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The Askey scheme for hypergeometric orthogonal polynomials viewed from asymptotic analysis

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Abstract

Many limits are known for hypergeometric orthogonal polynomials that occur in the Askey scheme. We show how asymptotic representations can be derived by using the generating functions of the polynomials. For example, we discuss the asymptotic representation of the Meixner–Pollaczek, Jacobi, Meixner, and Krawtchouk polynomials in terms of Laguerre polynomials. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

It is well known that the Hermite polynomials play a crucial role in certain limits of the classical orthogonal polynomials. For example, the ultraspherical (Gegenbauer) polynomials $C_n^\gamma(x)$, which are defined by the generating function (see [8, p.155]

$$(1-2xw+w^2)^{-\gamma} = \sum_{n=0}^{\infty} C_n^\gamma(x) w^n, \quad -1 \leqslant x \leqslant 1, \quad |w|<1, \tag{1}$$

have the well-known limit

$$\lim_{\gamma\to\infty} \gamma^{-n/2} C_n^\gamma(x/\sqrt{\gamma}) = \frac{1}{n!} H_n(x). \tag{2}$$

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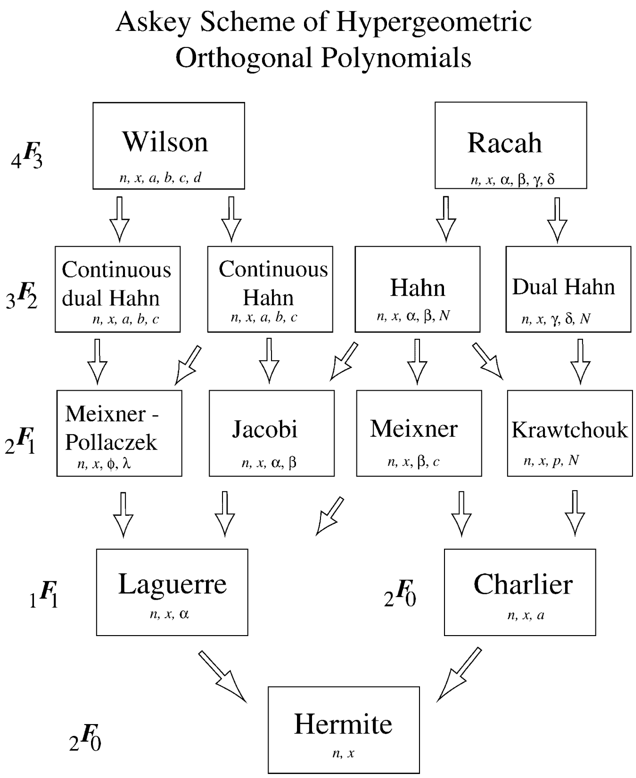

Fig. 1. The Askey scheme for hypergeometric orthogonal polynomials, with indicated limit relations between the polynomials.

For the Laguerre polynomials, which are defined by the generating function ([8, p. 155])

$$(1-w)^{-\alpha-1}\mathrm{e}^{-wx/(1-w)} = \sum_{n=0}^{\infty} L_n^{\alpha}(x)w^n, \quad |w|<1, \tag{3}$$

$\alpha, x \in \mathbb{C}$, a similar results reads

$$\lim_{\alpha\to\infty} \alpha^{-n/2} L_n^{\alpha}(x\sqrt{\alpha}+\alpha) = \frac{(-1)^n 2^{-n/2}}{n!} H_n(x/\sqrt{2}). \tag{4}$$

These limits give insight into the location of the zeros for large values of the limit parameter, and the asymptotic relation with the Hermite polynomials if the parameters γ and α become large and x is properly scaled.

Many methods are available to prove these and other limits. In this paper, we concentrate on asymptotic relations between the polynomials, from which the limits may follow as special cases.

In [3], many relations are given for hypergeometric orthogonal polynomials and their q-analogues, including limit relations between many polynomials. In Fig. 1, we show examples for which limit relations between neighboring polynomials are available, but many other limit relations are mentioned in [1–3,7].

