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Markov Chains and λ-Invariant Measures

EUGENE ALBERT*†

Department of Mathematics, Dartmouth College, Hanover, New Hampshire

Submitted by John G. Kemeny

INTRODUCTION AND SUMMARY

$P = (p_{ij})$ shall denote an infinite matrix with constant elements, and $P^n = (p_{ij}^{(n)})$ shall denote its powers, when they exist. Until Section VII, P will not be assumed stochastic.

This paper generalizes to the nonrecurrent case Derman's result [1] on the existence of an invariant measure for a recurrent Markov chain. The Cesaro-type limit ratio, $\lim_{n\to\infty} \sum_{s=1}^{n} p_{ii}^{(s)} / \sum_{s=1}^{n} p_{00}^{(s)}$, used by Derman cannot be the appropriate one for nonrecurrent chains since now $\lim_{n\to\infty} \sum_{s=1}^{n} p_{ii}^{(s)} < \infty$ and thus the first terms exert a permanent effect as $n \to \infty$, whereas under any proper summability method the effect of individual terms should approach zero. Also, as seen in Section II, for nonrecurrent chains it is necessary to form the ratios from elements of a row of P^n rather than from the diagonal, as done by Derman.

If the individual limit ratios $\lim_{n\to\infty} p_{0i}^{(n)}/p_{00}^{(n)}$ do happen to exist, they will certainly suffice. In the general case, however, it is necessary to have a ratio summability method which "smoothes" when the individual limit ratios do not exist and which does not have the defect, discussed above, of the Cesaro-type for nonrecurrent chains. This motivates the introduction of a general type of limit-ratio: the K-limit of Section I. Some special K-limits are mentioned, of which the Euler-type E-limit will be subsequently the most useful.

The concept of K-limit is applied to matrices in Section II, where the basic quantity λ is defined. Theorem 2.1 is the main theorem of this paper. It is to be emphasized that Theorem 2.1 is valid for matrix P where the p_{ij} may be negative or even complex-valued; we have not yet specialized to stochastic

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matrices. In fact, it can be seen from the proof that the theorem is actually valid for a "matrix" of uncountably many rows and columns; countability is not used. The example of Section III shows that something like our column-finiteness (Definition 2.1) is necessary for Theorem 2.1.

Section IV presents lemmas involving binomial coefficients which will be crucial for Theorem 5.1. Theorem 5.1 will permit us to transform from the often convenient d-limits of Section V, which arise in a quite natural manner in Markov chains of period d to the more structural E-limits. The d-limit is not a K-limit, and Theorem 2.1 does not apply to d-limits, while it does to E-limits. The example of Section VI further brings out the "accidental" and nonstructural nature of the d-limit.

Finally, we specialize to stochastic matrices in Section VII. The concept of λ-invariant measure is here defined and seen to coincide with the ordinary invariant measure when $\lambda = 1$. Intuitively, the λ-invariant measure represents a "pseudo-stationary" or "pseudo-equilibrium" distribution in the sense that if the states of the Markov chain were originally provided with large numbers of particles in proportion to the values of the components of the λ-invariant measure, and the particles then move from state to state, governed by the transition probabilities of the chain, then the ratios of the numbers of particles in the various states would remain constant, though there would be a constant "rate of decay" or drift of the particles to "outside," which is measured by λ.

The theory of Section VII even applies in a significant manner to a transient (nonessential) class of a finite Markov chain (the theory is trivial for a finite ergodic (essential) class, since in this case it is not necessary to take ratios as the limit probabilities are not zero). It is seen from its proof that Theorem 7.1 is actually valid for any substochastic matrix; nowhere is the condition $\sum p_{ij} = 1$ used. For a finite chain, the d-limit vector can be shown to always exist. It then follows from Theorem 7.1 that the E-limit vector exists and is a λ-invariant measure for the substochastic matrix corresponding to the finite transient class. λ will be less than 1 in this case; the "outside" towards which the particles escape in the preceding discussion is simply the ergodic classes of states.

Periodicity in Markov chains, without bringing in any essentially new structure, often proves to be quite a nuisance. As seen in Sections V-VII, the E-limit is useful in coping with periodicity. The assumption of the existence of the d-limit vector amounts to assuming the existence of the limit ratios $\lim p_{0j}^{(n)}/p_{00}^{(n)}$, except for "phase shifts," and the E-limit then smoothes correctly (which the d-limit cannot do) over the periodic positive and zero probabilities.

On the other hand, even when $d = 1$, it is well known that the d-limit vector may not exist (see Chung [2, p. 55]). In such cases, the E-limit can play a deeper role in smoothing than merely to compensate for periodicity. It is conjectured that the E-limit may be powerful enough to guarantee the existence of E-limit vectors for all Markov chains. If this is actually false, however, it would still be desirable to know whether there exists any "universal" K-limit, which will serve for all chains; or even if an appropriate K-limit may be found which may vary with the chain.

