ELSEVIER
Journal of Computational and Applied Mathematics 133 (2001) 341–353
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Quantum diffusions and Appell systems

Demosthenes Ellinas *
Department of Sciences, Section of Mathematics, Technical University of Crete, GR-73 100 Chania Crete, Greece

Received 16 November 1999; received in revised form 17 March 2000

Abstract

Within the algebraic framework of Hopf algebras, random walks and associated diffusion equations (master equations) are constructed and studied for two basic operator algebras of quantum mechanics i.e. the Heisenberg–Weyl algebra (hw) and its q-deformed version hw_q. This is done by means of functionals determined by the associated coherent state density operators. The ensuing master equations admit solutions given by hw and hw_q-valued Appell systems. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: Random walks; Hopf algebras; Appell systems

1. Introduction

We work in the general framework of the so-called quantum probability theory [36,42] and more specifically along the research line relating random walks, diffusions and Markov transition operators to Lie–Hopf algebras [32]. Our aim is to construct algebraic random walks and their diffusion limit in terms of master equations [39]. Two kinds of new results are obtained: first the Hopf algebraic formulation of random walks of [32–34] is extended to quantum random walks and their associated Lindblad-type master equations are constructed for two typical operator algebras and their solutions are studied in the framework of generalized Appell systems; second, from the physical application point of view the so-obtained master equations can be interpreted as describing the dissipative dynamics of a single (canonical and noncanonical) quantum oscillator interacting with a heathbath the properties of which are determined by choosing the underlying Hopf algebra structure and a related positive definite, linear and normalized functional given by a convex combination of selected density operators. It is obvious from the following exposition that the mathematical framework is rather broad and unifying so that it allows to construct and study the dynamics of

* Tel.: +30 821 37 269; fax: +30 821 69 522.
E-mail address: ellinas@science.tuc.gr (D. Ellinas).

PII: S0377-0427(00)00652-X

physically different random walks by making proper choices of the type of underlying algebras and functionals on them. Specifically, we work with two basic operator algebras of quantum mechanics [23] i.e. the Heisenberg–Weyl algebra (hw) and its q-deformed version hw_q [6,10,28,31], and use their Hopf-algebra-like structures for our construction (Section 2). The density of the two functionals needed is constructed by the association to those algebras coherent states vectors [27,37]. As the random walks take place on the manifold of these coherent states vectors it is important to investigate their geometrical features (Section 3). Then a limiting procedure leads to the master (diffusion) equations for the case of hw random walk (Section 5) and the case of hw_q random walk (Section 6), correspondingly. The solutions of the resulting master equations of motion for certain general elements of the respective operator algebras are obtained in terms of the associated operator-valued Appell systems [19–22]. Certain generalities of classical Appell systems are discussed in Section 4 [2,3,9,38,43]. Finally, some technicalities such as ordering formulae for generators of the two hw algebras [26], as well as some Baker–Campbell–Hausdorff decompositions formulae for the SU(1,1) group elements [40,44,45] are summarized in Appendices A and B.

2. Hopf algebras

A $*$-Hopf algebra [1] $\mathscr{A}=\mathscr{A}(\mu,\eta,\Delta,\varepsilon,S)$ over a field k is a vector space equipped with an algebra structure with homomorphic associative product map $\mu:\mathscr{A}\times\mathscr{A}\to\mathscr{A}$, and a homomorphic unit map $\eta:k\to\mathscr{A}$, that are related by $\mu\circ(\eta\otimes\mathrm{id})=\mathrm{id}=\mu\circ(\mathrm{id}\otimes\eta)$, together with a coalgebra structure with a homomorphic coassociative coproduct map $\Delta:\mathscr{A}\to\mathscr{A}\otimes\mathscr{A}$ and a homomorphic counit map $\varepsilon:\mathscr{A}\to k$, that are interrelated between by $(\varepsilon\otimes\mathrm{id})\circ\Delta=\mathrm{id}=(\mathrm{id}\otimes\varepsilon)\circ\Delta$. Both products satisfy the compatibility condition of bialgebra i.e. $(\mu\otimes\mu)\circ(\mathrm{id}\otimes\tau\otimes\mathrm{id})\circ(\Delta\otimes\Delta)=\Delta\circ\mu$, where $\tau(x\otimes y)=y\otimes x$ stands for the twist map. If η or ε is not defined in $\mathscr{A}$, we speak about nonunital or noncounital Hopf algebra.

Suppose that we have a functional $\phi:\mathscr{A}\to\mathbb{C}$, defined on $\mathscr{A}$, that is linear, positive semi-definite $\phi(a^*a)\geqslant 0$, and normalized $\phi(\mathbf{1})=1$, and let us define the operator $T_\phi:\mathscr{A}\to\mathscr{A}$ as $T_\phi=(\phi\otimes\mathrm{id})\circ\Delta$, then $\varepsilon\circ T_\phi=\phi$, namely the counit aids to pass from the operator to its associated functional. From this relation, we can define the convolution product $\psi*\phi$, between functionals as follows [33]:

$$\begin{aligned}\varepsilon\circ T_\psi T_\phi&=\varepsilon\circ(\psi\otimes\mathrm{id})\circ\Delta\circ(\phi\otimes\mathrm{id})\circ\Delta=(\phi\otimes\psi)\circ(\mathrm{id}\otimes\mathrm{id}\otimes\varepsilon)\circ(\mathrm{id}\otimes\Delta)\circ\Delta\\&=(\phi\otimes\psi)\circ\Delta=\phi*\psi,\end{aligned}\tag{1}$$

and in general $\varepsilon\circ T_\phi^n=\varepsilon\circ T_{\phi^{*n}}=\phi^{*n}$. These last relations imply that the transition operators form a discrete semigroup w.r.t. their composition with identity element $T_\varepsilon\equiv\mathrm{id}$ (due to the axioms of Hopf algebra) and generator T_ϕ, while the functionals form a dual semigroup w.r.t. the convolution with identity element e and generator ϕ, and that these two semigroups are homomorphic to each other.