In [4–6], we have given several asymptotic relations between polynomials and Hermite polynomials. In these first papers, we considered Gegenbauer polynomials, Laguerre polynomials, Jacobi polynomials, Tricomi–Carlitz polynomials, generalized Bernoulli polynomials, generalized Euler polynomials, generalized Bessel polynomials and Buchholz polynomials.

The method for all these cases is the same and we observe that the method also works for polynomials outside the class of hypergeometric polynomials, such as Bernoulli and Euler polynomials.

Our method is different from the one described in [1,2], where also more terms in the limit relation are constructed in order to obtain more insight in the limiting process. In these papers expansions of the form

$$P_n(x;\lambda) = \sum_{k=0}^{\infty} R_k(x;n)\lambda^{-k}$$

are considered, which generalizes the limit relation

$$\lim_{\lambda\to\infty} P_n(x;\lambda) = R_0(x;n),$$

and which gives deeper information on the limiting process. In [2], a method for the recursive computation of the coefficients $R_k(x;n)$ is designed.

In [7], similar methods are used, now in particular for limits between classical discrete (Charlier, Meixner, Krawtchouk, Hahn) to classical continuous (Jacobi, Laguerre, Hermite) orthogonal polynomials.

In current research, we investigate if other limits in the Askey scheme can be replaced by asymptotic results. Until now we verified all limits from the third level to the fourth (Laguerre and Charlier) and the fifth level (Hermite). Several limits are new, and all results have full asymptotic expansions.

2. Asymptotic representations

Starting point in our method is a generating series

$$F(x,w) = \sum_{n=0}^{\infty} p_n(x)w^n, \tag{5}$$

F is a given function, which is analytic with respect to w at $w=0$, and p_n is independent of w.

The relation (5) gives for p_n the Cauchy-type integral

$$p_n(x) = \frac{1}{2\pi i}\int_{\mathscr{C}} F(x,w)\,\frac{\mathrm{d}w}{w^{n+1}},$$

where $\mathscr{C}$ is a circle around the origin inside the domain where F is analytic (as a function of w).

We write

$$F(x,w) = \mathrm{e}^{Aw-Bw^2} f(x,w),$$

where A and B do not depend on w. This gives

$$p_n(x) = \frac{1}{2\pi i}\int_{\mathscr{C}} \mathrm{e}^{Aw-Bw^2} f(x,w)\,\frac{\mathrm{d}w}{w^{n+1}}. \tag{6}$$

Because f is also analytic (as a function of w), we can expand

$$f(x,w)=\mathrm{e}^{-Aw+Bw^2}F(x,w)=\sum_{k=0}^{\infty}c_k w^k, \tag{7}$$

that is,

$$f(x,w)=1+[p_1(x)-A]w+[p_2(x)-Ap_1(x)+B+\tfrac{1}{2}A^2]w^2+\cdots$$

if we assume that $p_0(x)=1$ (which implies $c_0=1$).

We substitute (7) into (6). The Hermite polynomials have the generating function

$$\mathrm{e}^{2xw-w^2}=\sum_{n=0}^{\infty}\frac{H_n(x)}{n!}w^n, \quad x,w\in\mathbb{C},$$

which gives the Cauchy-type integral

$$H_n(x)=\frac{n!}{2\pi i}\int_{\mathscr{C}}\mathrm{e}^{2xz-z^2}z^{-n-1}\,\mathrm{d}z, \tag{8}$$

where $\mathscr{C}$ is a circle around the origin and the integration is in positive direction. The result is the finite expansion

$$p_n(x)=z^n\sum_{k=0}^{n}\frac{c_k}{z^k}\frac{H_{n-k}(\xi)}{(n-k)!}, \quad z=\sqrt{B}, \quad \xi=\frac{A}{2\sqrt{B}}, \tag{9}$$

because terms with $k>n$ do not contribute in the integral in (6).

In order to obtain an asymptotic property of (9) we take A and B such that $c_1=c_2=0$. This happens if we take

$$A=p_1(x), \quad B=\tfrac{1}{2}p_1^2(x)-p_2(x).$$

As we will show, the asymptotic property follows from the behavior of the coefficients c_k if we take a parameter of the polynomial $p_k(x)$ large. We use the following lemma, and explain what happens by considering a few examples.

