \quad \text{and} \quad \partial \Omega = \{y \in K : \|y\| = r\}.
\]

Finally, we define an operator \( T : K \rightarrow K \) as

\[
(Ty)(t) = \lambda \int_t^{t + \omega} G(t, s) f \left( s, y(s - \tau(s)) \right) \, ds + \mu \sum_{t \leq t_k < t + \omega} G(t, t_k) I_k(t_k, y(t_k - \tau(t_k))), \tag{2.1}
\]

where

\[
G(t, s) = \frac{e^{-\int_t^s \frac{h(u, y(u))}{y(u)} \, du}}{1 - e^{-\int_0^{\omega} \frac{h(u, y(u))}{y(u)} \, du}}. \tag{2.2}
\]

Note from \((A_4)\) and \((2.2)\)

\[
\alpha \leq G(t, s) \leq \beta, \quad t \leq s \leq t + \omega. \tag{2.3}
\]
Lemma 2.1. Assume that (A₁)–(A₅) hold. Then $T: K \to K$ is well defined.

Proof. From (2.1), it is easy to verify that $(Ty)(t)$ is continuous in $(t_k, t_{k+1})$, $(Ty)(t_k^+)$ and $(Ty)(t_k^-)$ exist and $(Ty)(t_k^-) = (Ty)(t_k)$ for each $k \in \mathbb{Z}$. Moreover, for any $y \in K$

$$
(Ty)(t + \omega) = \lambda \int_{t + \omega}^{t + 2\omega} G(t + \omega, s) f(s, y(s - \tau(s))) ds + \mu \sum_{t + \omega \leq t_k < t + 2\omega} G(t + \omega, t_k) I_k(t_k, y(t_k - \tau(t_k)))
$$

$$
= \lambda \int_{t}^{t + \omega} G(t + u + \omega, y(u + \omega - \tau(u + \omega))) du + \mu \sum_{t \leq t_k < t + \omega} G(t, t_k) I_k(t_k, y(t_k - \tau(t_k)))
$$

$$
= (Ty)(t).
$$

From (A₃) and (A₅), it is easy to show that $\int_{t}^{t + \omega} f(s, y(s - \tau(s))) ds$ and $\sum_{t \leq t_k < t + \omega} I_k(t_k, y(t_k - \tau(t_k)))$ are constants because of the periodicity of $f(t, y(t - \tau(t)))$ and $\sum_{t \leq t_k < t + \omega} I_k(t_k, y(t_k - \tau(t_k)))$. Hence for any $y \in K$ and $t \in [0, \omega]$, we obtain

$$
(Ty)(t) \geq \alpha \left[ \lambda \int_{t}^{t + \omega} f(s, y(s - \tau(s))) ds + \mu \sum_{t \leq t_k < t + \omega} I_k(t_k, y(t_k - \tau(t_k))) \right]
$$

$$
= \sigma \beta \left[ \lambda \int_{t}^{t + \omega} f(s, y(s - \tau(s))) ds + \mu \sum_{t \leq t_k < t + \omega} I_k(t_k, y(t_k - \tau(t_k))) \right] \geq \sigma \|Ty\|.
$$

Therefore, $Ty \in K$. This completes the proof of the lemma.

Lemma 2.2. Assume that (A₁)–(A₅) hold. Then $T: K \to K$ is completely continuous.

Proof. We first show that $T$ is continuous. By (A₃) and (A₅) $f$ and $I_k$ are continuous in $y$, it follows that for any $\varepsilon > 0$, let $\delta > 0$ be small enough to satisfy that if $y_1, y_2 \in K$ with $|y_1 - y_2| < \delta$

$$
|f(s, y_1(s - \tau(s))) - f(s, y_2(s - \tau(s)))| < \frac{\varepsilon}{2\beta \lambda \omega}, \quad s \in \mathbb{R},
$$

and

$$
|I_k(t_k, y_1(t_k - \tau(t_k))) - I_k(t_k, y_2(t_k - \tau(t_k)))| < \frac{\varepsilon}{2\beta \mu} \rho, \quad k \in \mathbb{Z}.
$$

Thus,
\[
\|(Ty_1)(t) - (Ty_2)(t)\| \leq \lambda \beta \int_{t}^{t+\omega} \left| f(s, y_1(s - \tau(s))) - f(s, y_2(s - \tau(s))) \right| ds \\
+ \mu \beta \sum_{t \leq t_k < t+\omega} \left| I_k(t_k, y_1(t_k - \tau(t_k))) - I_k(t_k, y_2(t_k - \tau(t_k))) \right| \\
< \epsilon,
\]
which implies that \(T\) is continuous on \(K\).