We recall now two algebras and their structural maps that concern us here:

(i) *Heisenberg–Weyl algebra* hw: This is the algebra of the quantum mechanical oscillator and is generated by the creation, annihilation and the unit operator $\{a^\dagger,a,\mathbf{1}\}$, respectively, which satisfy the commutation relation (Lie bracket) $[a,a^\dagger]=\mathbf{1}$, while $\mathbf{1}$ commutes with the other elements. This algebra possesses a natural noncounital Hopf algebra structure (or bialgebra-like cf. [32], Chapter 3), with nth fold comultiplication involving n terms, defined as

$$\Delta^{(n-1)}a = n^{-1/2}(a\otimes\cdots\otimes\mathbf{1} + \mathbf{1}\otimes a\otimes\cdots\otimes\mathbf{1} + \mathbf{1}\otimes\cdots\otimes a),$$

$$\Delta^{(n-1)}a^{\dagger} = n^{-1/2}(a^{\dagger}\otimes\cdots\otimes\mathbf{1} + \mathbf{1}\otimes a^{\dagger}\otimes\cdots\otimes\mathbf{1} + \mathbf{1}\otimes\cdots\otimes a^{\dagger}),$$

$$\Delta\mathbf{1} = \mathbf{1}\otimes\mathbf{1} + \mathbf{1}\otimes\mathbf{1}. \tag{2}$$

Let us also define the so-called number operator $N = a^{\dagger}a$ with the following commutation relations with the generators of hw:

$$[a, a^{\dagger}] = \mathbf{1}, \quad [N, a^{\dagger}] = a^{\dagger}, \quad [N, a] = -a. \tag{3}$$

The module which carries the unique irreducible and infinite-dimensional representation of the oscillator algebra is the Hilbert–Fock space $\mathscr{H}_{\mathrm{F}}$ which is generated by a starting (or "vacuum") state vector $|0\rangle \in \mathscr{H}$ and is given as $\mathscr{H} = \{|n\rangle = ((a^{\dagger})^n/n!)|0\rangle,\ n \in \mathbb{Z}_+\}$.

(ii) *The q-deformed Heisenberg–Weyl algebra* hw_q: The q-deform Heisenberg–Weyl algebra is generated by the elements $\mathrm{hw}_q = \langle b, b^{\dagger}, q^N, q^{-N}, \mathbf{1}\rangle$ that satisfy the relations

$$bb^{\dagger} - q^{-1}b^{\dagger}b = q^N, \quad q^N q^{-N} = \mathbf{1},$$

$$q^N b q^{-N} = q^{-1}b, \quad q^N b^{\dagger} q^{-N} = qb^{\dagger}. \tag{4}$$

For real q the Fock representation space is spanned by the vectors $\{|n\rangle = ((b^{\dagger})^n/\sqrt{[n]_q!})|0\rangle,\ n \in \mathbb{Z}_+\}$, where $[n]_q = (q^n - q^{-n})/(q - q^{-1})$ and $[n]_q = [1]_q[2]_q\ldots[n]_q$. In the Fock space representation of this algebra, we have the additional relations $b^{\dagger}b = [N]_q$, $bb^{\dagger} = [N+1]_q$. This algebra has no satisfactory Hopf structure but still, as will be seen below, we can define algebraic random walks on it and study their diffusion limit. To this end let us make the transformations [6,10,28,31] $a_q = q^{N/2}b$ and $a_q^{\dagger} = b^{\dagger}q^{N/2}$, and obtain the resulting algebra

$$a_q a_q^{\dagger} - q^2 a_q^{\dagger} a_q = \mathbf{1}, \tag{5}$$

which is the new form of the hw_q algebra [41]. Although not an algebra homomorphism, we will use below the coassociative maps

$$\Delta a_q = a_q\otimes\mathbf{1} + \mathbf{1}\otimes a_q, \quad \Delta a_q^{\dagger} = a_q^{\dagger}\otimes\mathbf{1} + \mathbf{1}\otimes a_q^{\dagger}. \tag{6}$$

3. Functionals

Since the linear functionals on the operator algebras that we intend to build up random walks will be constructed by means of density operators given in terms of the so-called coherent states, we give here a brief introduction to the concept and collect some formulae. Let us consider a Lie group $\mathscr{G}$, with a unitary irreducible representation $T(g)$, $g \in \mathscr{G}$, in a complex Hilbert space $\mathscr{H}$. We select a reference vector $|\Psi_0\rangle \in \mathscr{H}$, to be called the "vacuum" state vector, and let $\mathscr{G}_0 \subset \mathscr{G}$ be its isotropy subgroup, i.e. for $h \in \mathscr{G}_0$, $T(h)|\Psi_0\rangle = \mathrm{e}^{\mathrm{i}\varphi(h)}|\Psi_0\rangle$. The map from the factor group $\mathscr{M} = \mathscr{G}/\mathscr{G}_0$ to the Hilbert space $\mathscr{H}$, introduced in the form of an orbit of the vacuum state under a factor group element, defines a CSV $|x\rangle = T(\mathscr{G}/\mathscr{G}_0)|\Psi_0\rangle$ labelled by points $x \in \mathscr{M}$ of the coherent state manifold. (cf. [27,37] and references therein).

What concerns us here is mostly the geometry of the CS manifold $\mathcal{M}$. This is due to the fact that the random walks and their diffusion limits that will be studied below will be given in terms of functionals associated with coherent states so that the random walks will be induced on the functions defined on $\mathcal{M}$ (passive description) or on the operators acting on the functions defined on $\mathcal{M}$ (active description). Although only the latter description will be studied here in terms of the quantum master equations, it should be obvious that the geometry of the background manifold $\mathcal{M}$ namely both the Riemannian and the symplectic geometry (the symplectic geometry especially in the case of nonstationary random walks), will manifest itself in the associated diffusion equations. Specifically, below it will be shown that the hw random walk takes place on the flat complex plane $\mathbb{C}$ with canonical symplectic structure, while the deformed hw_q random walk takes place on a q-deformed surface of revolution with modified, due to q-deformation, Riemannian and symplectic geometry. This fact provides a further motivation for studying random walks and diffusions within the present algebraic framework since in this way, we are able to study these phenomena taking place on nontrivial spaces. Detailed constructions and studies can be found elsewhere [11,14,15]; here we summarize some relevant information:

Let us first specialize to the HW group: The hw-CS is defined by the relation

$$|\alpha\rangle = \mathrm{e}^{\alpha a^\dagger - \bar{\alpha} a}|0\rangle = \mathcal{N}\mathrm{e}^{\alpha a^\dagger}|0\rangle = \mathrm{e}^{-1/2|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{7}$$

It is an (over)complete set of states with respect to the measure $\mathrm{d}\mu(\alpha) = 1/\pi e^{-|\alpha|^2}\,\mathrm{d}^2\alpha$ for the nonnormalized CS, and $\alpha \in \mathcal{M} = \mathrm{HW}/U(1) \approx \mathbb{C}$ is the CS manifold. Since $a|\alpha\rangle = \alpha|\alpha\rangle$, $\mathcal{M}$ is the flat canonical phase plane with the standard line element $\mathrm{d}s^2 = \mathrm{d}\alpha\,\mathrm{d}\bar{\alpha}$. Also, the symplectic 2-form $\omega = \mathrm{i}\mathrm{d}\alpha \wedge \mathrm{d}\bar{\alpha}$ is associated to the canonical Poisson bracket $\{f,g\} = i(\partial_\alpha^f\partial_{\bar{\alpha}}^g - \partial_{\bar{\alpha}}^f\partial_\alpha^g)$.