Lemma 2.1. *Let $\phi(w)$ be analytic at $w=0$, with Maclaurin expansion of the form*

$$\phi(w)=\mu w^n(a_0+a_1w+a_2w^2+\cdots),$$

where n is a positive integer and a_k are complex numbers that do not depend on the complex number $\mu, a_0\neq 0$. Let c_k denote the coefficients of the power series of $f(w)=\mathrm{e}^{\phi(w)}$, that is,

$$f(w)=\mathrm{e}^{\phi(w)}=\sum_{k=0}^{\infty}c_k w^k.$$

Then $c_0=1,\ c_k=0,\ k=1,2,\ldots,n-1$ and

$$c_k=\mathcal{O}(|\mu|^{\lfloor k/n\rfloor}), \quad \mu\to\infty.$$

Proof. The proof follows from expanding

$$\sum_{k=0}^{\infty} c_k w^k = e^{\phi(w)} = \sum_{k=0}^{\infty} \frac{[\phi(w)]^k}{k!}$$
$$= \sum_{k=0}^{\infty} \frac{\mu^k w^{kn}}{k!} (a_0 + a_1 w + a_2 w^2 + \cdots)^k,$$

and comparing equal powers of w. □

2.1. Ultraspherical polynomials

The generating function is given in (1), and we obtain

$$A = C_1^{\gamma}(x) = 2x\gamma, \quad B = \tfrac{1}{2}[C_1^{\gamma}(x)]^2 - C_2^{\gamma}(x) = \gamma(1-2x^2).$$

The expansion reads

$$C_n^{\gamma}(x) = z^n \sum_{k=0}^{n} \frac{c_k}{z^k} \frac{H_{n-k}(\xi)}{(n-k)!}, \tag{10}$$

where $z = \sqrt{\gamma(1-2x^2)}$, $\xi = x\gamma/z$. We have

$$c_0 = 1, \quad c_1 = c_2 = 0, \quad c_3 = \tfrac{2}{3}\gamma x(4x^2-3).$$

Higher coefficients follow from a recursion relation.

The function $f(x,w)$ of (7) has the form

$$f(x,w) = e^{\phi(x,w)}, \quad \phi(x,w) = \gamma w^3(a_0 + a_1 w + a_2 w^2 + \cdots).$$

By using Lemma 2.1 and $\xi = \mathcal{O}(\sqrt{\gamma})$ we conclude that the sequence $\{\phi_k\}$ with $\phi_k = c_k/z^k H_{n-k}(\xi)$ has the following asymptotic property:

$$\phi_k = \mathcal{O}(\gamma^{n/2+\lfloor k/3\rfloor - k}), \quad k = 0,1,2,\ldots .$$

This explains the asymptotic nature of the representation in (10) for large values of γ, with x and n fixed.

To verify the limit given in (2), we first write x in terms of ξ: $x = \xi/\sqrt{\gamma+2\xi^2}$. With this value of x we can verify that $c_k/z^k = o(1)$, $\gamma\to\infty$, and in fact we have the limit

$$\lim_{\gamma\to\infty} \frac{\gamma^n}{(\gamma+2x^2)^{n/2}} C_n^{\gamma}\left(\frac{x}{\sqrt{\gamma+2x^2}}\right) = \frac{1}{n!} H_n(x).$$

2.2. Laguerre polynomials

We take as generating function (see (3))

$$F(x,w) = (1+w)^{-\alpha-1} e^{wx/(1+w)} = \sum_{n=0}^{\infty} (-1)^n L_n^{\alpha}(x)\, w^n.$$

We have $A = x - \alpha - 1, \quad B = x - \frac{1}{2}(\alpha + 1)$, and we obtain

$$L_n^\alpha(x) = (-1)^n z^n \sum_{k=0}^{n} \frac{c_k}{z^k} \frac{H_{n-k}(\xi)}{(n-k)!}, \tag{11}$$

where $z = \sqrt{x - (\alpha+1)/2}$, $\xi = (x - \alpha - 1)/(2z)$. The first coefficients are

$$c_0 = 1, \quad c_1 = c_2 = 0, \quad c_3 = \tfrac{1}{3}(3x - \alpha - 1).$$

Higher coefficients follow from a recursion relation. The representation in (11) has an asymptotic character for large values of $|\alpha| + |x|$. It is not difficult to verify that the limit given in (4) follows from (11).