For recurrent chains, it can be seen that the Cesaro-limit can be used (the critical equation (1.1) is automatically satisfied in this case), and that λ will be 1; thus the present paper generalizes Derman [1]. The C-limit does not exist for transient chains, since (1.1) is not satisfied.

The results are applied in Section VIII to sums of independent random variables. Illustrative examples for Theorem 8.1 are given in Section IX.

The concluding section X generalizes the concept of time-reversed Markov chain to K-limits.

D. Vere-Jones [3] has obtained some results of a related nature to those given here, by using the concept of geometric ergodicity.

I. K-Limits

For a sequence of fractions, the statement "the limit exists" shall mean that the fractions are defined for n large enough and that the limit of the sequence obtained by deleting the finite number with zero denominators exists.

Definition 1.1. Let x_n, y_n, $n = 0, 1, 2, \cdots$, be numbers and $K(n, s)$ a function of the nonnegative integers n, s. Define

$$K\text{-}\lim_{n\to\infty} (x_n ; y_n) = \lim_{n\to\infty} \frac{\sum_{s=1}^{n} K(n, s)\, x_s}{\sum_{s=1}^{n} K(n, s)\, y_s}$$

provided that the limit exists and that, for every positive integer t

$$\lim_{n\to\infty} \frac{\sum K(n, s)\, x_{s+t}}{\sum K(n, s)\, y_{s+t}} = \lim_{n\to\infty} \frac{\sum K(n, s)\, x_s}{\sum K(n, s)\, y_s} \tag{1.1}$$

In most applications, it will be true that $K(n, s) \geq 0$ and that $K(n, s) = 0$ for $s > n$.

Definition 1.2. The Cesaro limit or C-lim is the K-limit, when $K(n, s) = 1$ for $s \leq n$ and $K(n, s) = 0$ for $s > n$.

Definition 1.3. The Euler limit or E-lim is the K-limit, when $K(n, s) = \binom{n}{s}$.

Note that when $K(n, s) = 1$ for $s = n$ and $K(n, s) = 0$ for $s \neq n$, we obtain the ordinary limit.

II. K-Limit Vectors

DEFINITION 2.1. A matrix is column-finite if every column has a finite number of nonzero elements.

DEFINITION 2.2. For a given $K(n, s)$, the K-limit (row) vector of matrix P is $u = (u_i)$, where $u_i = K\text{-}\lim_{n\to\infty} (p_{0i}^{(n)}; p_{00}^{(n)})$, if it exists.

DEFINITION 2.3. $K\text{-}\lim_{n\to\infty} (p_{ij}^{(n+1)}; p_{ij}^{(n)})$, if it exists, will be denoted by λ_{ij}. λ_{00} will be denoted simply by λ, or sometimes by K-limit λ when it is necessary to call attention to the specific K-limit used.

LEMMA 2.1. *Let P have K-limit vector u with $u_i \neq 0$, $u_j \neq 0$. Let λ_{0i} and λ_{0j} exist.* Then $\lambda_{0i} = \lambda_{0j}$.

PROOF OF LEMMA 2.1:

$$\frac{\sum K(n, s)\, p_{0j}^{(s+1)}}{\sum K(n, s)\, p_{0j}^{(s)}} = \frac{\dfrac{\sum K(n, s)\, p_{0j}^{(s+1)}}{\sum K(n, s)\, p_{00}^{(s+1)}}}{\dfrac{\sum K(n, s)\, p_{0i}^{(s+1)}}{\sum K(n, s)\, p_{00}^{(s+1)}}} \; \frac{\sum K(n, s)\, p_{0i}^{(s+1)}}{\sum K(n, s)\, p_{0i}^{(s)}} \; \frac{\dfrac{\sum K(n, s)\, p_{0i}^{(s)}}{\sum K(n, s)\, p_{00}^{(s)}}}{\dfrac{\sum K(n, s)\, p_{0j}^{(s)}}{\sum K(n, s)\, p_{00}^{(s)}}}$$

is true whenever the denominators are not 0. By hypothesis, the limits of each of the three factors on the right exist and are, in fact, u_j/u_i, λ_{0i}, and u_i/u_j, respectively. (Relation (1.1), with $t = 1$, must be used for the first factor.) Since λ_{0j} is the limit of the left side, the lemma is proved.