Next we show that \(T\) maps bounded sets into bounded sets. Indeed, let \(B \subset K\) be a bounded set. For any \(t \in R\) and \(y \in B\), by (2.1), we have

\[
\|(Ty)(t)\| \leq \lambda \int_{t}^{t+\omega} G(t, s) f(s, y(s - \tau(s))) ds + \mu \sum_{t \leq t_k < t+\omega} G(t, t_k) \left| I_k(t_k, y(t_k - \tau(t_k))) \right| \\
\leq \beta \left[ \lambda \int_{t}^{t+\omega} f(s, y(s - \tau(s))) ds + \mu \sum_{t \leq t_k < t+\omega} \left| I_k(t_k, y(t_k - \tau(t_k))) \right| \right] \\
= \beta \left[ \lambda \int_{0}^{\omega} f(s, y(s - \tau(s))) ds + \mu \sum_{0 \leq k < \rho} \left| I_k(t_k, y(t_k - \tau(t_k))) \right| \right].
\]

Since \(B\) is bounded, in view of the continuity of \(T\), it is follows from (2.4) that \(Ty\) is bounded and \(\{Ty : y \in B\}\) is uniformly bounded.

Finally, we show that the family of functions \(\{Ty : y \in B\}\) is equicontinuous on \([0, \omega]\). Let \(\theta_1, \theta_2 \in [0, \omega]\) with \(\theta_1 < \theta_2\). From (2.1), for any \(y \in B\), we have

\[
\|(Ty)(\theta_2) - (Ty)(\theta_1)\| \leq \lambda \left[ \int_{\theta_1+\omega}^{\theta_2+\omega} G(\theta_2, s) f(s, y(s - \tau(s))) ds \\
+ \int_{\theta_2}^{\theta_2+\omega} (G(\theta_2, s) - G(\theta_1, s)) f(s, y(s - \tau(s))) ds \\
- \int_{\theta_1}^{\theta_2} G(\theta_1, s) f(s, y(s - \tau(s))) ds \right] \\
+ \mu \left[ \sum_{\theta_1+\omega \leq t_k < \theta_2+\omega} G(\theta_2, t_k) \left| I_k(t_k, y(t_k - \tau(t_k))) \right| \\
+ \sum_{\theta_2 \leq t_k < \theta_1+\omega} (G(\theta_2, t_k) - G(\theta_1, t_k)) \left| I_k(t_k, y(t_k - \tau(t_k))) \right| \\
- \sum_{\theta_1 \leq t_k < \theta_2} G(\theta_1, t_k) \left| I_k(t_k, y(t_k - \tau(t_k))) \right| \right] \\
= \lambda \left[ \int_{\theta_1}^{\theta_2} (G(\theta_2, s) - G(\theta_1, s)) f(s, y(s - \tau(s))) ds \right]
\]
\[
\begin{align*}
&\frac{\theta_1 + \omega}{\theta_2} \left( G(\theta_2, s) - G(\theta_1, s) \right) f(s, y(s - \tau(s))) ds \\
&+ \mu \left[ \sum_{\theta_1 \leq t_k < \theta_2} (G(\theta_2, t_k) - G(\theta_1, t_k)) I_k(t_k, y(t_k - \tau(t_k))) \right] \\
&+ \sum_{\theta_2 \leq t_k < \theta_1 + \omega} (G(\theta_2, t_k) - G(\theta_1, t_k)) I_k(t_k, y(t_k - \tau(t_k))) \right].
\end{align*}
\]

Since for \( y \in B, t \in [0, \omega], 0 \leq k \leq \rho, f(t, y(t - \tau(t))) \) and \( I_k(t_k, y(t_k - \tau(t_k))) \) are uniformly bounded in \( y \), in view of (2.5), it is easy to see that when \( \theta_2 - \theta_1 \) tends to zero, \( |(Ty)(\theta_2) - (Ty)(\theta_1)| \) tends uniformly to zero in \( y \). Hence, \( \{Ty: y \in B\} \) is a family of uniformly bounded and equicontinuous functions on \([0, \omega]\). By Ascoli–Arzela theorem, the operator \( T \) is completely continuous. The proof of Lemma 2.2 is complete. \( \square \)

**Lemma 2.3.** Assume that (A1)–(A5) hold. The existence of positive \( \omega \)-periodic solution of (1.1) is equivalent to that of non-zero fixed point of \( T \) in \( K \).