3. Expansions in terms Laguerre polynomials

We give examples on how to use Laguerre polynomials for approximating other polynomials. The method for the Hermite polynomials demonstrated in the previous section can be used in a similar way.

Lemma 3.1. *Let the polynomials $p_n(x)$ be defined by the generating function*

$$F(x,w) = \sum_{n=0}^{\infty} p_n(x) w^n,$$

where $F(x,w)$ is analytic in $w = 0$ and $F(x,0) = 1$. Let

$$f(x,w) = e^{-Aw/(Bw-1)}(1 - Bw)^{C+1} F(x,w),$$

and let the coefficients $c_k(x)$ be defined by the expansion

$$f(x,w) = \sum_{k=0}^{\infty} c_k(x) w^k, \quad c_0 = 1, \tag{12}$$

where A, B and C do not depend on w. Then $p_n(x)$ can be represented as the finite sum

$$p_n(x) = B^{n/2} \sum_{k=0}^{n} \frac{c_k(x)}{B^{k/2}} L_{n-k}^{(C)}(\xi), \quad \xi = \frac{A}{B}, \tag{13}$$

where $L_n^\alpha(x)$ are the Laguerre polynomials.

Proof. The polynomials $p_n(x)$ can be written as

$$p_n(x) = \frac{1}{2\pi i} \int_{\mathscr{C}} e^{Aw/(Bw-1)}(1 - Bw)^{-C-1} f(x,w) \frac{dw}{w^{n+1}},$$

where $\mathscr{C}$ is a circle around the origin in the domain where $F(x,w)$ is analytic (as a function of w). By substituting the expansion of $f(x,w)$ and using the generating function (3) of the Laguerre polynomials the proof follows. □

This time, A, B and C can be chosen such that $c_1=0,\ c_2=0,\ c_3=0$. These coefficients are given by

$$\begin{aligned}
c_1 &= p_1 - BC - B + A,\\
c_2 &= p_2 - p_1BC - p_1B + p_1A - ABC + \tfrac{1}{2}(B^2C^2 + B^2C + A^2),\\
c_3 &= p_3 - p_2BC - p_2B + p_2A - p_1ABC - \tfrac{1}{6}(B^3C^3 + B^3C + A^3)\\
&\quad + \tfrac{1}{2}(p_1B^2C^2 + p_1B^2C + p_1A^2 + AB^2C^2 - AB^2C - BA^2C + A^2B).
\end{aligned}$$

We see that the equations $c_1=0,\ c_2=0,\ c_3=0$ for solving for A, B and C are nonlinear. However, solving $c_1=0,\ c_2=0$ for A and C gives

$$A = B(C+1) - p_1, \quad C = \frac{p_1^2 - 2p_2 + 2p_1B - B^2}{B^2},$$

and with these values c_3 becomes

$$c_3 = p_3 - p_2p_1 + \tfrac{1}{3}(p_1B^2 + p_1^3 + 2p_1^2B - 4p_2B),$$

and $c_3=0$ is a quadratic equation for B.

As follows from the above representation of C, this quantity will depend on x. This gives an expansion for $p_n(x)$ in terms of Laguerre polynomials $L_k^C(\xi)$ with the order depending on x. When studying properties of $p_n(x)$ (for example investigating the zeros) this may not be very desirable. In that case, we can always take $C=\alpha$ (not depending on x), and concentrate on two equations $c_1=0,\ c_2=0$ for solving A and B. This gives

$$A = \sqrt{p_1^2 - (\alpha+1)(2p_2 - p_1^2)}, \quad B = \frac{p_1 + A}{\alpha+1}. \tag{14}$$

The order α may be chosen conveniently, without requiring $c_3=0$.

For large values of certain parameters in $p_n(x)$ expansion (13) may have an asymptotic property when taking $c_1=c_2=c_3=0$, but also when only $c_1=0$ or $c_1=c_2=0$. In the following section, we give four examples, for one level of the Askey scheme, namely for the Meixner–Pollaczek, Jacobi, Meixner, and Krawtchouk polynomials.