Next, we turn to the hw_q case: The definition of the hw_q-CS reads [11,14,15]

$$|\alpha\rangle_q = \mathrm{e}_q^{\alpha a_q^\dagger}|0\rangle = \mathrm{e}^{\alpha A_q^\dagger}|0\rangle = \sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{[n]!}}|n\rangle, \tag{8}$$

where $[n] = (q^{2n}-1)/(q^2-1)$. The states are first defined in terms of the q-deformed exponential function $\mathrm{e}_q^x = \sum_{n\geqslant 0}(x^n/[n]!)$ and the q-creation operator and then equivalently by exponentiation of the operator $A_q^\dagger = (N/[N])a_q^\dagger$, that satisfies with the hw_q elements the hw algebra relations [11,14,15]

$$[a_q, A_q^\dagger] = \mathbf{1}, \quad [A_q, a_q^\dagger] = \mathbf{1}. \tag{9}$$

The q-CS is an (over)complete set of states [8,24,25] with respect to the measure $\mathrm{d}\mu(\alpha)_q = 1/\pi(\mathrm{e}_q^{|\alpha|^2})^{-1}$ $d_q^2\alpha$, and w.r.t. the Jackson q-integral [18]. If $q = \mathrm{e}^\lambda$, then since $a_q|\alpha\rangle_q = \alpha|\alpha\rangle_q$, the q-CS manifold $\mathcal{M}$ is a nonflat surface of revolution with q-deformed induced curvature with curvature scalar $R = \lambda^2 12(1 + 2|\alpha|^2 + \mathcal{O}(\lambda^3))$. Also, the symplectic 2-form ω is modified by the q-deformation as $\omega = \{\mathrm{i} - (\lambda^2/2)|\alpha|^2(|\alpha|^2 + 2) + \mathcal{O}(\lambda^3)\}\,\mathrm{d}\alpha \wedge \mathrm{d}\bar{\alpha}$ [11,14,15].

The density operator (state) ρ will be used below to determine functionals of some Hopf operator algebras $\mathcal{A}$, so here we introduce the general concept and give its construction in terms of convex combinations of projectors of coherent states. Assume a Hilbert vector space $\mathcal{H}$ that carries a unitary irreducible representation of $\mathcal{A}$ of finite or infinite dimension. The set

$$\mathcal{S} = \{\rho \in \mathrm{End}(\mathcal{H}) \colon \rho \geqslant 0, \rho^\dagger = \rho, \mathrm{tr}\,\rho = 1\}, \tag{10}$$

namely the set of nonnegative, Hermitian, trace-one operators acting on $\mathscr{H}$ form a convex subspace of End($\mathscr{H}$), which is the convex hull of the set

$$\mathscr{S}_P = \{\rho \in \mathscr{S}, \rho^2 = \rho\} \equiv \mathscr{H}/U(1), \tag{11}$$

namely of the set of pure density operators (states), that are in one-to-one correspondence with the state vectors of $\mathscr{H}$. Two kinds of ρ density operators that will be used in the sequel are constructed by hw-CS and hw$_q$-CS. Explicitly, from the pure density operators $|\pm\alpha\rangle\langle\pm\alpha| \in \mathscr{S}_P$ and the q-deformed ones $|\alpha\rangle_{qq}\langle\alpha| \equiv |\alpha\rangle\langle\alpha|_q \in \mathscr{S}_P$, we form convex combination belonging to the convex hull of $\mathscr{S}_P$, i.e.

$$\begin{aligned} \rho &= p|\alpha\rangle\langle\alpha| + (1-p)|-\alpha\rangle\langle-\alpha|, \\ \rho_q &= p|\alpha\rangle\langle\alpha|_q + (1-p)|-\alpha\rangle\langle-\alpha|_q. \end{aligned} \tag{12}$$

4. Appell systems

Classical Appell polynomials [2,3,9,38,43] on the real line are polynomials $\{h_n(x);\ n \in \mathbb{N}\}$ of degree n that satisfy the condition $(\mathrm{d}/\mathrm{d}x)h_n(x) = nh_n(x)$. A class of such systems is the shifted moment sequences $h_n(x) = \int_{-\infty}^{\infty}(x+y)^n p(\mathrm{d}y)$, for some positive real measure p with finite moments. The class of Appell polynomials includes cases such as the divided sequences, the Bernoulli polynomials and the Hermite polynomials, which correspond to the Gaussian measure $p = p(\mathrm{d}y) = (1/\sqrt{2\pi})\mathrm{e}^{-y^2/2}\,\mathrm{d}y$. Some important properties of the Appell polynomial sets that have been investigated are the following: Hermite polynomials are the only Appell polynomials associated to the ordinary derivative operator that are also orthogonal [2,3,9,38,43]; similarly, Charlier polynomials are the only Appell systems associated to the difference operator that are also orthogonal [2,3,9,38,43] (d), while the Rogergs q-Hermite polynomials are the only Appell systems associated to the Askey–Wilson q-derivative operator that are orthogonal too [2,3,9,38,43].