LEMMA 2.2. *Let P be column-finite and have K-limit vector u with $u_i \neq 0$ for some i. Then λ_{0i} exists.*

PROOF OF LEMMA 2.2:

$$\frac{\sum_{s=1}^{n} K(n, s)\, p_{0i}^{(s+1+t)}}{\sum_{s=1}^{n} K(n, s)\, p_{0i}^{(s+t)}} = \frac{\sum_{s=1}^{n} K(n, s) \sum_{j} p_{0j}^{(s+t)} p_{ji}}{\sum_{s=1}^{n} K(n, s)\, p_{0i}^{(s+t)}}.$$

Since only a finite number of the p_{ji} are not 0, for fixed i, by the hypothesis of column-finiteness, the right side can be written as

$$\frac{\sum_j \sum_s K(n, s)\, p_{0j}^{(s+t)}\, p_{ji}}{\sum_s K(n, s)\, p_{0i}^{(s+t)}} = \sum_j \frac{\dfrac{\sum K(n, s)\, p_{0j}^{(s+t)}}{\sum K(n, s)\, p_{00}^{(s+t)}}}{\dfrac{\sum K(n, s)\, p_{0i}^{(s+t)}}{\sum K(n, s)\, p_{00}^{(s+t)}}}\, p_{ji}.$$

Now taking limits and again using the fact that there is only a finite number of nonzero terms in the summation because of the column-finiteness, the order of limit and summation can be interchanged, and we have

$$\lambda_{0i} = K\text{-}\lim_{n\to\infty} (p_{0i}^{(n+1)}; p_{0i}^{(n)}) = \sum_j \frac{u_j}{u_i}\, p_{ji},$$

proving the lemma.

THEOREM 2.1. *Let matrix P be column-finite and let its K-limit vector u exist. Then, for every i,* $\sum_j u_j p_{ji} = \lambda u_i$, *with* λ *as in Definition 2.3.*

PROOF OF THEOREM 2.1. Since $u_0 = 1 \neq 0$, Lemma 2.2 guarantees the existence of $\lambda_{00} = \lambda$. Now if $u_i \neq 0$, $\lambda_{0i} = \lambda$ by Lemma 2.1. Thus

$$\lambda u_i = \lambda_{0i} u_i = \lim_{n\to\infty} \frac{\sum K(n, s)\, p_{0i}^{(s+1)}}{\sum K(n, s)\, p_{0i}^{(s)}} \lim_{n\to\infty} \frac{\sum K(n, s)\, p_{0i}^{(s)}}{\sum K(n, s)\, p_{00}^{(s)}}$$

and

$$\lambda u_i = \lim_{n\to\infty} \frac{\sum K(n, s)\, p_{0i}^{(s+1)}}{\sum K(n, s)\, p_{00}^{(s)}}\,. \tag{2.1}$$

If, however, $u_i = 0$, then $\lambda u_i = 0$. We have

$$\lim_{n\to\infty} \frac{\sum K(n, s)\, p_{0i}^{(s+1)}}{\sum K(n, s)\, p_{00}^{(s)}} = \lim_{n\to\infty} \frac{\sum K(n, s)\, p_{0i}^{(s+1)}}{\sum K(n, s)\, p_{00}^{(s+1)}} \lim_{n\to\infty} \frac{\sum K(n, s)\, p_{00}^{(s+1)}}{\sum K(n, s)\, p_{00}^{(s)}}\,.$$

But the first factor on the right is u_i (using here relation (1.1), with $t = 1$). The product is then $0(\lambda) = 0$, proving that (2.1) is valid whether or not $u_i = 0$.

Now

$$\sum_j u_j p_{ji} = \sum_j \left[\lim_{n\to\infty} \frac{\sum_{s=1}^n K(n, s)\, p_{0j}^{(s)}}{\sum_{s=1}^n K(n, s)\, p_{00}^{(s)}} \right] p_{ji} = \lim_{n\to\infty} \sum_j \frac{\sum_s K(n, s)\, p_{0j}^{(s)}\, p_{ji}}{\sum_s K(n, s)\, p_{00}^{(s)}}$$

$$= \lim_{n\to\infty} \frac{\sum K(n, s)\, p_{0i}^{(s+1)}}{\sum K(n, s)\, p_{00}^{(s)}}\,,$$

since by column-finiteness, for any fixed i, only a finite number of the p_{ji}

will be nonzero, thereby permitting interchange of order of limit and summation. Thus we have proved

$$\sum_j u_j p_{ji} = \lim_{n\to\infty} \frac{\sum K(n, s)\, p_{0i}^{(s+1)}}{\sum K(n, s)\, p_{00}^{(s)}}. \tag{2.2}$$

Comparing (2.1) and (2.2) now proves the theorem.

III. Example. The Role of Column-Finiteness in Theorem 2.1

We shall consider below a certain nonrecurrent stochastic renewal-type matrix which is not column-finite. We shall prove that P has a K-limit vector u (in fact, the K-limit here will be simply the ordinary limit). However, u will not satisfy $uP = \lambda u$; indeed, there will be no positive vector v whatsoever, such that $vP = \alpha v$, for any α. This generalizes Derman [4], who considered only the case $\alpha = 1$.