4. Expanding Meixner–Pollaczek into Laguerre polynomials

For the Meixner–Pollaczek polynomials we have the generating function

$$F(x,w) = (1 - e^{i\phi}w)^{-\lambda+ix}(1 - e^{-i\phi}w)^{-\lambda-ix} = \sum_{n=0}^{\infty} P_n^{(\lambda)}(x;\phi)w^n. \tag{15}$$

The expansion for the Meixner–Pollaczek polynomials reads

$$P_n^{(\lambda)}(x;\phi) = \sum_{k=0}^{n} B^{n-k}c_kL_{n-k}^{(C)}(\xi), \quad \xi = A/B, \tag{16}$$

where the coefficients c_k follow from (12) with $F(x,w)$ given in (15).

We write $x + i\lambda = re^{i\theta},\ \theta\in[0,\pi],\ r \geqslant 0$, and consider $r\to\infty$; the asymptotic results hold uniformly with respect to θ.

4.1. One free parameter

First, we consider a simple case by taking $B=1$ and $C=\alpha$, and solve $c_1=0$ for A. This gives

$$A=\alpha+1-2\lambda\cos\phi-2x\sin\phi.$$

The first coefficients c_k are given by

$$c_0=1,\quad c_1=0,\quad c_2=x\sin 2\phi+\lambda\cos 2\phi-2(x\sin\phi+\lambda\cos\phi)+\tfrac{1}{2}\alpha,$$

and the remaining ones can be obtained from the recursion

$$\begin{aligned}(k+1)c_{k+1} &= 2(1+\cos\phi)kc_k+[\alpha+1-2\lambda+4(\cos\phi-1)(\lambda\cos\phi+x\sin\phi)\\ &\quad +2(1-k)(1+2\cos\phi)]c_{k-1}+[4\lambda+2(1+\cos\phi)(k-2)\\ &\quad -2(\alpha+1)\cos\phi]c_{k-2}+(\alpha+4-k-2\lambda)c_{k-3}.\end{aligned}\tag{17}$$

The asymptotic property follows from the fact that, as in Lemma 2.1, the function $f(x,w)$ can be written as $f(x,w)=\exp[\psi(w)]$, where $\psi(w)=rw^2(a_0+a_1w+\cdots)$. Hence, the coefficients c_k have the asymptotic behaviour $c_k=\mathcal{O}(r^{\lfloor k/2\rfloor})$, as $r\to\infty$. The first-term approximation can be written as

$$P_n^{(\lambda)}(x;\phi)=L_n^{(\alpha)}(\xi)+\mathcal{O}(r^{n-1}),\quad \xi=A,\quad r\to\infty.$$

In this case, a limit for large values of r (or λ or x) cannot obtained from the above representations. We can obtain a limit by putting $\lambda=(\alpha+1)/2$. Then, as follows from the recursion relation (18), we have $c_k=\mathcal{O}(\phi^2)$ as $\phi\to 0$, and we obtain the limit of the Askey scheme

$$\lim_{\phi\to 0} P_n^{(\alpha+1)/2}[(\alpha+1)(1-\cos\phi)-\xi)/(2\sin\phi);\phi]=L_n^{(\alpha)}(\xi).$$

This includes the limit of the Askey scheme (cf. [3])

$$\lim_{\phi\to 0} P_n^{(\alpha+1)/2}(-\xi/(2\phi);\phi)=L_n^{(\alpha)}(\xi).$$

4.2. Two free parameters

Next, we solve $c_1=0$, $c_2=0$ for A and B, with $C=\alpha$. This gives (cf. (14))

$$\begin{aligned}A &= \sqrt{4(\lambda\cos\phi+x\sin\phi)^2-2(\alpha+1)(\lambda\cos 2\phi+x\sin 2\phi)},\\ B &= \frac{2(\lambda\cos\phi+x\sin\phi)+A}{\alpha+1},\end{aligned}$$

and the first-term approximation can be written as

$$P_n^{(\lambda)}(x;\phi)=L_n^{(\alpha)}(\xi)+\mathcal{O}(r^{n-2}),\quad \xi=A/B,$$

as $r\to\infty$, uniformly with respect to θ.