The following Hopf algebraic reformulation of the real line Appell systems (i.e. nonpolynomials necessarily) motivates their generalization to more general spaces. Let $\mathscr{A} = \mathbb{R}[[X]]$ be the algebra of the real formal power series generated by pointwise multiplication $fg(x) = f(x)g(x)$, $f, g \in \mathscr{A}$. Then $\mathscr{A}$ becomes a Hopf algebra with comultiplication $(\Delta f)(x, y) = f(x + y)$ and counit $\varepsilon(\mathrm{id}) = 1$, $\varepsilon(X) = 0$, where id is the identity function and $X(x) = x$ stands for the coordinate function. For a given functional $\phi : \mathscr{A} \to \mathbb{C}$ and a chosen basis (x^n), $n \in \mathbb{Z}_+$ in $\mathscr{A}$, it is easy to verify that the relation $h_n(x) = (\phi \otimes \mathrm{id}) \circ \Delta x^n = T_\phi x^n$ defines an Appell system and is equivalent to the preceding definition. Specifically, for $\phi = \int_{-\infty}^{\infty} p(\mathrm{d}y)$ with p the Gaussian measure, we obtain the Hermite polynomials if we make the identifications $x \otimes 1 \equiv x$ and $1 \otimes x \equiv \mathrm{i}y$. This algebraic definition has been used extensively to introduce Appell systems in noncommuting algebras [19–22]. Here we will utilize it to define below Appell systems on two important operator algebras of Quantum Mechanics i.e. the Heisenberg algebra and the q-deformed Heisenberg algebra and to show that the resulting operator-valued Appell systems are solutions of quantum master equations that are constructed respectively as limits of random walks defined on these algebras.

5. Diffusion on $\mathbb{C}$

Let $\phi(\cdot)=\mathrm{Tr}\,\rho(\cdot)\equiv\langle\rho,\cdot\rangle$, a functional defined on the enveloping Heisenberg–Weyl algebra $\mathscr{U}(\mathrm{hw})$, where $\rho=p|\alpha\rangle\langle\alpha|+(1-p)|-\alpha\rangle\langle-\alpha|$, i.e. the ρ density operator is given as a convex sum of pure state density operators. The action of the transition operator $T_\phi=(\phi\otimes\mathrm{id})\circ\Delta$ on the generating monomials of $\mathscr{U}(\mathrm{hw})$ (where we ignore the numerical factors in the comultiplication of Eq. (2)) reads as

$$\begin{aligned} T_\phi((a^\dagger)^m a^n) &= (\phi\otimes\mathrm{id})\circ\Delta((a^\dagger)^m a^n) \\ &= \sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m}{i}\binom{n}{j}[p\alpha^{*i}\alpha^j+(1-p)(-\alpha)^i(-\alpha)^j](a^\dagger)^{m-i}a^{n-j} \\ &= p(a^\dagger+\alpha^*)^m(a+\alpha)^n+(1-p)(a^\dagger-\alpha^*)^m(a-\alpha)^n. \end{aligned} \tag{13}$$

For a general element $f(a,a^\dagger)\in\mathscr{U}(\mathrm{hw})$ that is normally ordered, namely the annihilation operator a is placed to the right of the creation operator $a^\dagger$, denoted by $\hat{f}(a,a^\dagger)=\sum_{m,n\geqslant 0}c_{mn}(a^\dagger)^m a^n$, the action of the linear operator T_ϕ becomes

$$T_\phi(\hat{f}(a,a^\dagger))=p\hat{f}(a+\alpha,a^\dagger+\alpha^*)+(1-p)\hat{f}(a-\alpha,a^\dagger-\alpha^*). \tag{14}$$

By means of the CS eigenvector property and the normal ordering of the f element, we also compute the value of functional viz

$$\phi(\hat{f}(a,a^\dagger))=p\hat{f}(\alpha,\alpha^*)+(1-p)\hat{f}(-\alpha,-\alpha^*). \tag{15}$$

Let us consider the displacement operator $D_\alpha=\mathrm{e}^{\alpha a^\dagger-\alpha^* a}$ which acts with the group adjoint action on any element f of the $\mathscr{U}(\mathrm{hw})$ algebra viz. [27,37]

$$AdD_a(f)=Ad\mathrm{e}^{\alpha a^\dagger-\alpha^* a}(f)=Ad\mathrm{e}^{ad(\alpha a^\dagger-\alpha^* a)}(f)=D_\alpha f D_\alpha^\dagger, \tag{16}$$

where $ad(X)f=[X,f]$ and $ad(X)ad(X)f=[X,[X,f]]$ and similarly for higher powers, stands for the Lie algebra adjoint action that is defined in terms of the Lie commutator. Explicitly, the action of the displacement operator on the generators of $\mathscr{U}(\mathrm{hw})$ reads as $AdD_{\pm\alpha}(a)=a\mp\alpha$ and $AdD_{\pm\alpha}(a^\dagger)=a^\dagger\mp\alpha^*$. By means of these expressions, we rewrite the action of the preceding transition operator as

$$T_\phi(\hat{f}(a,a^\dagger))=[pAdD_{-\alpha}+(1-p)AdD_\alpha]\hat{f}. \tag{17}$$

Next, we want to compute the limiting transition operator

$$\begin{aligned} T_t\equiv T_{\phi_t} &\equiv \lim_{n\to\infty} T_\phi^n \\ &= \lim_{n\to\infty}[p(1+ad(-\alpha a^\dagger+\alpha^* a)+\tfrac{1}{2}adad(-\alpha a^\dagger+\alpha^* a)+\cdots) \\ &\quad +(1-p)(1+ad(\alpha a^\dagger-\alpha^* a)+\tfrac{1}{2}adad(\alpha a^\dagger-\alpha^* a)+\cdots]^n. \end{aligned} \tag{18}$$

If we introduce the parameters $t\in\mathbb{R}$ and $c,\gamma\in\mathbb{C}$ by means of the relations,

$$2\alpha\left(p-\frac{1}{2}\right)=\frac{tc}{n},\qquad \frac{\alpha^2}{2}=\frac{t\gamma}{n}, \tag{19}$$

and then take $\alpha \to 0$, $n \to \infty$, with t, c, γ fixed, we use the limit $(1 + (Z/n))^n \to e^Z$, to arrive at the limiting Markov operator $T_t = e^{tad\mathscr{L}}$, where

$$\mathscr{L} = -ca^\dagger + c^* a + \gamma(a^\dagger)^2 - \gamma^* a^2 - |\gamma|(a^\dagger a + aa^\dagger). \tag{20}$$