**Proof.** If \( y \in K \) and \( Ty = y \) with \( y \neq 0 \), then for any \( t \neq t_k \),

\[
y'(t) = \frac{d}{dt} \left[ \lambda \int_t^{t+\omega} G(t, s) f(s, y(s - \tau(s))) ds \right]
\]

\[
= \lambda \left[ G(t, t + \omega) f(t + \omega, y(t + \omega - \tau(t + \omega))) - G(t, t) f(t, y(t - \tau(t))) \right]
\]

\[
+ \frac{h(t, y(t))}{y(t)} (Ty)(t)
\]

\[
= h(t, y(t)) - \lambda f(t, y(t - \tau(t))).
\]

For any \( t = t_j, j \in \mathbb{Z} \), we have from (2.1) that

\[
y(t_j^+) = \lambda \int_{t_j}^{t_j+\omega} G(t_j^+, s) f(s, y(s - \tau(s))) ds
\]

\[
+ \mu \sum_{t_j^+ \leq t_k < t_j + \omega} G(t_j^+, t_k) I_k(t_k, y(t_k - \tau(t_k)))
\]

\[
y(t_j) = \lambda \int_{t_j}^{t_j+\omega} G(t_j, s) f(s, y(s - \tau(s))) ds + \mu \sum_{t_j \leq t_k < t_j + \omega} G(t_j, t_k) I_k(t_k, y(t_k - \tau(t_k))).
\]

Hence

\[
y(t_j^+) - y(t_j) = \lambda \int_{t_j}^{t_j+\omega} \left[ G(t_j^+, s) - G(t_j, s) \right] f(s, y(s - \tau(s))) ds.
\]
Thus \( y(t) \) is a positive periodic solution of (1.1).

Conversely, suppose that \( y(t) \) is a positive periodic solution of (1.1). Then for any \( t \neq t_k \), it follows from (1.1) that

\[
\frac{d}{dt} \left[ y(t) \exp \left( - \int_0^t \frac{h(s,y(s))}{y(s)} \, ds \right) \right] = \lambda \exp \left( - \int_0^t \frac{h(s,y(s))}{y(s)} \, ds \right) f(t, y(t - \tau(t))).
\]

Integrating (2.6) from \( t \) to \( t + \omega \), in view of (1.1), we obtain

\[
y(t + \omega) \exp \left( - \int_0^{t+\omega} \frac{h(s,y(s))}{y(s)} \, ds \right) - y(t) \exp \left( - \int_0^t \frac{h(s,y(s))}{y(s)} \, ds \right) = \lambda \int_t^{t+\omega} \exp \left( - \int_0^s \frac{h(u,y(u))}{y(u)} \, du \right) f(s, y(s - \tau(s))) \, ds
\]

\[
+ \sum_{t \leq t_k < t + \omega} \exp \left( - \int_0^{t_k} \frac{h(u,y(u))}{y(u)} \, du \right) \left( y\left(t + \tau(t_k)\right) - y(t_k)\right).
\]

Since \( y(t + \omega) = y(t) \), it follows from (2.7) that

\[
y(t) = \lambda \int_t^{t+\omega} G(t,s) f\left(s, y\left(s - \tau(s)\right)\right) \, ds + \mu \sum_{t \leq t_k < t + \omega} G(t,t_k) I_k\left(t_k, y(t_k - \tau(t_k))\right),
\]

which implies that \((Ty)(t) = y(t)\). The proof of Lemma 2.3 is complete. \( \square \)

3. Existence of positive periodic solutions

In this section, we establish the existence of positive periodic solutions of (1.1) by using the following Krasnoselskii’s fixed index theorem on cones.