Let $P = (p_{ij})$, $i, j = 0, 1, \cdots$, be a nonrecurrent, irreducible renewal-type stochastic matrix, so that, denoting $p_{k,k+1}$ by q_k, we have $0 < q_k < 1$, $p_{k0} = 1 - q_k$, and $p_{kj} = 0$ when $j \neq 0, j \neq k + 1$.

Suppose that, for some $\alpha > 0$, and some vector $v = (v_i)$, with $v_0 = 1$ and $v_i > 0$ for all i, it is true that

$$\alpha v_k = \sum_j v_j p_{jk}, \qquad \text{for every} \qquad k. \tag{3.1}$$

From (3.1), letting $k = 0$, we obtain

$$\alpha v_0 = \sum_j v_j (1 - q_j) \tag{3.2}$$

and, for $k > 0$,

$$v_k = \frac{q_0 q_1 \cdots q_{k-1}}{\alpha^k}. \tag{3.3}$$

Substituting now from (3.3) in (3.2),

$$\alpha = (1 - q_0) + \frac{q_0(1 - q_1)}{\alpha} + \frac{q_0 q_1 (1 - q_2)}{\alpha^2} + \cdots$$

or

$$0 = (1 - \alpha)\left(1 + \frac{q_0}{\alpha} + \frac{q_0 q_1}{\alpha^2} + \cdots\right)$$

But since $0 < q_i < 1$ and $\alpha > 0$, this is not possible unless $\alpha = 1$. However, according to Derman [4], for $\alpha = 1$, there exists a positive vector satisfying (3.1) only if P represents a recurrent chain. Thus, if P is renewal-type and not recurrent, there exists no positive vector satisfying (3.1) for any $\alpha > 0$.

We consider now the following specific choice for P. Let $q_0 = 2/3$, $q_{k+1} = (4q_k - 1)/3q_k$ for $k = 0, 1, \cdots$. We shall prove that

$$p_{00}^{(n)} = \frac{2^{n-1}}{3^n}. \tag{3.4}$$

Using the probability $f_{00}^{(n)}$ of first return to 0 after n steps, we have

$$f_{00}^{(n+1)} = p_{01}p_{12} \cdots p_{n-1,n}p_{n0} = p_{01}p_{12} \cdots p_{n-1,0} \frac{p_{n-1,n}p_{n0}}{p_{n-1,0}}$$
$$= p_{01}p_{12} \cdots p_{n-1,0}q_{n-1}(1 - q_n)/(1 - q_{n-1}).$$

Substituting now $q_n = (4q_{n-1} - 1)/3q_{n-1}$ from the inductive definition of q_n and simplifying, we obtain

$$f_{00}^{(n+1)} = p_{01}p_{12} \cdots p_{n-1,0}(\tfrac{1}{3}) = \frac{f_{00}^{(n)}}{3},$$

proving inductively that $f_{00}^{(n)} = 3^{-n}$ for $n > 0$, since $f_{00}^{(1)} = 1 - q_0 = 1/3$.

Now using the relation

$$p_{00}^{(k+1)} = f_{00}^{(k+1)} + f_{00}^{(k)}p_{00}^{(1)} + \cdots + f_{00}^{(1)}p_{00}^{(k)}$$

together with $f_{00}^{(n)} = 3^{-n}$, (3.4) can be proven by induction.

Since now $\sum_n p_{00}^{(n)} < \infty$ follows from (3.4), P is nonrecurrent.

It is seen from (3.4) that $p_{00}^{(n-i)}/p_{00}^{(n)}$ is independent of n, for fixed i. Since

$$\frac{p_{0i}^{(n)}}{p_{00}^{(n)}} = p_{00}^{(n-i)}p_{01}p_{12} \cdots \frac{p_{i-1,i}}{p_{00}^{(n)}},$$

it follows that $p_{0i}^{(n)}/p_{00}^{(n)}$ is independent of n, and that therefore

$$\lim_{n\to\infty} \frac{p_{0i}^{(n)}}{p_{00}^{(n)}} = u_i$$

certainly exists, and $u_i > 0$. It can similarly be seen that

$$\lim_{n\to\infty} \frac{p_{00}^{(n+1)}}{p_{00}^{(n)}} = \lambda$$

exists and, in fact, $\lambda = 2/3$.

IV. Some Lemmas Involving Binomial Coefficients

For later application to the Euler limit, we state two lemmas, the second of which is a generalization of Grossman [5]. Grossman considered only the special case where $x_n = 1$, for all n. Proofs will not be given here, but can be found in the author's doctoral dissertation at the University of Virginia (1961). The first is quite simple and straightforward. The second is extremely long and cumbersome; it would be interesting to know if a shorter proof could be found.

It is understood that, in the binomial symbol $\binom{a}{b}$, a and b are integers with $a > 0$, and $\binom{a}{b} = 0$ if $b < 0$, or if $b > a$.