As an alternative, we solve $c_1=0$, $c_2=0$ for A and C, with $B=1$. This gives

$$\begin{aligned}A &= 2[x(\sin\phi-\sin 2\phi)+\lambda(\cos\phi-\cos 2\phi)],\\ C &= 2[x(2\sin\phi-\sin 2\phi)+\lambda(2\cos\phi-\cos 2\phi)]-1.\end{aligned}$$

and the first-term approximation can be written as

$$P_n^{(\lambda)}(x;\phi) = [L_n^{(\alpha)}(\xi) + \mathcal{O}(r^{n-2})], \quad \xi = A, \quad \alpha = C.$$

as $r \to \infty$, uniformly with respect to θ.

Solving $A = \xi$, $C = \alpha$ for x and λ, we obtain

$$\begin{aligned} \lambda &= (1-\cos\phi)\xi + \tfrac{1}{2}(\alpha+1)(2\cos\phi - 1), \\ x &= \frac{2(\xi-\alpha-1)\cos^2\phi + (\alpha+1-2\xi)\cos\phi + \alpha + 1 - \xi}{2\sin\phi}. \end{aligned} \tag{18}$$

Then $c_3 = \frac{2}{3}(\alpha+1-2\xi)(1-\cos\phi)$ and $c_k = \mathcal{O}(\phi^2)$ as $\phi \to 0$, which follows from deriving a recursion relation for c_k.

Using these values of x and λ, we obtain the limit

$$\lim_{\phi\to 0} P_n^{(\lambda)}(x;\phi) = L_n^{(\alpha)}(\xi).$$

4.3. Three free parameters

We solve $c_1 = 0$, $c_2 = 0$, $c_3 = 0$ for A, B and C. This gives

$$A = \frac{2\sin\phi(x\sin\phi + \lambda\cos\phi)\sqrt{x^2+\lambda^2}}{x\sin 2\phi + \lambda\cos 2\phi + \sin\phi\sqrt{x^2+\lambda^2}} = \frac{2r\sin\phi\sin\frac{1}{2}(\theta+\phi)}{\sin\frac{1}{2}(\theta+3\phi)},$$

$$B = \frac{x\sin 2\phi + \lambda\cos 2\phi + \sin\phi\sqrt{x^2+\lambda^2}}{x\sin\phi + \lambda\cos\phi} = \frac{\sin\frac{1}{2}(\theta+3\phi)}{\sin\frac{1}{2}(\theta+\phi)},$$

$$C + 1 = 2\frac{x\sin 2\phi + \lambda\cos 2\phi + 2\sin\phi\sqrt{x^2+\lambda^2}}{B^2} = \frac{2r[\sin(\theta+2\phi) + 2\sin\phi]}{B^2}.$$

The first coefficients c_k are given by

$$\begin{aligned} c_0 &= 1, \quad c_1 = c_2 = c_3 = 0, \\ c_4 &= \frac{r}{2}\{\sin(\theta+4\phi) + [\sin\phi - \sin(\theta+2\phi)]B^2\}. \end{aligned}$$

The first-term approximation can be written as

$$P_n^{(\lambda)}(x;\phi) = B^n[L_n^{(C)}(\xi) + \mathcal{O}(r^{n-3})], \quad \xi = \frac{A}{B} = \frac{2r\sin\phi}{B^2},$$

as $r \to \infty$, uniformly with respect to θ.

5. Jacobi, Meixner and Krawtchouk to Laguerre

We give the results for one free parameter only.

5.1. Jacobi to Laguerre

Let $R(w)=\sqrt{1-2xw+w^2}$. The generating function reads

$$F(x,w)=\frac{2^{\alpha+\beta}(1+R-w)^{-\alpha}(1+R+w)^{-\beta}}{R}=\sum_{n=0}^{\infty}P_n^{(\alpha,\beta)}(x)w^n.$$

As in Lemma 3.1, we define coefficients c_k, and the expansion reads

$$P_n^{(\alpha,\beta)}(x)=\sum_{k=0}^{n}B^{n-k}c_kL_{n-k}^{(C)}(\xi),\quad \xi=A/B.$$