**Theorem K.** (See [5,8,12].) Let \( E \) be a Banach space and \( K \) be a cone in \( E \). For \( r > 0 \), define \( K_r = \{ y \in K : \|y\| < r \} \). Assume that \( T : K_r \to K \) is completely continuous such that \( Ty \neq y \) for \( y \in \partial K_r = \{ y \in K : \|y\| = r \} \).

(i) If \( \|Ty\| \geq \|y\| \) for \( y \in \partial K_r \), then \( i(T, K_r, K) = 0 \).

(ii) If \( \|Ty\| \leq \|y\| \) for \( y \in \partial K_r \), then \( i(T, K_r, K) = 1 \).
In the sequel, we will use the following notations and assumption (A6). For $r > 0$

\[
\begin{align*}
\bar{f}_r &= \max_{0 < y \leq r} \bar{f}(y), & f_r &= \min_{0 < y \leq r} f(y), & \widetilde{f}_r &= \max_{0 < y \leq r} \widetilde{f}(y), & L_r &= \min_{0 < y \leq r} L(y), \\
\bar{I}_r &= \max_{0 < y \leq r} \bar{I}(y), & \bar{I}_r &= \min_{0 < y \leq r} \bar{I}(y), & f^0 &= \limsup_{y \to 0^+} \bar{f}(y), & f^\infty &= \limsup_{y \to \infty} \bar{f}(y), \\
\bar{I}^0 &= \limsup_{y \to 0^+} \bar{I}(y), & \bar{I}^\infty &= \limsup_{y \to \infty} \bar{I}(y), & I_0 &= \liminf_{y \to 0^+} f(y), & I_\infty &= \liminf_{y \to \infty} f(y),
\end{align*}
\]

where

\[
\begin{align*}
\bar{f}(y) &= \max_{0 \leq t \leq \omega} \frac{f(t, y)}{p(t)y}, & f(y) &= \min_{0 \leq t \leq \omega} \frac{f(t, y)}{p(t)y}, & \widetilde{I}(y) &= \max_{0 \leq t \leq \omega, 0 \leq k < \rho} \frac{I_k(t, y)}{q(t)y}, & I(y) &= \min_{0 \leq t \leq \omega, 0 \leq k < \rho} \frac{I_k(t, y)}{q(t)y}.
\end{align*}
\]

(A6) \(p\) and \(q : R \to R_+\) are positive bounded Lebesgue measurable \(\omega\)-periodic functions and is bounded away from zero.

In the sequel, we set

\[
P = \int_0^\omega p(t) \, dt \quad \text{and} \quad Q = \sum_{0 \leq t_k < \omega} q(t_k).
\]

**Theorem 3.1.** Assume that (A1)–(A6) hold and \(f^0, f^\infty, I^0\) and \(I_\infty\) are positive constants,

\[
\beta(\lambda f^0 P + \mu I^0 Q) < 1 \quad (3.1)
\]

and

\[
\alpha \sigma(\lambda f^\infty P + \mu I_\infty Q) > 1, \quad (3.2)
\]

then (1.1) has a positive \(\omega\)-periodic solution.

**Proof.** From (3.1) we can choose \(\epsilon > 0\) so that

\[
\beta\left[\lambda(f^0 + \epsilon)P + \mu(I^0 + \epsilon)Q\right] < 1.
\]

Thus there exists \(r > 0\) such that

\[
\begin{align*}
f(t, y) &\leq (f^0 + \epsilon)p(t)y \quad \text{and} \\
I_k(t, y) &\leq (I^0 + \epsilon)q(t)y \quad \text{for } 0 \leq y \leq r, \ t \in [0, \omega], \ 0 \leq k < \rho.
\end{align*}
\]

Hence

\[
\begin{align*}
f(t, y(t)) &\leq (f^0 + \epsilon)p(t)y(t) \quad \text{and} \\
I_k(t, y(t)) &\leq (I^0 + \epsilon)q(t)y(t) \quad \text{for } y \in \partial \Omega_r, \ t \in [0, \omega], \ 0 \leq k < \rho.
\end{align*}
\]