Lemma 4.1. *Let d be a positive integer and let x_n be a sequence of non-negative numbers, with infinitely many not zero. Let M be a nonnegative integer and a be any integer. Then*

$$\lim_{n\to\infty} \frac{\sum_{s=M}^{\infty} \binom{n}{sd+a} x_s}{\sum_{s=0}^{\infty} \binom{n}{sd+a} x_s} = 1.$$

Lemma 4.2. *Let x_n be a sequence of nonnegative numbers, only finitely many of which are zero, such that* $\lim_{n\to\infty} x_{n+1}/x_n = X$ *exists and $X \neq 0$. Let d be a positive integer and a be any integer. Then*

$$\lim_{n\to\infty} \frac{\sum_{s=0}^{\infty} \binom{n}{sd+a} x_s}{\sum_{s=0}^{\infty} \binom{n}{sd} x_s} = X^{-a/d}.$$

V. d-Limits

Definition 5.1. Let d be a positive integer and let x_n, y_n be numbers such that $x_n = 0$ except possibly for $n \equiv a \pmod{d}$ and $y_n = 0$ except possibly for $n \equiv b \pmod{d}$, with $0 \leq a < d$, $0 \leq b < d$. Define the d-$\text{limit}_{n\to\infty}(x_n; y_n)$ as $\lim_{s\to\infty} x_{sd+a}/y_{sd+b}$, $s = 0, 1, 2, \cdots$, provided that the fractions are defined for large s, and that the limit exists.

Theorem 5.1. *Let x_n, y_n be sequences of nonnegative numbers as in the above definition. Let d-lim $(x_n; y_n)$ and d-lim $(x_{n+d}; x_n)$ both exist. Denote d-lim $(x_{n+d}; x_n)$ by X and let $X > 0$. Let r be any integer. Then E-lim $(x_n; y_n)$*

and E-lim $(x_{n+r}; x_n)$ *exist and, in fact,*

$$E\text{-lim}\,(x_n; y_n) = X^{(b-a)/d}(d\text{-lim}\,(x_n; y_n))$$

and

$$E\text{-lim}\,(x_{n+r}; x_n) = X^{r/d}.$$

PROOF OF THEOREM 5.1. Let t be any positive integer. By Lemma 4.1,

$$\lim_{n\to\infty} \frac{\sum_{s=d-r}^{\infty} \binom{n}{s} x_{s+r+t}}{\sum_{s=0}^{\infty} \binom{n}{s} x_{s+r+t}} = 1.$$

Therefore

$$\lim_{n\to\infty} \frac{\sum_{s=0}^{\infty} \binom{n}{s} x_{s+r+t}}{\sum_{s=0}^{\infty} \binom{n}{s} x_{s+t}} = \lim_{n\to\infty} \frac{\sum_{s=d-r}^{\infty} \binom{n}{s} x_{s+r+t}}{\sum_{s=0}^{\infty} \binom{n}{s} x_{s+t}}$$

$$= \lim_{n\to\infty} \frac{\sum_{s=0}^{\infty} \binom{n}{s+d-r} x_{s+d+t}}{\sum_{s=0}^{\infty} \binom{n}{s} x_{s+t}}$$

$$= \lim_{n\to\infty} \frac{\sum_{s} \binom{n}{s+d-r} x_{s+d+t}}{\sum_{s} \binom{n}{s} x_{s+d+t}} \lim_{n\to\infty} \frac{\sum_{s} \binom{n}{s} x_{s+d+t}}{\sum_{s} \binom{n}{s} x_{s+t}}$$

$$= X^{(r-d)/d} X = X^{r/d},$$

using Lemmas 4.1 and 4.2. Also,

$$\lim_{n\to\infty} \frac{\sum_{s} \binom{n}{s} x_s}{\sum_{s} \binom{n}{s} y_s} = \lim_{n\to\infty} \frac{\sum_{s} \binom{n}{sd+a} x_{sd+a}}{\sum \binom{n}{sd+b} y_{sd+b}}$$

$$= \lim_{n\to\infty} \frac{\sum \binom{n}{sd+a} x_{sd+a}}{\sum \binom{n}{sd+b} x_{sd+a}} \lim_{n\to\infty} \frac{\sum \binom{n}{sd+b} x_{sd+a}}{\sum \binom{n}{sd+b} y_{sd+b}}$$

$$= X^{(b-a)/d}(d\text{-lim}\,(x_n; y_n)),$$

using Lemmas 4.1 and 4.2. Finally, it can be easily verified that Eq. (1.1) is again satisfied here.