By construction T_t is the time evolution operator for any element f of $\mathscr{U}(\mathrm{hw})$ i.e. $f_t = T_t(f)$ and forms a continuous semigroup $T_t T_{t'} = T_{t+t'}$ under composition. This yields the diffusion equation obeyed by f_t, which will be taken to be normally ordered hereafter. By time derivation of the equation

$$\phi_t(\hat{f}) = \langle \rho, \hat{f}_t \rangle = \langle \rho, e^{tad\mathscr{L}} \hat{f} \rangle = \langle e^{-tad\mathscr{L}^\dagger} \rho, \hat{f} \rangle = \langle \rho_t, \hat{f} \rangle, \tag{21}$$

we obtain the diffusion equation $(d/dt)\hat{f}_t = \mathscr{L}\hat{f}_t$, as well as the dual one satisfied by the ρ density operator viz. $(d/dt)\rho_t = \mathscr{L}^\dagger \rho_t$. To simplify and eventually solve the ensuing equations, we will assume here that the parameter γ introduced above is a complex variable with random argument of zero average and constant nonzero magnitude. Then if we average over random γ the equations of motion only the term proportional to the amplitude of γ will be retained. If in addition, we consider the case of a symmetric random walk i.e. $p = \frac{1}{2}$, $c = 0$ the equation of motion becomes

$$\frac{d}{dt}\hat{f}_t = -2|\gamma|[a^\dagger \hat{f}_t a + a\hat{f}_t a^\dagger - N\hat{f} - \hat{f}(N+1)]. \tag{22}$$

This is a quantum master equation of the Lindblad type [29,30] which will be shown to admit a solution in terms of an operator-valued Appell system associated with the generator of that equation. [1] We may introduce the following operators [4,5,12]:

$$K_+ f = a^\dagger f a, \quad K_- f = a f a^\dagger, \quad K_0 f = \tfrac{1}{2}(a^\dagger a f + f a a^\dagger), \tag{23}$$

and $K_c f = [a^\dagger a, f]$. These operators acting on the elements f of the enveloping algebra $\mathscr{U}(\mathrm{hw})$, generate the su(1,1) Lie algebra defined by the commutation relations

$$[K_-, K_+] = 2K_0, \quad [K_0, K_\pm] = \pm K_\pm, \tag{24}$$

where K_c is the central element (Casimir operator) of the algebra. In terms of these operators, the quantum master equation (22) is cast in the form

$$\frac{d}{dt}\hat{f}_t = -2|\gamma|(-2K_0 + K_+ + K_-)\hat{f}_t. \tag{25}$$

Use of the disentangling theorem (Baker–Campbell–Hausdorff formula) of a general SU(1,1) group element (cf. Appendix A), allows to express the solution of the quantum master equation in the form

$$\hat{f}_t = \exp(A_+ K_+)\exp(\ln A_0 K_0)\exp(A_- K_-)(\hat{f}) = \exp(B_- K_-)\exp(\ln B_0 K_0)\exp(B_+ K_+)(\hat{f}), \tag{26}$$

if the normally or, respectively, antinormally ordered BCH decomposition is used. In the above, $\hat{f} = \sum_{mn \geqslant 0} c_{mn}(a^\dagger)^m a^n$ stands for the initial time operator which can be a general element of the enveloping algebra $\mathscr{U}(\mathrm{hw})$. Specifically, in the case of normally ordered decomposition with initial operator taken as $\hat{f} = (a^\dagger)^m a^n$ the solution of the quantum master equation is obtained by means of the actions issued in Eq. (23) and by the antinormal-to-normal reordering relations among the

[1] It should be mentioned that this is a well-known quantum master equation and that its derivation and solution has been well studied from the mathematical and the physical points of view cf. Refs. [4,5,12,39].

generators of the $\mathscr{U}(\mathrm{hw})$ algebra (cf. Appendix B). An arduous but straightforward calculation yields the normal ordered solution:

$$\hat{f}_t = \exp(A_+K_+)\exp(\ln A_0K_0)\exp(A_-K_-)((a^\dagger)^s a^t)$$

$$= \sum_{k\geqslant 0}\sum_{l\geqslant 0}\sum_{m\geqslant 0}\sum_{i=0}^{\min(k,s)}\sum_{j=0}^{\min(k+t-i,k)}\sum_{u=0}^{l}\sum_{v=0}^{u}\sum_{q=0}^{l-u}\sum_{w=0}^{v}\sum_{f=0}^{\min(q,x)}\sum_{h=0}^{\min(y+q-f,w)}$$

$$\times\frac{A_-^k}{k!}\frac{\bar{A}_0^l}{l!}\frac{A_+^m}{m!}d^i_{k,s}d^j_{k+t-i,k}d^f_{q,x}d^h_{y+q-f,w}\bar{d}_{l-u,q}\bar{d}_{v,w}\frac{1}{2^l}\binom{l}{u}\binom{u}{v}(a^\dagger)^{x+w+m-f-h}a^{y+q+m-f-h}, \tag{27}$$

where $x = s + k + q - i - j$, $y = t + k + w - i - j$ and $\bar{A}_0 = \ln A_0$, with $A_0 = 1/(1-4|\gamma|t)$ and $A_\pm = -2|\gamma|t/(1-2|\gamma|t)$. A similar solution can be obtained for the antinormal BCH decomposition. We can therefore state the results in the following.

Proposition 1. *The solution of the quantum master equation* $(\mathrm{d}/\mathrm{d}t)\hat{f}_t = \mathscr{L}\hat{f}_t$ *where the generator* $\mathscr{L}(\hat{f}_t) = -2|\gamma|[a^\dagger\hat{f}_t a + a\hat{f}_t a^\dagger - N\hat{f} - \hat{f}(N+1)]$ *of Lindblad type generates the semigroup of Markov transition operators* $T_t = \mathrm{e}^{t\mathscr{L}}$ *acting on the enveloping algebra* $\mathscr{U}(\mathrm{hw})$, *is given by the associated* $\mathscr{U}(\mathrm{hw})$-*valued Appell system which in its normally ordered form is given by Eq.* (27).

We note also that the dual master equation satisfied by the density operator can easily be solved along the above lines in terms of the associated Appell system.