By using (2.1) we obtain that
\[(Ty)(t) \leq \beta \left[ \lambda \int_{t}^{t+\omega} f(s, y(s - \tau(s))) \, ds + \mu \sum_{t \leq t_{k} < t+\omega} I_{k}(t_{k}, y(t_{k} - \tau(t_{k}))) \right] \]

This yields

\[\|Ty\| \leq \|y\| \quad \text{for} \quad y \in \partial \Omega_{r}.\]

On the other hand, choose \(\epsilon_{1} > 0\) so that \(f_{\infty} - \epsilon_{1} > 0, I_{0} - \epsilon_{1} > 0\) and from (3.2)

\[\alpha \sigma \left[ \lambda (f_{\infty} - \epsilon_{1}) P + \mu (I_{\infty} - \epsilon_{1}) Q \right] > 1.\]

It is easy to see that there is \(N > r\) such that

\[f(t, y) \geq (f_{0} - \epsilon_{1}) p(t) y \quad \text{and} \quad I_{k}(t, y) \geq (I_{0} - \epsilon_{1}) q(t) y \quad \text{for} \quad y \geq N, \ t \in [0, \omega], \ 0 \leq k < \rho.\]

Let \(r_{1} = \max \{2r, \frac{N}{\sigma} \}\) and it follows that \(y(t) \geq \sigma \|y\| \geq N\) for \(y \in \partial \Omega_{r_{1}}, \ t \in [0, \omega].\) From (2.1) we have

\[(Ty)(t) \geq \alpha \left[ \lambda \int_{t}^{t+\omega} f(s, y(s - \tau(s))) \, ds + \mu \sum_{t \leq t_{k} < t+\omega} I_{k}(t_{k}, y(t_{k} - \tau(t_{k}))) \right] \]

This implies

\[\|Ty\| > \|y\| \quad \text{for} \quad y \in \partial \Omega_{r_{1}}.\]

It follows from Theorem K that

\[i(T, \Omega_{r}, K) = 1 \quad \text{and} \quad i(T, \Omega_{r_{1}}, K) = 0.\]

Thus \(i(T, \Omega_{r_{1}} \setminus \Omega_{r}, K) = -1\) and \(T\) has a fixed point in \(\Omega_{r_{1}} \setminus \Omega_{r},\) which is a positive \(\omega\)-periodic solution of (1.1). This proves Theorem 3.1. \(\square\)

**Remark 3.1.** When \(\mu = 0,\) (1.1) becomes a non-impulsive equation with deviating arguments. Thus from Theorem 3.1 we have the following result.
Corollary 3.1. Assume that (A1) with \( \mu = 0 \), (A2)–(A4), (A6) hold and \( f^0, f_\infty \) are positive constants. Moreover, if
\[
\frac{1}{\alpha \sigma f_\infty P} < \lambda < \frac{1}{\beta f^0 P},
\]
then (1.1) has a positive \( \omega \)-periodic solution.

Similarly, we can prove the following theorem.

Theorem 3.2. Assume that (A1)–(A6) hold and \( f^0, f_\infty, I_0, I_\infty \) are positive constants, \( \beta (\lambda f_\infty P + \mu I_\infty Q) < 1 \) and
\[
\alpha \sigma (\lambda f^0 P + \mu I_\infty Q) > 1,
\]
then (1.1) has a positive \( \omega \)-periodic solution.

Corollary 3.2. Assume that (A1) with \( \mu = 0 \), (A2)–(A4), (A6) hold and \( f^0, f_\infty \) are positive constants. Moreover, if
\[
\frac{1}{\alpha \sigma f_\infty P} < \lambda < \frac{1}{\beta f^0 P},
\]
then (1.1) has a positive \( \omega \)-periodic solution.

Theorem 3.3. Assume that (A1)–(A6) hold and
\[
\alpha \sigma [\lambda f_r P + \mu I_r Q] > 1.
\] (3.3)

(i) If
\[
f_0 = I_0 = 0 \quad \text{or} \quad f_\infty = I_\infty = 0,
\] (3.4)
then (1.1) has a positive \( \omega \)-periodic solution.

(ii) If
\[
f_0 = I_0 = f_\infty = I_\infty = 0,
\] (3.5)
then (1.1) has two positive \( \omega \)-periodic solutions.