VI. Example Illustrating the Various Types of Limit Ratios

Let

$$x_n = 1/2, 0, 1/8, 0, 1/32, 0, \cdots$$

and

$$y_n = 0, 1/4, 0, 1/16, 0, 1/64, \cdots.$$

The ordinary $\text{limit}_{n\to\infty} (x_n/y_n)$ does not exist because of the alternating zero denominators. The C-lim $(x_n; y_n)$ does not exist, since

$$\lim_{n\to\infty} \frac{\sum_{s=0}^{n} x_s}{\sum_{s=0}^{n} y_s} = 2,$$

while

$$\lim_{n\to\infty} \frac{\sum_{s=0}^{n} x_{s+1}}{\sum_{s=0}^{n} y_{s+1}} = \tfrac{1}{2},$$

violating (1.1). The d-lim$(x_n; y_n)$ does exist and is 2. The quantity X of Theorem 5.1 is here 1/4; $a = 0$, $b = 1$, and $d = 2$. Hence, by that theorem, E-lim $(x_n; y_n) = \sqrt{1/4}\,(2) = 1$. Note that if the sequences x_n, y_n are altered by deleting just the first term of each, the d-limit changes from 2 to 1/2, while the E-limit remains 1.

VII. Stochastic Matrices

Let P be now a stochastic matrix corresponding to an irreducible Markov chain with an infinite number of states, having stationary transition probabilities and period d. For $d = 1$, we have the aperiodic case.

See Feller [6] or Chung [2] for the terminology and the facts stated below without proof.

Definition 7.1. Denote by a_{ij} the unique integer such that $0 \leq a_{ij} < d$, and $p_{ij}^{(n)} = 0$ if $n \not\equiv a_{ij}(d)$.

Note that $a_{ii} = 0$.

Definition 7.2. The d-limit (row) vector of P is the vector $u = (u_i)$ where $u_i = d\text{-lim}_{n\to\infty} (p_{0i}^{(n)}; p_{00}^{(n)})$, if this limit exists.

Using the fact that $p_{ij}^{(n)} > 0$ for $n \equiv a_{ij}(d)$ and n large enough, we see that the denominators of the fractions involved in the definition of the d-limit vector will ultimately be nonzero.

DEFINITION 7.3. A nonnegative vector u which satisfies $\sum_j u_j p_{ji} = \lambda u_i$ for every i will be called a λ-invariant measure for P.

THEOREM 7.1. *Let P be a column-finite irreducible stochastic matrix of period $d \geq 1$ for which the d-limit row vector exists. Then the d-limit vector is positive and the quantity X of Theorem 5.1 exists and is positive, so that this theorem can be applied. Hence the E-limit row vector and E-limit λ exist and are positive, and by Theorem 2.1, the E-limit vector is a λ-invariant measure of P.*

PROOF OF THEOREM 7.1. We prove first that the d-limit row vector is positive. Denote a_{0i} simply by a. Then

$$d\text{-}\lim_{n\to\infty} (p_{0i}^{(n)}; p_{00}^{(n)}) = \lim_{s\to\infty} \frac{p_{0i}^{(sd+a)}}{p_{00}^{(sd)}} .$$

Since P is irreducible, there exists an integer N such that $p_{0i}^{(Nd+a)} = \alpha > 0$. Then

$$p_{0i}^{(sd+Nd+a)} \geq p_{00}^{(sd)} p_{0i}^{(Nd+a)} = \alpha p_{00}^{(sd)}$$

and

$$\frac{p_{0i}^{(sd+Nd+a)}}{p_{00}^{(sd)}} \geq \alpha, \qquad \text{for all} \quad s. \tag{7.1}$$

Assume now the only other possibility: that d-lim $(p_{0i}^{(n)}; p_{00}^{(n)}) = 0$. Then there exists an integer $M > N$ such that $p_{0i}^{(sd+a)}/p_{00}^{(sd)} < \alpha$ for $s = M = N + t$ or $p_{0i}^{(td+Nd+a)}/p_{00}^{(sd)} < \alpha$, contradicting (7.1) and thus proving the positivity of the d-limit vector.

We can now prove the existence and positivity of X as follows.

$$\frac{p_{00}^{(sd+d)}}{p_{00}^{(sd)}} = \frac{\sum_j p_{0j}^{(sd)} p_{j0}^{(d)}}{p_{00}^{(sd)}} ,$$

the summation being taken over all indices (states). As previously remarked, $p_{00}^{(sd)} > 0$ for s large enough. We take now the limit of both sides as $s \to \infty$. Since P and hence P^d is column-finite, there are only a finite number of nonzero terms in the summation, and we can interchange the order of limit and sum, giving

$$\lim_{s\to\infty} \frac{p_{00}^{(sd+d)}}{p_{00}^{(sd)}} = \sum_j p_{j0}^{(d)} \left[\lim_{s\to\infty} \frac{p_{0j}^{(sd)}}{p_{00}^{(sd)}} \right] .$$

Now $\lim_{s\to\infty} p_{0j}^{(sd)}/p_{00}^{(sd)}$ exists for all j and is either 0, if $a_{0j} \neq 0$, or $d\text{-lim}\,(p_{0j}^{(n)}; p_{00}^{(n)})$ which exists by hypothesis, if $a_{0j} = 0$. This proves the existence of $\lim_{s\to\infty} p_{00}^{(sd+d)}/p_{00}^{(sd)}$, which is $X = d\text{-lim}\,(p_{00}^{(n+d)}; p_{00}^{(n)})$.