6. q-diffusion

Let $\phi_\phi(\cdot) = \mathrm{Tr}\,\rho_q(\cdot) \equiv \langle\rho_q,\cdot\rangle$, a functional defined on the enveloping q-Heisenberg–Weyl algebra $\mathscr{U}_q(\mathrm{hw})$, where $\rho_q = p|\alpha\rangle\langle\alpha|_q + (1-p)|-\alpha\rangle\langle-\alpha|_q$ is the ρ density operator given as a convex sum of pure state q-density operators. The action of transition operator $T^q_\phi = (\phi_q\otimes\mathrm{id})\circ\Delta$ on the monomials of $\mathscr{U}_q(\mathrm{hw})$, with Δ map given in Eq. (6) reads as

$$T_{\phi_q}((a^\dagger)^m_q a^n_q) = (\phi_q\otimes\mathrm{id})\circ\Delta((a^\dagger)^m_q a^n_q)$$

$$= \sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m}{i}\binom{n}{j}[p\alpha^{*i}\alpha^j + (1-p)(-\alpha)^i(-\alpha)^j](a^\dagger)^{m-i}_q a^{n-j}_q$$

$$= p(a^\dagger_q + \alpha^*)^m(a_q+\alpha)^n + (1-p)(a^\dagger_q - \alpha^*)^m(a_q-\alpha)^n. \tag{28}$$

On an element $f(a_q, a^\dagger_q)$ of the enveloping algebra $\mathscr{U}_q(\mathrm{hw})$ that is normally ordered, namely the annihilation operator a_q is placed to the right of the creation operator $a^\dagger_q$, that is expressed as $\hat{f}(a_q,a^\dagger_q) = \sum_{m,n\geqslant 0} c_{mn}(a^\dagger)^m_q a^n_q$, the action of the linear operator T_{ϕ_q} becomes

$$T_{\phi_q}(\hat{f}(a_q,a^\dagger_q)) = p\hat{f}(a_q+\alpha, a^\dagger_q+\alpha^*) + (1-p)\hat{f}(a_q-\alpha, a^\dagger_q-\alpha^*). \tag{29}$$

By means of the q-CS eigenvector property and the normal ordering of the element f, we also compute the value of functional viz.

$$\phi_q\hat{f}(a_q,a^\dagger_q) = p\hat{f}(\alpha_q,\alpha^*_q) + (1-p)\hat{f}(-\alpha,-\alpha^*). \tag{30}$$

Let us now consider the displacement operator $D_\alpha^q = e^{\alpha A_q^\dagger - \alpha^* a_q}$, which acts with the following adjoint action on any element f of the $\mathscr{U}_q(\mathrm{hw})$ algebra, $AdD_\alpha^q(f) = Ade^{\alpha A_q^\dagger - \alpha^* a_q}(f) = e^{ad(\alpha A_q^\dagger - \alpha^* a_q)}(f) = D_\alpha^q f D_{-\alpha}^q$. We should emphasize at this point that $D_\alpha^{q\dagger} \neq D_{-\alpha}^q$. This is an important difference from the preceding undeformed case with $q=1$, which stems from the fact that though Eq. (5) is valid the two involved operators are not Hermitian conjugate to each other. This fact would not permit us to proceed with the construction of quantum diffusion equation in a manner analogous to the $q=1$ case. Instead here, we will restrict the space of solutions of the resulting q-master equation from the whole algebra $\mathscr{U}_q(\mathrm{hw})$ to the commuting subalgebra generated either by monomials of the creation operator $\{(a^\dagger)_q^m, m \in \mathbb{Z}_+\}$ or of the annihilation operator $\{a_q^m, m \in \mathbb{Z}_+\}$ alone. Notice, however, that such a choice would be undesirable from the physical point of view since it would not allow us to study Hermitian solutions of the ensuing master equation.

Then the explicit action of the q-displacement operator on the generators of $\mathscr{U}_q(\mathrm{hw})$ reads as $AdD_{\pm\alpha}^q(a_q) = a_q \mp \alpha$ and $AdD_{\mp\alpha}^{q\dagger}(a_q^\dagger) = a_q^\dagger \mp \alpha^*$. By means of these expressions, we rewrite the action of the preceding q-transition operator on an analytic formal power series $f(a_q)$ as

$$T_{\phi_q}(f(a_q)) = [pAdD_{-\alpha}^q + (1-p)AdD_\alpha^q](f(a_q)). \tag{31}$$

We wish to compute the limiting transition operator

$$\begin{aligned} T_t^q &\equiv T_{\phi_t^q} \equiv \lim_{n\to\infty} (T_{\phi_q})^n \\ &= \lim_{n\to\infty} [p(1 + ad(-\alpha A_q^\dagger + \alpha^* a_q) + \tfrac{1}{2} adad(-\alpha A_q^\dagger + \alpha^* a_q) + \cdots) \\ &\quad + (1-p)(1 + ad(\alpha A_q^\dagger - \alpha^* a_q) + \tfrac{1}{2} adad(\alpha A_q^\dagger - \alpha^* a_q) + \cdots]^n. \end{aligned} \tag{32}$$

If we introduce the parameters $t \in \mathbb{R}$ and $c, \gamma \in \mathbb{C}$ by means of the same relations (19) as in the $q=1$ case, then we will obtain the limiting q-transition operator $T_t^q = e^{tad\mathscr{L}_q}$, where $\mathscr{L}_q = -cA_q^\dagger + c^* a_q + \gamma (A_q^\dagger)^2 - \gamma^* a_q^2 - |\gamma|(A_q^\dagger a_q + a_q A_q^\dagger)$.

To simplify this q-master equation, we will assume as in the undeformed case that the parameter γ is a complex variable with random argument of zero average and constant nonzero magnitude. Then if we average over random γ the equation of motion then only terms proportional to the amplitude of γ will be retained. If in addition we consider the case of a symmetric random walk i.e. $p = \frac{1}{2}, c = 0$ the equation of motion becomes

$$\frac{\mathrm{d}}{\mathrm{d}t} f_t = -2|\gamma|[A_q^\dagger f_t a + a f_t A_q^\dagger - N f_t - f_t (N+1)]. \tag{33}$$

This is a q-quantum master equation of the Lindblad type [29,30] which will be shown to admit a solution in terms of an operator-valued Appell system associated with the generator of that equation [7,13,35]. [2] We may introduce as in the preceding undeformed case the following operators:

$$K_+ f = A_q^\dagger f a_q \quad K_- f = a_q f A_q^\dagger \quad K_0 f = \tfrac{1}{2}(A_q^\dagger a_q f + f a_q A_q^\dagger) \tag{34}$$

[2] Our study of this q-deformed master equation will be constrained here to show only the potentialities of the Hopf algebraic formalism, therefore its physical interpretation as a dissipative dynamical equation or questions such as whether or not due to q-deformation its dynamics violates the uncertainty principle, or what is the form of the associated Fokker–Planck equation and how to obtain its stationary solutions, will not be addressed. Also, questions as to the possible relations of that random walk to others that have been investigated recently, with q-deformed Heisenberg–Weyl algebras and nonlinear coherent states, cf. [7,35], Ref. [13] will have to be deferred till a next communication.