Proof. From (2.1) and (3.3), for \( y \in K, t \in [0, \omega], 0 \leq k < \rho \), we have that
\[
(Ty)(t) \geq \alpha \left[ \lambda \int_0^\omega f(s, y(s - \tau(s))) \, ds + \mu \sum_{0 \leq t_k < \omega} I_k(t_k, y(t_k - \tau(t_k))) \right]
\geq \alpha \sigma \left[ \lambda f_r \int_0^\omega p(s) \, ds + \mu I_r \sum_{0 \leq t_k < \rho} q(t_k) \right] \|y\|
\geq \alpha \sigma [\lambda f_r P + \mu I_r Q] \|y\| > \|y\|.
\]
This yields
\[
\|Ty\| > \|y\| \quad \text{for} \quad y \in \partial \Omega_r.
\]
(i) If \( f_0 = I_0 = 0 \) hold, then we can choose \( 0 < r_1 < r \) so that \( f(t, y) \leq \epsilon p(t) y, I_k(t, y) \leq \epsilon q(t) y \) for \( 0 \leq y \leq r_1, t \in [0, \omega] \), \( 0 \leq k < \rho \), where constant \( \epsilon > 0 \) satisfies \( \epsilon \beta(\lambda P + \mu Q) < 1 \). Thus \( f(t, y(t)) \leq \epsilon p(t) y(t), I_k(t, y(t)) \leq \epsilon q(t) y(t) \) for \( y \in \partial \Omega r_1, 0 \leq t \leq \omega, 0 \leq k < \rho \). By using (2.1), we have

\[
(Ty)(t) \leq \epsilon \beta \left[ \lambda \int_0^\omega p(t) \, dt + \mu \sum_{0 \leq k < \omega} q(t_k) \right] \|y\| < \|y\| \quad \text{for} \quad y \in \partial \Omega r_1, \ t \in [0, \omega].
\]

This implies

\[
\|Ty\| < \|y\| \quad \text{for} \quad y \in \partial \Omega r_1.
\]

It follows from Theorem K that

\[
i(T, \Omega r, K) = 0 \quad \text{and} \quad i(T, \Omega_{r_1}, K) = 1.
\]

Hence \( i(T, \Omega_r \setminus \hat{\Omega}_{r_1}, K) = -1 \) and \( T \) has a fixed point in \( \Omega_r \setminus \hat{\Omega}_{r_1} \), that is, (1.1) has a positive \( \omega \)-periodic solution.

If \( f_\infty = I_\infty = 0 \), then there is \( N > 0 \) such that \( f(t, y) \leq \epsilon_1 p(t) y \) and \( I_k(t, y) \leq \epsilon_1 q(t) y \) for \( y \geq N, 0 \leq t \leq \omega, 0 \leq k < \rho \), where \( \epsilon_1 > 0 \) satisfies \( \epsilon_1 \beta(\lambda P + \mu Q) < 1 \).

Let \( r_2 = \max\{2r, N\sigma\} \) and it follows that \( y(t) \geq \sigma \|y\| > N \) for \( y \in \Omega_{r_1}, t \in [0, \omega], 0 \leq k < \rho \).

Thus

\[
f(t, y(t)) \leq \epsilon_1 p(t) y(t) \quad \text{and} \quad I_k(t, y(t)) \leq \epsilon_1 q(t) y(t)
\]

for \( y \in \Omega r_2, t \in [0, \omega], 0 \leq k < \rho \).

From (2.1), we find

\[
(Ty)(t) \leq \epsilon_1 \beta(\lambda P + \mu Q) \|y\| < \|y\| \quad \text{for} \quad y \in \partial \Omega r_2, \ t \in [0, \omega].
\]

Thus

\[
\|Ty\| < \|y\| \quad \text{for} \quad y \in \partial \Omega r_2.
\]

Again, by Theorem K, we can obtain

\[
i(T, \Omega r, K) = 0 \quad \text{and} \quad i(T, \Omega_{r_2}, K) = 1.
\]

Hence \( i(T, \Omega_{r_2} \setminus \hat{\Omega}_r, K) = -1 \) and (1.1) has a positive \( \omega \)-periodic solution.