To prove $X > 0$, we proceed as follows.

There exists an integer k such that $a_{0k} = 0$ and $p_{k0}^{(d)} \neq 0$, since otherwise, for every j, either $a_{0j} \neq 0$ or $p_{j0}^{(d)} = 0$. For any s and any j, it would then be true that $p_{0j}^{(sd)} p_{j0}^{(d)} = 0$, since $p_{0j}^{(sd)} = 0$ if $a_{0j} \neq 0$. But then

$$p_{00}^{(sd+d)} = \sum_j p_{0j}^{(sd)} p_{j0}^{(d)} = 0,$$

and this is false for large s; so such a k must exist.

Now

$$\frac{p_{00}^{(sd+d)}}{p_{00}^{(sd)}} \geq \frac{p_{0k}^{(sd)} p_{k0}^{(d)}}{p_{00}^{(sd)}}.$$

Since $a_{0k} = 0$, then

$$d\text{-}\lim_{n\to\infty} (p_{0k}^{(n)}; p_{00}^{(n)}) = \lim_{s\to\infty} \frac{p_{0k}^{(sd)}}{p_{00}^{(sd)}}.$$

But we have already proved that the d-limit vector is positive, and

$$\lim_{s\to\infty} \frac{p_{00}^{(sd+d)}}{p_{00}^{(sd)}} \geq \left[\lim_{s\to\infty} \frac{p_{0k}^{(sd)}}{p_{00}^{(sd)}}\right] p_{k0}^{(d)} > 0,$$

completing the proof of the theorem.

VIII. Application to Sums of Independent Random Variables

Let S_n be the sum of n independent identically distributed integer-valued random variables, which take on values a_i with probability p_i. Using the notation of Kemeny [7], we define $f(s) = \sum p_i s^{a_i}$, $g(s) = sf'(s)/f(s)$, and $h(t) = g^{-1}(t)$. Then, without any essential change in the proof, equation (2) of Kemeny can be extended to the case where the Markov chain corresponding to S_n is irreducible and has arbitrary period d and becomes, taking $t = 0$, $a = i$, and $a' = 0$,

$$\lim_{n\to\infty} \frac{\Pr[S_{n+b} = i]}{\Pr[S_n = 0]} = \frac{[h(0)]^{-i}}{[f(h(0))]^{-b}}, \tag{8.1}$$

where now n is restricted by $n \equiv 0 \pmod d$ and $b \equiv a_{0i} \pmod d$, with a_{0i} as in Definition 7.1. The quantity on the left is then seen to be a d-limit ratio, and changing the notation slightly, we have, for $b = a_{0i}$,

$$d\text{-}\lim_{n\to\infty} (p_{0i}^{(n)}; p_{00}^{(n)}) = \lim_{s\to\infty} \frac{p_{0i}^{(sd+a_{0i})}}{p_{00}^{(sd)}} = \frac{[f(h(0))]^{a_{0i}}}{[h(0)]^{i}}$$

and, for $i = 0$ and $b = d$,

$$d\text{-}\lim_{n\to\infty} (p_{00}^{(n+d)}; p_{00}^{(n)}) = \lim_{s\to\infty} \frac{p_{00}^{(sd+d)}}{p_{00}^{(sd)}} = [f(h(0))]^d.$$

If now the chain is column-finite, which means that only a finite number of the a_i correspond to nonzero p_i, then Theorems 7.1 and 5.1 can be applied, and we have

$$X = d\text{-}\lim (p_{00}^{(n+d)}; p_{00}^{(n)}) = [f(h(0))]^d$$

$$\lambda = E\text{-}\lim (p_{00}^{(n+1)}; p_{00}^{(n)}) = X^{1/d} = f(h(0)$$

$$u_i = E\text{-}\lim (p_{0i}^{(n)}; p_{00}^{(n)}) = X^{-a_{0i}/d}[d\text{-}\lim (p_{0i}^{(n)}; p_{00}^{(n)})]$$

$$= \frac{[f(h(0))]^{-a_{0i}/d}\,[f(h(0))]^{a_{0i}/d}}{[h(0)]^i} = [h(0)]^{-i}.$$

We have thus proved:

THEOREM 8.1. *For an irreducible chain of period d determined by the sum of independent identically distributed integer valued random variables which take on only a finite number of distinct values with nonzero probabilities, the E-limit row vector and E-limit λ exist and are given by $u_i = [h(0)]^{-i}$ and $\lambda = f(h(0))$. The E-limit vector (a geometric sequence) is then a λ-invariant measure, by Theorem 2.1.*

IX. EXAMPLES OF λ-INVARIANT MEASURES

A. Example where $\lambda < 1$: Consider the one-dimensional generalized random walk where the particle moves either one step to the left or two steps to the right each time, with equal probabilities of 1/2. Applying Theorem 8.1, we have here

$$d = 3, \qquad p_{-1} = p_2 = \tfrac{1}{2}, \qquad f(s) = \frac{1 + s^3}{2s}, \qquad g(s) = \frac{2s^3 - 1}{s(s^3 + 1)},$$

so that $h(0) = g^{-1}(0)$ is obtained by solving $2s^3 - 1 = 0$. Thus $h(0) = 2^{-1/3}$, $u_i = 2^{i/3}$, and $\lambda = (3/2)^{5/3} < 1$.