We consider $\alpha+\beta\to\infty$, and solve $c_1=0$ for A, with $B=1$ and $C=\alpha$. This gives

$$A=\tfrac{1}{2}(\alpha+\beta+2)(1-x).$$

The first coefficients c_k are given by

$$c_0=1,\quad c_1=0,\quad c_2=\tfrac{1}{8}[-\alpha+3\beta-2(\alpha+3\beta+4)x+(3\alpha+3\beta+8)x^2].$$

The first-term approximation can be written as

$$P_n^{(\alpha,\beta)}(x)=L_n^{(C)}(\xi)+\mathcal{O}(\gamma^{n-1}),\quad \gamma=\alpha+\beta,\quad \xi=\tfrac{1}{2}(\alpha+\beta+2)(1-x).$$

A limit can be obtained by writing $x=1-2\xi/(\alpha+\beta+2)$. Then we have $c_k=\mathcal{O}(1/\beta)$ as $\beta\to\infty$ for $k\geqslant 2$, and we obtain

$$\lim_{\beta\to\infty}P_n^{(\alpha,\beta)}[1-2\xi/(2+\alpha+\beta)]=L_n^{(\alpha)}(\xi),$$

which includes the limit of the Askey scheme (cf. [3])

$$\lim_{\beta\to\infty}P_n^{(\alpha,\beta)}(1-2\xi/\beta)=L_n^{(\alpha)}(\xi).$$

5.2. Meixner to Laguerre

The generating function reads

$$F(w)=\left(1-\frac{w}{c}\right)^{x}(1-w)^{-\beta-x}=\sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}M_n(x;\beta,c)w^n,$$

and we define c_k as in Lemma 3.1. The expansion reads

$$M_n(x;\beta,c)=\sum_{k=0}^{n}B^{n-k}c_kL_{n-k}^{(C)}(\xi),\quad \xi=A/B.$$

We solve $c_1=0$ for A, with $B=1$ and $C=\alpha$. This gives

$$A=\frac{(\alpha-\beta+1)c+(1-c)x}{c}.$$

The first coefficients c_k are given by

$$c_0=1,\quad c_1=0,\quad c_2=\frac{(1+\alpha-\beta)c^2+(2c-c^2-1)x}{2c^2}.$$

The first-term approximation can be written as

$$M_n(x;\beta,c) = L_n^{(\alpha)}(\xi) + \mathcal{O}(\beta^{n-1}), \quad \xi = \frac{(\alpha-\beta+1)c + (1-c)x}{c}.$$

A limit can be obtained by putting $\beta = \alpha + 1$ and writing $x = c\xi/(1-c)$. Then we have $c_2 = (c-1)\xi/(2c)$, and $c_k = \mathcal{O}(1-c)$ as $c \to 1$ for $k \geqslant 2$. We obtain the limit of the Askey scheme (cf. [3])

$$\lim_{c\to 1} M_n(c\xi/(1-c);\alpha+1,c) = \frac{L_n^{(\alpha)}(\xi)}{L_n^{(\alpha)}(0)}.$$

5.3. Krawtchouk to Laguerre

Let $q := (1-p)/p$. The generating function reads

$$F(w) = (1-qw)^x(1+w)^{N-x} = \sum_{n=0}^{N} \binom{N}{n} K_n(x;p,N)w^n,$$

and we define c_k as in Lemma 3.1. The expansion reads

$$\binom{N}{n} K_n(x;p,N) = \sum_{k=0}^{n} B^{n-k} c_k L_{n-k}^{(C)}(\xi), \quad \xi = A/B.$$

We solve $c_1 = 0$ for A, with $B = 1$ and $C = \alpha$. This gives $A = \alpha + 1 - N + (1+q)x$. The first coefficients c_k are given by

$$c_0 = 1, \quad c_1 = 0, \quad c_2 = \tfrac{1}{2}[1 + \alpha - 3N + (3 + 2q - q^2)x].$$

The first-term approximation can be written as

$$\binom{N}{n} K_n(x;p,N) = L_n^{(C)}(\xi) + \mathcal{O}(N^{n-1}), \quad \xi = \alpha + 1 - N + (1+q)x, \quad N \to \infty.$$

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