(ii) If (3.5) holds, it is easy to see from the above proof that \( T \) has a fixed point \( y_1 \) in \( \Omega_r \setminus \hat{\Omega}_{r_1} \) and a fixed point \( y_2 \) in \( \Omega_{r_2} \setminus \hat{\Omega}_r \) such that

\[
r_1 < \|y_1\| < r < \|y_2\| < r_2.
\]

Therefore, (1.1) has two positive \( \omega \)-periodic solutions. The proof of Theorem 3.3 is complete. \( \square \)

**Corollary 3.3.** Assume that \( (A_1) \) with \( \mu = 0, (A_2)-(A_4), (A_6) \) hold and \( \lambda > \frac{1}{\alpha \sigma \underline{I}_{x,T}} \).

(i) If (3.4) is satisfied, then (1.1) has a positive \( \omega \)-periodic solution.

(ii) If (3.5) is satisfied, then (1.1) has two positive \( \omega \)-periodic solutions.

A slight modification in the proof of Theorem 3.3 shows that the following result is also true. We omit its proof in order to avoid repetition.
Theorem 3.4. Assume that (A1)–(A6) hold and
\[ \beta(\lambda \bar{f}_r P + \mu \bar{I}_r Q) < 1. \] (3.6)

(i) If
\[ f_0 = I_0 = \infty \text{ or } f_\infty = I_\infty = \infty, \] (3.7)
then (1.1) has a positive \( \omega \)-periodic solution.

(ii) If
\[ f_0 = I_0 = f_\infty = I_\infty = \infty, \] (3.8)
then (1.1) has two positive \( \omega \)-periodic solutions.

Corollary 3.4. Assume that (A1) with \( \mu = 0 \), (A2)–(A4), (A6) hold and \( \lambda < \frac{1}{\beta \bar{f}_r P} \).

(i) If (3.7) is satisfied, then (1.1) has a positive \( \omega \)-periodic solution.

(ii) If (3.8) is satisfied, then (1.1) has two positive \( \omega \)-periodic solutions.

4. Non-existence of positive periodic solution

In this section, our aim is to obtain some sufficient conditions for non-existence of positive \( \omega \)-periodic solutions of (1.1).

Theorem 4.1. Assume that (A1)–(A6) hold. If
\[ I_0 > 0 \text{ and } I_\infty > 0, \] (4.1)
then there exists \( \mu_0 > 0 \) such that for all \( \mu > \mu_0 \), (1.1) has no positive \( \omega \)-periodic solution.

Proof. Let \( \xi_0 < \min\{I_0, I_\infty\} \), \( \xi_0 > 0 \). From (4.1) there exist positive constants \( r_1 \) and \( r_2 \) with \( r_1 < r_2 \) so that
\[ I_k(t, y) \geq \xi_0 q(t) y \text{ for } y \in [0, r_1], \ t \in [0, \omega], \ 0 \leq k < \rho, \]
and
\[ I_k(t, y) \geq \xi_0 q(t) y \text{ for } y \in [r_2, \infty), \ t \in [0, \omega], \ 0 \leq k < \rho. \]

Let
\[ \xi = \min \left\{ \xi_0, \min_{r_1 \leq y \leq r_2} \min_{0 \leq t \leq \omega, 0 \leq k < \rho} \frac{I_k(t, y)}{q(t) y} \right\}. \]

Thus
\[ I_k(t, y) \geq \xi q(t) y \text{ for } y \in [0, \infty), \ t \in [0, \omega], \ 0 \leq k < \rho. \]

Suppose, for the sake of contradiction, that \( \tilde{y}(t) \) is a positive \( \omega \)-periodic solution of (1.1). Let \( \mu_0 = \frac{1}{\alpha \sigma Q \xi} \), then by Lemma 2.3 we have
\[ \tilde{y}(t) = (T \tilde{y})(t) \geq \alpha \sigma \mu Q \xi \| \tilde{y} \| > \| \tilde{y} \|. \]

This yields
\[ \| \tilde{y} \| > \| \tilde{y} \|, \]
which is a contradiction. The proof is complete. \( \square \)
Similarly, we can prove the following results.

**Theorem 4.2.** Assume that \((A_1)-(A_6)\) hold. If
\[ I^0 < \infty \quad \text{and} \quad I^\infty < \infty, \]
then there exists \(\mu_1 > 0\) such that for all \(\mu < \mu_1\), (1.1) has no positive \(\omega\)-periodic solution.