and $K_c f = [A_q^\dagger a_q, f]$. These operators acting on the elements f of the enveloping algebra $\mathscr{U}_q(\mathrm{hw})$, generate the SU(1, 1) Lie algebra defined as in Eq. (24). In terms of these operators, the q-quantum master equation (33) is cast in the form

$$\frac{\mathrm{d}}{\mathrm{d}t} f_t = -2|\gamma|(-2K_0 + K_+ + K_-) f_t. \tag{35}$$

Use of the disentangling theorem (Baker–Campbell–Hausdorff formula) of a general SU(1, 1) group element (cf. Appendix A), allows to express the solution of the quantum q-master equation in the form

$$\hat{f}_t = \exp(A_+K_+)\exp(\ln A_0K_0)\exp(A_-K_-)(\hat{f}) = \exp(B_-K_-)\exp(\ln B_0K_0)\exp(B_+K_+)(\hat{f}), \tag{36}$$

if the normally or, respectively, the antinormally ordered BCH decomposition is used. In the above we choose $f = \sum_{n\geqslant 0} c_n a_q^n$, to stand for the initial time operator which can be a general element of the subalgebra of $\mathscr{U}_q(\mathrm{hw})$ that is generated by the q-annihilation operator. Specifically, in the case of normally ordered decomposition with initial operator taken as $f = a_q^t$ the solution of the quantum q-master equation is obtained by means of the actions given in Eq. (34). A straightforward calculation yields the solution:

$$\begin{aligned} f_t &= \exp(A_+K_+)\exp(\ln A_0K_0)\exp(A_-K_-)(a^t) \\ &= \sum_{k\geqslant 0}\sum_{l\geqslant 0}\sum_{m\geqslant 0}\sum_{r=0}^{l} \frac{A_-^k}{k!}\frac{\bar{A}_0^l}{l!}\frac{A_+^m}{m!}\frac{1}{2^m}\binom{l}{r}(a^\dagger)_q^{k+m}(N+k)^{l-r}(N+k+t+1)^r a_q^{k+t+m}, \end{aligned} \tag{37}$$

where the A's have the same values as before. A similar solution can be obtained for the antinormal BCH decomposition. We can therefore state the results in the following.

Proposition 2. *The solution of the quantum q-master equation* $(\mathrm{d}/\mathrm{d}t)f_t = \mathscr{L}_q f_t$, *where the operator* $\mathscr{L}_q(f_t) = -2|\gamma|[A_q^\dagger f_t a_q + a_q f_t A_q^\dagger - N f_t - f_t(N+1)]$ *of Lindblad type generates the semigroup of q-Markov transition operators* $T_t^q = \mathrm{e}^{t\mathscr{L}_q}$ *acting on the enveloping algebra* $\mathscr{U}_q(\mathrm{hw})$, *is given by the associated* $a_q^\dagger a_q$*-valued Appell system which is given by Eq.* (37).

We note also that the dual q-master equation satisfied by the density operator can easily be solved along the above lines in terms of the associated Appell system.

7. Discussion

A novel way for constructing quantum master equations has been provided with solutions given by certain sets of operator-valued functions that constitute a generalization of the concept of classical Appell polynomials. This entire approach is algebraic and utilizes concepts and tools from the powerful structure of Hopf algebra. For that construction a linear functional on the Hopf algebra is needed. The choice of that functional is determined by projection operators written in terms of coherent states that span 2D manifolds with nontrivial geometry embedded in the representation Hilbert space of the underlying algebra; this offers a chance to investigate random walks associated with nontrivial geometries. It would also be an interesting task to appropriately interpret physically the resulting quantum master equations as dissipative dynamical equation of a single quantum system. In closing, we note that the prospect of such a framework is rich enough to allow for random walks constructed on e.g. noncommuting spaces with braided/smash structure [16] or on Lie groups, quantum groups and quantum modules and comodules. The kinds of Appell systems resulting in those cases might provide new challenges to the theory of special functions. Some of these issues, however, will be taken up in a forthcoming communication [17].

Acknowledgements

Discussions with I. Tsohantjis are gratefully acknowledged.

Appendix A.

The disentangling theorem (Baker–Campbell–Hausdorff formula) [23] of a general SU(1,1) group element [40,44,45] $g(a_+,a_0,a_-)$ in the normal $\{K_+^a K_0^b K_-^c\colon a,b,c\in\mathbb{Z}_+\}$, and antinormal $\{K_-^a K_0^b K_+^c\colon a,b,c\in\mathbb{Z}_+\}$ ordering of the generators of the enveloping algebra $\mathscr{U}(su(1,1))$ reads, respectively, as

$$\begin{aligned} g(a_+,a_0,a_-) &= \exp(\alpha_+K_+ + \alpha_0K_0 + \alpha_-K_-)\\ &= \exp(A_+K_+)\exp(\ln A_0K_0)\exp(A_-K_-),\\ &= \exp(B_-K_-)\exp(\ln B_0K_0)\exp(B_+K_+), \end{aligned} \tag{38}$$

where $A_\pm(a_0)=((a_\pm/\phi)\sinh\phi)/(\cosh\phi-(a_0/2\phi)\sinh\phi)$, $A_0=(\cosh\phi-(a_0/2\phi)\sinh\phi)^{-2}$ and $B_\pm(a_0)=-A_\pm(-a_0)$, $B_0=(\cosh\phi+(a_0/2\phi)\sinh\phi)^2$, with $\phi^2=((\alpha_0/2)^2-a_+a_-)$. The relation between the two types of ordered decompositions is based on the formulae $A_\pm=(B_0B_\pm)/(1-B_0B_+B_-)$, $A_0=B_0/(1-B_0B_+B_-)^2$, and $B_\pm=A_\pm/(A_0-A_+A_-)$, $B_0=1/A_0(A_0-A_+A_-)^2$.

Appendix B.

Relations among ordered basic monomials of the enveloping algebra $\mathscr{U}(\mathrm{hw})$ [26]. From antinormal to normal ordering:

$$a^i(a^\dagger)^j=\sum_{l=0}^{\min(i,j)} d^l_{i,j}(a^\dagger)^{j-l}a^{j-l}=\sum_{l=0}^{\min(i,j)} l!\binom{i}{l}\binom{j}{l}(a^\dagger)^{j-l}a^{j-l}. \tag{39}$$

From number operator to normal ordering:

$$N^k=\sum_{l=1}^{k} c_{k,l}(a^\dagger)^l a^l, \tag{40}$$

where $\bar{c}_{k+1,l}=\bar{c}_{k,l-1}+l\bar{c}_{k,l}$, and these coefficients are recognized as the Stirling numbers of second kind.