B. Example where $\lambda = 1$ even though chain is nonrecurrent: Consider the symmetric random walk in three dimensions. By equation (12), p. 153 of Polya [8], $\lim_{n\to\infty} n^{3/2}p_{00}^{(2n)} = c$ with c a constant. Thus

$$\lim \left(\frac{n+1}{n}\right)^{3/2} \frac{p_{00}^{(2n+2)}}{p_{00}^{(2n)}} = 1,$$

and

$$\lim_{n\to\infty} \frac{p_{00}^{(2n+2)}}{p_{00}^{(2n)}} = d\text{-lim}\,(p_{00}^{(n+d)}; p_{00}^{(n)}) = 1.$$

Hence by Theorem 8.1,

$$E\text{-lim}\,(p_{00}^{(n+1)}; p_{00}^{(n)}) = \lambda = 1.$$

It can be also easily seen that $u_i \equiv 1$ is the λ-invariant measure.

X. Reverse Matrix

Definition 10.1. Consider a Markov chain given by stochastic matrix P and starting, at time 0, in state 0. The K-limit conditional probability of state j given state i after the next step is defined to be K-lim $(p_{0j}^{(n)}p_{ji}; p_{0i}^{(n+1)})$, if it exists, and will be denoted by $\hat{p}_{ij}$.

Note that $p_{0j}^{(n)}p_{ji}/p_{0i}^{(n+1)}$ is, if the denominator is not zero, the ordinary conditional probability of state j after the nth transition, given state i after the $(n+1)$st.

Theorem 10.1. *Let P be stochastic. Let the K-limit row vector u and the K-limit λ exist and let $u > 0$, $\lambda > 0$. Then, for all i, j, $\hat{p}_{ij}$ exists and equals $u_jp_{ji}/u_i\lambda$. If, in addition, u is a λ-invariant measure for P, then $\hat{P} = (\hat{p}_{ij})$ is stochastic.*

Proof of Theorem 10.1.

$$\hat{p}_{ij} = \underset{n\to\infty}{K\text{-lim}}\,(p_{0j}^{(n)}p_{ji}; p_{0i}^{(n+1)}) = p_{ji}\,\underset{n\to\infty}{K\text{-lim}}\,(p_{0j}^{(n)}; p_{0i}^{(n+1)})$$

$$= p_{ji}\lim \frac{\sum_s K(n, s)\,p_{0j}^{(s)}}{\sum_s K(n, s)\,p_{0i}^{(s+1)}}$$

$$= p_{ji}\lim_{n\to\infty} \frac{\dfrac{\sum K(n, s)\,p_{0j}^{(s)}}{\sum K(n, s)\,p_{00}^{(s)}}}{\dfrac{\sum K(n, s)\,p_{0i}^{(s)}}{\sum K(n, s)\,p_{00}^{(s)}}}\;\frac{\sum K(n, s)\,p_{0i}^{(s)}}{\sum K(n, s)\,p_{0i}^{(s+1)}}$$

$$= p_{ji}(u_j/u_i)\,(1/\lambda), \qquad \text{since} \qquad u_i \neq 0 \quad \text{and} \quad \lambda \neq 0.$$

If now $\sum u_jp_{ji} = \lambda u_i$ for all i, then

$$\sum_j \hat{p}_{ij} = \sum_j \frac{u_jp_{ji}}{u_i\lambda} = \frac{1}{u_i\lambda}\sum_j u_jp_{ji} = \frac{\lambda u_i}{u_i\lambda} = 1,$$

and $\hat{P}$ is stochastic.

COROLLARY. *Let P be a column-finite stochastic matrix which has K-limit row vector u with $u > 0$. Then $\hat{P}$ is stochastic.*

PROOF OF COROLLARY. The existence of λ and the λ-invariance of u follow from Theorem 2.1. The fact that $\lambda > 0$ also follows from $\sum_j u_j p_{ji} = \lambda u_i$ since $u_i > 0$ for all i and $p_{ji} \geq 0$.

REMARK. Under the conditions of Theorem 10.1 or its Corollary, $\hat{P}$ is stochastic and thus again defines a Markov chain. This new chain may be considered as the time-reversed chain, and generalizes the usual concept of time-reversal (see Feller [6]).

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