**Theorem 4.3.** Assume that \((A_1)-(A_6)\) hold and
\[ \text{each of } f_0, f_\infty, I_0, I_\infty \text{ is not zero.} \] \(\tag{4.2}\)
Then there exists \(\eta > 0\) such that for all \(\lambda, \mu\) satisfying
\[ \alpha \sigma (\lambda P + \mu Q) > 1, \] \(\tag{4.3}\)
(1.1) has no positive \(\omega\)-periodic solution.

**Proof.** Let \(\eta_0 < \min\{f_0, I_0, f_\infty, I_\infty\}, \eta_0 > 0\). From (4.2) there exist positive constants \(r_1\) and \(r_2\) with \(r_1 < r_2\) so that
\[ f(t, y) \geq \eta_0 p(t)y, \quad I_k(t, y) \geq \eta_0 q(t)y \quad \text{for } y \in [0, r_1], \ t \in [0, \omega], \ 0 \leq k < \rho, \]
and
\[ f(t, y) \geq \eta_0 p(t)y, \quad I_k(t, y) \geq \eta_0 q(t)y \quad \text{for } y \in [r_2, \infty), \ t \in [0, \omega], \ 0 \leq k < \rho. \]
Let
\[ \eta = \min \left\{ \eta_0, \min_{r_1 \leq y \leq r_2} \min_{0 \leq t \leq \omega} \frac{f(t, y)}{p(t)y}, \min_{0 \leq t \leq \omega, 0 \leq k < \rho} \frac{I_k(t, y)}{q(t)y} \right\}. \]
Thus
\[ f(t, y) \geq \eta p(t)y, \quad I_k(t, y) \geq \eta q(t)y \quad \text{for } y \in [0, \infty), \ t \in [0, \omega], \ 0 \leq k < \rho. \]
Suppose, for the sake of contradiction, that \(\tilde{y}(t)\) is a positive \(\omega\)-periodic solution of (1.1). By Lemma 2.3, we obtain
\[ \tilde{y}(t) = (T \tilde{y})(t) \geq \alpha \sigma (\lambda P + \mu Q) \eta \|\tilde{y}\| > \|\tilde{y}\|. \]
This leads to the fact
\[ \|\tilde{y}\| > \|\tilde{y}\|, \]
which is a contradiction. The proof of Theorem 4.3 is complete. \(\square\)

Similarly, we can obtain the following result. We omit its proof to avoid repetition.

**Theorem 4.4.** Assume that \((A_1)-(A_6)\) hold and
\[ \text{each of } f^0, f^\infty, I^0, I^\infty \text{ is not infinite.} \]
Then there exists \(\epsilon > 0\) such that for all \(\lambda, \mu\) satisfying
\[ \beta (\lambda P + \mu Q) < 1, \]
(1.1) has no positive \(\omega\)-periodic solution.
5. Discussion

Theorems 4.1 and 4.2 imply that impulsive effects can destroy the existence of periodic solutions of the corresponding non-impulsive equations. For example, if (A1)–(A6) hold and $\mu = 0$, $f_0 = f_\infty = 0$, $\lambda > \frac{1}{\omega \sigma I P}$, by Theorem 3.3, (1.1) has two positive $\omega$-periodic solutions. But if (A1)–(A6) hold and (4.1) is satisfied, then for all sufficiently large $\mu > 0$ and $\lambda > 0$, (1.1) has no positive $\omega$-periodic solutions.

Conversely, we can also establish the results which are similar to Theorems 4.1–4.3 imply that the impulsive effects do not influence for the existence of the positive $\omega$-periodic solutions of (1.1).

Similarly, we can also study the existence of positive $\omega$-periodic solutions for the following equation under the same assumptions:

$$y'(t) = -h(t, y(t)) + \lambda f(t, y(t - \tau(t))), \quad t \neq t_k,$$

$$y(t_{k+}) - y(t_k) = \mu I_k(t_k, y(t_k - \tau(t_k))), \quad k \in \mathbb{Z}.$$

The method of this paper on impulsive equations is not only restricted to scalar equation, but also it can be used for systems of impulsive functional differential equations. A detailed applications of this method of such systems will be considered in a subsequent investigation.

References