From number operator to antinormal ordering:

$$N^k = \sum_{l=1}^{k} \bar{d}_{k,l} a^l (a^\dagger)^l, \tag{41}$$

where $\bar{d}_{k+1,l} = \bar{d}_{k,l-1} - (l+1)\bar{d}_{k,l}$, with $\bar{d}_{0,0} = 1$.

Relations among ordered basic monomials of the enveloping algebra $\mathcal{U}_q(\mathrm{hw})$ [26]. From antinormal to normal ordering:

$$a_q^i (a^\dagger)_q^j = \sum_{l=0}^{\min(i,j)} \bar{b}_{i,j}^l (a^\dagger)_q^{j-l} a_q^{j-l} = \sum_{l=0}^{\min(i,j)} q^{l(l-i-j)+ij} [l]! \begin{bmatrix} i \\ l \end{bmatrix}_q \begin{bmatrix} j \\ l \end{bmatrix}_q (a^\dagger)_q^{j-l} a_q^{j-l}. \tag{42}$$

We note that for $q \to 1$ the $\bar{b}_{i,j}^l \to d_{i,j}^l$. From normal to antinormal ordering:

$$(a^\dagger)^i a^j = \sum_{l=0}^{\min(i,j)} b_{i,j}^l a^{j-l} (a^\dagger)^{i-l} = \sum_{l=0}^{\min(i,j)} (-)^l q^{l(l-i-j)-ij)} [l]! \begin{bmatrix} i \\ l \end{bmatrix}_q \begin{bmatrix} j \\ l \end{bmatrix}_q a^{j-l} (a^\dagger)^{i-l}. \tag{43}$$

References

[1] E. Abe, Hopf Algebras, CUP, Cambridge, 1997.
[2] W.A. Al-Salam, math.CA/9402212.
[3] A. Angelesco, Bulet. Soc. din Cluj 1 (1921) 44.
[4] S.M. Barnett, Ph.D Thesis, University of London, 1985.
[5] S.M. Barnett et al., in: E.R. Pike, S. Sarkar (Eds.), Frontiers in Quantum Optics, Hilger, Bristol, 1966.
[6] L.C. Biedenham, J. Phys. A 22 (1989) L873.
[7] M. Bozejko, R. Speicher, Comm. Math. Phys. 137 (1991) 519.
[8] A.J. Bracken et al., J. Phys. A 24 (1991) 1379.
[9] L. Carlitz, Portugalie Math. 20 (1961) 43.
[10] M. Chaichian, P.P. Kulish, Phys. Lett. B 234 (1990) 72.
[11] M. Chaichian, D. Ellinas, P.P. Kulish, Phys. Rev. Lett. 65 (1990) 980.
[12] E.D. Davies, Quantum Theory of Open System, Academic Press, New York, 1973.
[13] R.L. de Matos Filho, W. Vogel, Phys. Rev. A 54 (1996) 4560.
[14] D. Ellinas, J. Phys. A 26 (1993) L543.
[15] D. Ellinas, in: Coherent States: Past, Present and Future, D.H. Feng et al. (Eds.), World Scientific, Singapore, 1994, p. 139.
[16] D. Ellinas, I. Tsohantjis, Brownian motion on smash line, LANL archive: quant ph/0001048, J. Non-Linear Math. Phys. Suppl. Issue: Proceedings of NEEDS'99, February 2001.
[17] D. Ellinas, unpublished.
[18] H. Exton, q-Hypergeometric Functions and Applications, Horwood, Chichester, 1983.
[19] P. Feinsilver, R. Schott, J. Theor. Prob. 5 (1992) 251.
[20] P. Feinsilver, U. Franz, R. Schott, J. Theor. Prob. 10 (1997) 797.
[21] U. Franz, R. Schott, J. Phys. A 31 (1998) 1395.
[22] U. Franz, R. Schott, J. Math. Phys. 39 (1998) 2748.
[23] R. Gilmore, Lie Groups, Lie Algebras and Some of Their Applications, Wiley, New York, 1974.
[24] R.W. Gray, C.A. Nelson, J. Phys. A 23 (1990) L945.
[25] B. Jurčo, Lett. Math. Phys. 21 (1991) 51.
[26] J. Katriel, G. Duchamp, J. Phys. A 28 (1995) 7209 and references therein.
[27] J.R. Klauder, B.-S. Skagerstam, Coherent States, World Scientific, Singapore, 1986.
[28] P.P. Kulish, E.V. Damaskinsky, J. Phys. A 23 (1990) L983.
[29] G. Lindblad, Comm. Math. Phys. 48 (1976) 119.

[30] G. Lindblad, Non-Equilibrium Entropy and Irreversibility, Reidel, Dordrecht, 1983.
[31] A.J. MacFarlane, J. Phys. A 22 (1989) 4581.
[32] S. Majid, Foundations of Quantum Groups Theory, Cambridge University Press, 1955 (Chapter 5).
[33] S. Majid, Int. J. Mod. Phys. 8 (1993) 4521–4545.
[34] S. Majid, M.J. Rodriguez-Plaza, J. Math. Phys. 33 (1994) 3753–3760.
[35] V.I. Mank'o, R.V. Mendes, Phys. Lett. A 180 (1993) 39.
[36] P.A. Meyer, Quantum Probability for Probabilists, Lecture Notes in Mathematics, Vol. 1538, Springer, Berlin, 1993.
[37] A. Perelomov, Generalized Coherent States and their Applications, Springer, Berlin, 1986.
[38] E. Rainville, Special Functions, MacMillan, New York, 1967.
[39] H. Risken, The Fokker–Planck Equation, Springer, Berlin, 1996.
[40] M.A. Santiago, A.N. Vaidya, J. Phys. A 9 (1976) 897.
[41] R.M. Santilli, Lie-Admissible Approach to the Hadronic Structure, Vol. I, II, Hadronic Press, Nonantum 1978, 1983. In this works deformations of the ***hw*** algebra and earlier references can be found.
[42] M. Schürmann, White Noise on Bialgebras, Lecture Notes in Mathematics, Vol. 1544, Springer, Berlin, 1993.
[43] J. Shobat, Amer. J. Math. 58 (1936) 453.
[44] R.M. Wilcox, J. Math. Phys. 8 (1967) 962.
[45] W. Witschel, J. Phys. A 7 (1974) 1847.